SPECTRAL SHIFT FUNCTION AND RESONANCES FOR THE SCHRÖDINGER OPERATOR WITH NON-DECAYING POTENTIALS

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1. Introduction

This is a survey of some old and new results of the author. Some of them will be published elswhere. Consider the Schrödinger operators in $L^2(\mathbb{R}^n)$

$$H_j(h) = -h^2\Delta + V_j(x), \quad j = 1, 2, \tag{1.1}$$

where h is a small parameter and V_1, V_2 are real-valued bounded smooth potentials with difference $V_2(x) - V_1(x)$ of order $\mathcal{O}(|x|^{-\rho})$ as $|x| \to \infty$ for some $\rho > n$. The spectral shift function (SSF for short) correspondingly to the operators $(H_2(h), H_1(h))$ is defined as a distribution, $\xi(\lambda, h)$, on $\mathbb{R}_\lambda$ by the relation

$$\langle \xi'(\cdot, h), f\rangle := -\operatorname{tr}\left(f(H_2(h)) - f(H_1(h))\right), \ \forall f \in C_0^\infty(\mathbb{R}; \mathbb{R}), \tag{1.2}$$

with a normalization condition $\xi(\lambda, h) = 0$ for $\lambda < \inf\Big(\sigma(H_1(h)) \cup \sigma(H_2(h))\Big)$. The SSF plays an important role in perturbation theory for self-adjoint operators. When $f(H_1(h)) = 0$, it coincides with the eigenvalue counting function of H_2 for $\lambda \in \operatorname{supp} f$. It was introduced in a special case by I. Lifshitz [22] and generalized by M. Krein in [20]. The background of the SSF theory can be found in [34].

In the last thirty years, the asymptotic behavior of the SSF of the Schrödinger operator with a long-range or short-range potential has intensively been studied in different aspects. In the semi-classical regime, $h \searrow 0$, the Weyl type asymptotics of $\xi(\cdot, h)$ with sharp remainder estimate has been obtained (see [7, 8, 30] and the references given therein). On the other hand, a complete asymptotic expansion in powers of h of $\xi(\cdot, h)$ has been obtained for non-trapping energies λ (see [4, 5, 6, 30, 32]) i.e. for energies at which any hamiltonian flow of the underlying classical mechanics tends to ∞ as time tends to $\pm\infty$. Similar results are well-known for the SSF in the high energy regime, $h = 1$ and $\lambda \to \infty$ (see [7, 8, 25, 28, 30, 31, 32]). In [31, 32], it was established that the leading terms of the asymptotic behavior of $\xi(\lambda, 1)$ as $\lambda \to +\infty$ only depends on the average value of $V_2 - V_1$. The proof of all the above results follows from a beautiful local trace formula in the configuration space due to D. Robert (see Theorem 1.10 in [32]). However, the proof of this local trace formula, based on the construction of a long time parametrix for time-dependent Schrödinger equation, involves the decay assumptions for both potentials V_1 and V_2.

The relation between the asymptotics of the SSF and resonances was first investigated by R. Melrose [24], and then by many authors with successive extensions (see [26] and the references given therein). All these works use the scattering theory. In [33], J. Sjöstrand proposed a new

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approach based on the complex scaling of operators. The scattering determinant is replaced by $D(z,h) = \det(I + K(z,h))$, where $K(z,h)$ is a trace class operator whose zeros are the resonances (see section 4). Applying this approach, V. Bruneau and V. Petkov established in [4] a representation of the derivative of SSF as a sum of harmonic measures related to resonances.

There are only few works treating the SSF of the Schrödinger operator with non-decaying potentials, such as those homogeneous of degree zero, periodic or even of logarithmic decay.

In [9], the first author established a trace formula relating the SSF and the resonances of the periodic Schrödinger operator with slowly varying perturbation $W(hx)$. Using the Peierls substitution method, he reduced the spectral study of the perturbed operator to the study of the semiclassical operator $E(hD_x) + W(x)$ for a band function $E(k)$ describing the Floquet spectrum of the non-perturbed operator. Unfortunately, this method fails at high energy, since the band functions are not smooth due to the degeneracy of the Floquet eigenvalues.

The spectral and scattering theory for Schrödinger operator with a homogeneous potential of degree zero was investigated in [14] (see also [15] and the references there). The asymptotics of the number of eigenvalues for a perturbation of such an operator below the essectial spectrum was studied in [29]. To our best knowledge, the SSF has not been treated.

The aim of this paper is to fill this gap. We consider Schrödinger operators with non decaying potentials including in particular homogeneous ones of degree zero.

In the first sections, we study the high energy asymptotics of the SSF. In section 2, we compute the trace formulas (Theorem 2.1) and the explicit coefficients of all order of the weak asymptotic expansion in powers of λ^{-1} of $\xi'(\lambda, 1)$ as $\lambda \to \infty$ (Corollary 2.2). The k-th coefficient is given by the integral of the difference of a polynomial of degree k with respect to the potential and its derivatives as suggested in [31, 32]. For this we only use a standard pseudodifferential calculus combined with some commutator formulas for $H_0 = -\Delta$ (see section 6 for the proof, and also [23, 27]).

In section 3, we give a strong sense to this expansion for potentials homogeneous of degree zero, or those analytic and bounded in a complex sector at infinity (Corollary 3.3). For such a potential V, say for homogeneous one, the operator $H = -\Delta + V(x)$ is unitarily equivalent to $H_\theta := U_\theta H U_{-\theta} = -e^{-2\theta}\Delta + V(x)$, where $U(\theta)f(x) = e^{n\theta/2}f(e^\theta x)$ is a dilation operator in $L^2(\mathbb{R}^n)$ for real θ. The operator $-e^{-2\theta}\Delta + V(x)$ is analytic for $\theta \in \mathbb{C}$. The uniqueness of analytic continuation implies the invariance of the SSF under complex dilation. Since the resolvent is continued analytically to the lower half plane after a complex dilation with $\Im\theta > 0$, a representation formula of ξ' in terms of the resolvent (Lemma 3.4) enables us to show the analyticity and a polynomial estimate of ξ' as well as its derivatives (Theorem 3.2). It is now classical to deduce the strong full asymptotic expansion from the weak one using these estimates and Lemma 3.5.

Next we study the semiclassical asymptotics. We will restrict our attention to potentials V homogeneous of degree zero at infinity (i.e., there exists W independent of $|x|$ such that $|V(x) - W(x)| \to 0$ as $|x| \to \infty$). The essential spectrum of $U_\theta H(h) U_{-\theta}$ is the union of the semi-axis $t + e^{-2\theta}\overline{\mathbb{R}_+}$ over t in the range $W(S^{n-1})$, a band in the lower half plane intersecting with $\mathbb{R}$ on $W(S^{n-1})$ when $\Im\theta > 0$. In section 4, we consider the SSF for λ above the range $W(S^{n-1})$, and generalize the result of V. Bruneau and V. Petkov [5] proving a representation of $\xi'(\lambda, h)$ in terms of the resonances (Theorem 4.1). We apply this result to establish a

Weyl-type formula for the SSF with optimal remainder estimate $\mathcal{O}(h^{1-n})$. Moreover, under resonance free domain condition, we give a complete asymptotic expansion of $\xi'(\lambda, h)$.

Finally, in section 5, we consider λ in the range $W(S^{n-1})$ and prove a semi-classical Mourre estimate away from critical values of $W_{|S^{n-1}}$. This is a semiclassical version of S. Agmon, J. C.-Sampedro and I. Herbst [1] (Appendix C). The proof is based on a construction of an escape function for Schrödinger operators adapted to a homogeneous potential.

Notations: Throughout this paper, h is an asymptotic positive parameter going to zero. We use $f_h = \mathcal{O}(h^N)$ to denote an h-dependent function that is bounded in magnitude by an expression $C_N h^N$, where the implied constant C_N is independent of h but may depend on parameters independent of h. Similarly, we use $f_h = \mathcal{O}(h^\infty)$ or $f_h \equiv 0$ to denote the estimate $|f| \le C_N h^N$ for every N. For any quantity a_j defined for each $j = 1, 2$ concerning H_j, we sometimes denote their difference $a_2 - a_1$ by $[a_\cdot]_1^2$.

2. Weak Asymptotics

In this and the next sections, we study the high-energy asymptotics of $\xi(\lambda) = \xi(\lambda, 1)$. Let H_1, H_2 denote the operators (1.1) with $h = 1$ with potentials satisfying

(A1) V_j are real-valued smooth functions and there exists $\rho > n$ such that for all $\alpha \in \mathbb{N}^n$

$$\partial_x^\alpha V_j(x) = \mathcal{O}(1),\ j = 1, 2, \quad \partial_x^\alpha (V_2(x) - V_1(x)) = \mathcal{O}(|x|^{-\rho}) \ \text{ as } |x| \to \infty. \tag{2.1}$$

The following result follows from the standard h-pseudodifferential operators calculus (see chapters 7-8 in [11]).

Theorem 2.1. *Assume* **(A1)**. *Then the following full asymptotic expansion holds as $h \searrow 0$:*

$$\operatorname{tr}\left[f(h^2 H_\cdot)\right]_1^2 \sim \sum_{k=1}^{\infty} c_k(f)\, h^{2k-n}, \tag{2.2}$$

for every $f \in C_0^\infty(]0, +\infty[; \mathbb{R})$, with

$$c_k(f) = \frac{n\kappa_0}{k!(2\pi)^n} \int_0^\infty f^{(k)}(r^2) r^{n-1} dr \int_{\mathbb{R}^n} P_k(x) dx. \tag{2.3}$$

Here κ_0 is the measure of the unit ball in $\mathbb{R}^n$, and

$$P_1(x) = [V_\cdot]_1^2 := V_2 - V_1, \quad P_k(x) = \left[\mathcal{P}_k(\{D^\alpha V_\cdot\}_{|\alpha| \le 2k-4})\right]_1^2 \text{ for } k \ge 2,$$

where $\mathcal{P}_j$ is a universal polynomial of degree j. In particular,

$$\mathcal{P}_2 = V^2, \quad \mathcal{P}_3 = V^3 - \frac{1}{2} V \Delta V, \tag{2.4}$$

$$\mathcal{P}_4 = V^4 + \frac{3}{5} V \Delta^2(V) + \frac{4}{5} V^2(\Delta V) - \frac{2}{5} V \Delta(V^2) + \frac{2}{5} V |\nabla V|^2. \tag{2.5}$$

For f in $C_0^\infty(]0, +\infty[)$, a change of variable and integration by parts yield

$$\int_0^\infty f^{(k)}(r^2) r^{n-1} dr = \frac{(-1)^k}{2} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2} - k)} \int_{-\infty}^\infty f(\lambda) \lambda^{\frac{n}{2} - k - 1} d\lambda, \tag{2.6}$$

with the convention that $\Gamma(-m)^{-1} = 0$ for $m \in \mathbb{N} := \{0, 1, \ldots\}$, and hence

$$c_k(f) = -a_k \langle \lambda^{\frac{n}{2} - k - 1}, f \rangle, \tag{2.7}$$

where

$$a_k = \omega_k \int_{\mathbb{R}^n} P_k(x)dx, \quad \omega_k = \frac{(-1)^{k+1}}{2} \frac{n\kappa_0}{k!(2\pi)^n} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-k)}. \tag{2.8}$$

On the other hand, from (1.2) we have

$$\operatorname{tr}\left[f(h^2H.)\right]_1^2 = -\int_{-\infty}^{+\infty} \xi'(\lambda, H_2, H_1)\, f(h^2\lambda)\, d\lambda = -\left\langle \frac{1}{h^2}\xi'\left(\frac{\cdot}{h^2}\right), f(\cdot)\right\rangle.$$

As a consequence of Theorem 2.1 and (2.7), we have the weak asymptotics of $\xi'(\lambda)$ as $\lambda \to +\infty$.

Corollary 2.2. *For λ large enough, the asymptotic expansion*

$$\xi'(\lambda) \sim \sum_{k=1}^{\infty} a_k\, \lambda^{\frac{n}{2}-k-1} \tag{2.9}$$

holds in the sense of distribution, where a_k are given by (2.8). In particular, modulo $\mathcal{O}(\lambda^{-\infty})$, $\xi'(\lambda)$ is a polynomial of degree $\frac{n}{2}-2$ when $n \geq 4$ is even.

3. High energy Asymptotics

In this section, we suppose the following analyticity condition in addition to **(A1)** that

(A2) There exist $c > 0$ such that the functions : $(\theta, x) \ni]-c, c[\times\mathbb{R}^n \to V_j(e^\theta x)$, $j = 1, 2$ have an analytic extention on θ to a complex disk $D(c) := \{\theta \in \mathbb{C}, |\theta| \leq c\}$, and the estimate (2.1) holds for $x \mapsto V_j(e^\theta x)$ uniformly for all $\theta \in D(c)$.

Remark 3.1. *The above condition can be relaxed. In fact, it suffices to assume that : $\theta \to V_j(e^\theta x)$, has an analytic extension on $\theta \in D(c)$ uniformly for $|x| > C$. In that case we have to use in the proof of the below results the distortion analytic method. Here, the condition* **(A2)** *allows us to use the dilation analytic method which is more simpler for the exposition.*

Our main results of this section are the followings :

Theorem 3.2. *Under* **(A1)** *and* **(A2)***, there exists λ_0 such that $\xi(\lambda)$ is an analytic function in $]\lambda_0, +\infty[$ and for every $N \in \mathbb{N}$ there exists C_N such that for $m > n/2$ we have*

$$|\xi^{(N+1)}(\lambda)| \leq C_N \lambda^{m-N-1}, \tag{3.1}$$

uniformly for $\lambda \in [\lambda_0, +\infty[$.

Corollary 3.3. *Under* **(A1)** *and* **(A2)***, we have for every integer N*

$$\lim_{\lambda\to+\infty} \lambda^{N+1-\frac{n}{2}}\left[\xi'(\lambda) - \sum_{k=1}^{N} a_k\, \lambda^{\frac{n}{2}-k-1}\right] = 0, \tag{3.2}$$

where the coefficients a_k are given by (2.8).

3.1. **Proof of Theorem 3.2.** Let H_j, $j = 1, 2$ be two operators satisfying (A2). Fix an integer $m > n/2$ so that the operator $G(z) := \Big[(z - H_j)^{-1}(H_j - z_0)^{-m}\Big]_1^2$ is of trace class (we recall the notation $[a_j]_1^2 = a_2 - a_1$). To see this, we write

$$G(z) = \big((z - H_2)^{-1} - (z - H_1)^{-1}\big)(H_2 - z_0)^{-m} + (z - H_1)^{-1}\Big[(H_j - z_0)^{-m}\Big]_1^2 = I + II. \tag{3.3}$$

The condition (2.1) implies that $(V_2 - V_1)(H_2 - z_0)^{-m}$ is of trace class for $m > n/2$. Therefore, $I = (z - H_1)^{-1}(V_2 - V_1)(H_2 - z_0)^{-m}(z - H_2)^{-1}$ is of trace class. Now, the $m-1$-th derivatives of the resolvent identity implies that $(z_0 - H_1)^{-m} - (z_0 - H_2)^{-m}$ is a linear combination of terms of the form $(z_0 - H_1)^{-j}(V_2 - V_1)(z_0 - H_2)^{-(m+1-j)}$ with $1 \leq j \leq m$. This shows that II is also of trace class.

Let z_0 be in $\rho(H_1) \cap \rho(H_2) \cap \mathbb{R}$ and introduce the function

$$\sigma_\pm(z) = (z - z_0)^m \mathrm{tr}\Big[(z - H.)^{-1}(H. - z_0)^{-m}\Big]_1^2, \quad \pm\Im z > 0. \tag{3.4}$$

First, we give a representation formula of $\xi'(\lambda)$ in terms of σ_+.

Lemma 3.4. *In the sense of distribution, we have*

$$\xi'(\lambda) = \frac{1}{\pi}\Im\sigma_+(\lambda + i0).$$

More precisely, for all $f \in C_0^\infty(\mathbb{R})$, we have

$$\langle \xi', f\rangle = \lim_{\epsilon \searrow 0} \frac{1}{\pi}\int f(\lambda)\Im\sigma_+(\lambda + i\epsilon)d\lambda,$$

where the limit is taken in the sense of distribution.

Proof. Let $f \in C_0^\infty(\mathbb{R})$ and let $\tilde{f} \in C_0^\infty(\mathbb{C})$ be an almost analytic extension of f. According to the formula (**??**), we have

$$\mathrm{tr}\Big[f(H.)\Big]_1^2 = -\frac{1}{\pi}\int \bar{\partial}_z\tilde{f}(z)(z - z_0)^m \times \mathrm{tr}\Big[(H. - z_0)^{-m}(z - H.)^{-1}\Big]_1^2 L(dz). \tag{3.5}$$

Since we have $\sigma_\pm(z) = \mathcal{O}(|\Im z|^{-1})$ and $\bar{\partial}_z\tilde{f} = \mathcal{O}(|\Im z|^\infty)$, we may write the right hand side of the above identity as

$$\langle \xi', f\rangle = -\mathrm{tr}\Big[f(H.)\Big]_1^2 = \lim_{\epsilon\searrow 0}\frac{1}{\pi}\Big(\int_{\Im z > 0}\bar{\partial}_z\tilde{f}(z)\sigma_+(z + i\epsilon)L(dz) + \int_{\Im z < 0}\bar{\partial}_z\tilde{f}(z)\sigma_-(z - i\epsilon)L(dz)\Big).$$

The function $\sigma_+(z + i\epsilon)$ (resp. $\sigma_-(z - i\epsilon)$) is holomorphic on the complex domain $\{z \in \mathbb{C} : \Im z > 0\}$ (resp. $\{z \in \mathbb{C} : \Im z < 0\}$). Thus applying the Green's formula we obtain

$$\langle \xi', f\rangle = \lim_{\epsilon\searrow 0}\frac{1}{2\pi i}\int f(\lambda)\Big(\sigma_+(\lambda + i\epsilon) - \sigma_-(\lambda - i\epsilon)\Big)d\lambda.$$

Using the above formula and the fact that $\sigma_-(\lambda - i\epsilon) = \overline{\sigma_+(\lambda + i\epsilon)}$ we get the lemma. □

Now we prove Theorem 3.2. For $\theta \in \mathbb{R}$ set for $j = 1, 2$,

$$H_{j,\theta} = -e^{-2\theta}\Delta + V_j(e^\theta x).$$

The operator $(z-H_j)^{-1}(H_j-z_0)^{-m}$ is unitarily equivalent to $(z-H_{j,\theta})^{-1}(H_{j,\theta}-z_0)^{-m}$ for real θ. Consequently, the cyclicity of the trace yields

$$\sigma_+(z)=(z-z_0)^m\mathrm{tr}\Big[(z-H_{\cdot,\theta})^{-1}(H_{\cdot,\theta}-z_0)^{-m}\Big]_1^2, \tag{3.6}$$

for all $z\in\mathbb{C}_+=\{z\in\mathbb{C};\Im z>0\}$ and $\theta\in D(c)\cap\mathbb{R}$.

Fix $\delta>0$, and let $z\in\mathbb{C}_\delta=\{z\in\mathbb{C};\Im z\geq\delta\}$. Since $H_{j,\theta}$ extends to an analytic type A family of operators on $D(c)$ and $z\in\mathbb{C}_\delta$, the right hand side of (3.6) extends by analytic continuation in θ to the disc $D(c')$ for small enough $c'>0$. For $\theta\in D(c')$ with $\Im\theta<0$, both terms of (3.6) are analytic on $\mathbb{C}_+$ and consequently (3.6) remains true for all z in $\mathbb{C}_+$.

From now on, we fix $\theta=-i\eta$, $\eta>0$ in $D(c')$. Set $\mathcal{A}_{a,A}:=\{z\in\mathbb{C};\Re z>A,\Im z>-a\}$ for positive numbers a and A. The following estimate holds uniformly on $\mathcal{A}_{a,1}$ for some positive constant a:

$$\|(-e^{-2\theta}\Delta-z)^{-1}\|\leq\sup_{\xi\in\mathbb{R}^n}\Big(|e^{-2\theta}|\xi|^2-z|^{-1}\Big)\leq C\eta^{-1}(\Re z)^{-1}.$$

Using **(A2)** and the above estimate, we see that

$$H_{j,\theta}-z=(-e^{-2\theta}\Delta-z)\Big(I+(-e^{-2\theta}\Delta-z)^{-1}V_j(e^\theta x)\Big),$$

is invertible for $z\in\mathcal{A}_{a,A}$ with sufficiently large A. Moreover, uniformly on $z\in\mathcal{A}_{a,A}$,

$$\mathcal{A}_{a,A}\ni z\to(H_{j,\theta}-z)^{-1}\text{is holomorphic, and }\|(H_{j,\theta}-z)^{-1}\|=\mathcal{O}((\Re z)^{-1}), \tag{3.7}$$

On the other hand, a classical result on trace class operators (see for instance [11]) shows that

$$\|(z_0-H_{j,\theta})^{-m}[V_{\cdot,\theta}]_1^2\|_{\mathrm{tr}}=\mathcal{O}(1), \tag{3.8}$$

and hence, again by taking the derivatives of the resolvent identity, we have

$$\|\left[(z_0-H_{\cdot,\theta})^{-m}\right]_1^2\|_{\mathrm{tr}}=\mathcal{O}(1). \tag{3.9}$$

Next, we write $\sigma_+(z)=\sigma_+^1(z)+\sigma_+^2(z)$, where

$$\begin{aligned}\sigma_+^1(z)&=\mathrm{tr}\Big((z-z_0)^m(z-H_{1,\theta})^{-1}\Big[(H_{\cdot,\theta}-z_0)^{-m}\Big]_1^2\Big),\\ \sigma_+^2(z)&=\mathrm{tr}\Big((z-z_0)^m\Big[(z-H_{\cdot,\theta})^{-1}\Big]_1^2(H_{2,\theta}-z_0)^{-m}\Big)\\ &=\mathrm{tr}\Big[(z-z_0)^m(z-H_{1,\theta})^{-1}[V_{\cdot,\theta}]_1^2(H_{2,\theta}-z_0)^{-m}(z-H_{2,\theta})^{-1}\Big].\end{aligned}$$

From (3.7), (3.8) and (3.9) we deduce that the RHS are holomorphic in $\mathcal{A}_{a,A}$ which implies that $\xi'(\lambda)=\frac{1}{\pi}\Im(\sigma_+^1(\lambda+i0)+\sigma_+^2(\lambda+i0))$ is analytic in $]\lambda_0,+\infty[$ for a large constant λ_0.

On the other hand, the estimates (3.7), (3.8), (3.9) and the fact that $|\lambda-z_0|=\mathcal{O}(\lambda^m)$ imply that $|\sigma_+^1(\lambda+i\varepsilon)|,|\sigma_+^2(\lambda+i\varepsilon)|=\mathcal{O}(\lambda^{m-1})$, uniformly for $\lambda>\lambda_0\gg1$ and $\varepsilon\in[0,\varepsilon_0[$ for some ε_0 sufficiently small. Consequently,

$$\xi'(\lambda)=\frac{1}{\pi}\Im\sigma_+(\lambda+i0)=\frac{1}{\pi}\Im(\sigma_+^1(\lambda+i0)+\sigma_+^2(\lambda+i0))=\mathcal{O}(\lambda^{m-1}).$$

This ends the proof of Theorem 3.2 for $N=0$. For $N\geq1$ we take derivatives of $\sigma_+^{\cdot}(z)$ with respect to z and repeat the same arguments as above.

3.2. **Proof of Corollary 3.3.** The proof of Corollary 3.3 is a simple consequence of Theorem 3.2 and the following lemma. Let $\mathcal{F}_h\psi$ be the semiclassical Fourier transform:

$$\mathcal{F}_h\psi(x) = \frac{1}{2\pi h}\int_{\mathbb{R}} e^{itx/h}\psi(t)dt.$$

Lemma 3.5. *Let $\psi \in C_0^\infty(\mathbb{R})$, and let f_h be a C^∞ function in $\mathbb{R}$, depending on a parameter $h \in (0,1]$. We suppose that, there exist $m \in \mathbb{R}$ and $\delta \in [0,1[$ such that for all $k \in \mathbb{N}$,*

$$(\frac{\partial}{\partial x})^k f_h(x) = \mathcal{O}(h^{-m-k\delta}) \text{ as } \mu \to +\infty \text{ uniformly for } x \in \mathbb{R}. \tag{3.10}$$

Then for all $N \in \mathbb{N}$, there exists $h_N > 0$ such that :

$$\mathcal{F}_h\psi * f_h(x) = \sum_{k=0}^{N} \frac{(-ih)^k}{k!}\psi^{(k)}(0)(\frac{\partial}{\partial x})^k f_h(x) + \mathcal{O}(h^{N(1-\delta)+m}), \tag{3.11}$$

uniformly for $x \in \mathbb{R}$ and $h \in (0, h_N]$. In particular, if $\psi = 1$ near zero, then

$$\mathcal{F}_h\psi * f_h(x) = f_h(x) + \mathcal{O}(h^\infty). \tag{3.12}$$

Proof. By a change of variable, we have

$$\mathcal{F}_h\psi * f_h(x) = \int_{\mathbb{R}} \mathcal{F}_1\psi(t) f_h(x-ht)dt. \tag{3.13}$$

Applying Taylor's formula to the function $t \longmapsto f_h(x-ht)$ at $t = 0$, and using (3.10), we get

$$f_h(x-ht) = \sum_{k=0}^{N-1} f_h^{(k)}(x)\frac{(-th)^k}{k!} + \mathcal{O}(h^{N(1-\delta)-m}t^N). \tag{3.14}$$

Inserting the above equality in (3.13) and using the fact that $\int_{\mathbb{R}}(-it)^k\mathcal{F}_1\psi(t)dt = \psi^{(k)}(0)$ we obtain (3.11). □

Now we pass to the prove of Corollary 3.3. Let $g \in C_0^\infty\left(]\frac{1}{2},\frac{3}{2}[\right)$ be equal to 1 near one. For $h > 0$, we set $f_h(x) := g(x)\xi'(\frac{x}{h^2})$. Using Lemma 3.2, we see that the function f_h satisfies all the assumptions in Lemma 3.5 with $\delta = 0$. Let $\psi \in C_0^\infty(\mathbb{R})$ be as in Lemma 3.5 with $\psi = 1$ near zero. According to Lemma 3.5, we have

$$\mathcal{F}_h\psi * f_h(x) = f_h(x) + \mathcal{O}(h^\infty). \tag{3.15}$$

On the other hand, a simple calculation shows that

$$\begin{aligned} h^{-2}\mathcal{F}_h\psi * f_h(x) = h^{-2}\int_{\mathbb{R}} \mathcal{F}_h\psi(x-t)g(t)\xi'(t/h^2)dt &= \int_{\mathbb{R}} \mathcal{F}_h\psi_h\left(x-th^2\right)g(h^2t)\xi'(t)dt \\ &= \langle \xi', \mathcal{F}_h\psi\left(x-.h^2\right)g(h^2.)\rangle \\ &= \operatorname{tr}\left[\mathcal{F}_h\psi\left(x-h^2H_2\right)g(h^2H_2) - \mathcal{F}_h\psi\left(x-h^2H_1\right)g(h^2H_1)\right]. \end{aligned} \tag{3.16}$$

For $0 < h << 1, h^2H_.$ is an $h-$pseudodifferential operator. According to [31, 32] (see also chapters 11-12 in [11]), the right hand side of the last equality has a complete asymptotic expansion in powers of h^2. Combining this with (3.12), we get

$$h^{-2}\mathcal{F}_h\psi * f_h(x) = h^{-2}f_h(x) + \mathcal{O}(h^\infty) = h^{-n}\sum_{j=1}^{\infty} a_j(x)h^{2j} + \mathcal{O}(h^\infty).$$

Taking $x = 1$ and $\lambda = \mu^2$ we obtain

$$\xi'(\lambda) = \lambda^{\frac{n}{2}} \sum_{j=1}^{\infty} a_j(1)\lambda^{-j-1} + \mathcal{O}(\lambda^{-\infty}).$$

We recall that $f_h(x) = g(x)\xi'(x/h^2)$ and $g(1) = 1$. This ends the proof of Theorem 3.3. The explicit formula of a_j is given by (2.8) in Theorem 2.1.

Remark 3.6. *Theorem 3.2 remains true if V_1 is an homogeneous potential of degree zero and smooth on $\mathbb{R}^n \setminus \{0\}$ (i.e. a pur homogeneous potential of degree zero). In fact, according to the the proof of Theorem 3.2, one need only that the operator $L^2(\mathbb{R}^n) \ni u \to V_{1,\theta}(x)u(x) \in L^2(\mathbb{R}^n)$ is analytic with respect to $\theta \in D(c)$ and uniformly bounded.*

4. Semiclassical asymptotics

In this section we consider the semiclassical Schrödinger operators $H_1(h)$ and $H_2(h)$ given in (1.1). To simplify the presentation, let us assume, throughout this section, that the potentials are homogeneous of degree zeero at infinity. More precisely, in addition to the conditions **(A1)**, **(A2)** we suppose

(A3) There exists a homogeneous function W of degree zero such that

$$\lim_{|x|\to\infty} \Big(V_1(x) - W(x)\Big) = 0. \tag{4.1}$$

The essential spectrums of the operators $H_1(h)$ and $H_2(h)$ coincide with the semi-axis $[\min_{\omega\in S^{n-1}} W(\omega), +\infty)$. For $\Im\theta \neq 0$, we have

$$\sigma_{\rm ess}(H_{1,\theta}(h)) = \sigma_{\rm ess}(H_{2,\theta}(h)) = \mathbb{S}_\theta := \{e^{-2\theta}s + t; s \geq 0 \text{ and } t \in W(S^{n-1})\}. \tag{4.2}$$

In fact, we easily see $\sigma_{\rm ess}(-e^{-2\theta}h^2\Delta + W(x)) = \mathbb{S}_\theta$ by Weyl's criterion, and (4.2) follows from (4.1) and Theorem 5.35 in [21] [1]. Consequently, $W(S^{n-1}) = \sigma_{\rm ess}(H_{1,\theta}(h)) \cap \mathbb{R}$ is included in the essential spectrum of the distorted hamiltonian $H_{j,\theta}(h)$. For this reason, we will exclude the energies in $W(S^{n-1})$ in this section.

Fix an interval $J \subset \mathbb{R}$ with $\inf J > \max_{\omega\in S^{n-1}} W(\omega)$. Let $\operatorname{Res} H_j(h)$ denote the set of resonances, i.e. the eigenvalues of the $H_{j,\theta}(h)$ in the lower half complex plane near I.

Theorem 4.1. *Under **(A1)**-**(A3)**, there exist an h-independent open complex neighborhood Ω of J and a holomorphic function $r(z, h)$ in Ω satisfying*

$$|r(z, h)| \leq Ch^{-n} \tag{4.3}$$

such that for h small enough and $\lambda \in J$, it holds that

$$\xi'(\lambda, h) = \Im r(\lambda, h) + \left[\sum_{\omega\in \operatorname{Res} H_\cdot(h)\cap\Omega} \frac{-\Im\omega}{|\lambda-\omega|^2} + \sum_{\omega\in\sigma_{pp}(H_\cdot(h))\cap\Omega} \delta(\lambda-\omega) \right]_1^2. \tag{4.4}$$

[1] Theorem 5.35 [21] : Let T be a closed operator on a Hilbert space $\mathcal{H}$ and let A be a relatively T-compact operator. Then $\sigma_{\rm ess}(T) = \sigma_{\rm ess}(T + A)$.

As stated in the introduction, for the Schrödinger operator with long-range perturbations decaying at infinity (i.e., $V_1 = 0$), the proof of the above theorem is due to V. Bruneau and V. Petkov [4]. The main ingredient in the proof of Theorem 4.1 is Proposition 4.2.

By Lemma 3.4, we have

$$\xi'(\lambda, h) = \frac{1}{\pi}\Im\sigma_+(\lambda + i0, h),$$

where

$$\sigma_+(z, h) = (z - z_0)^m \mathrm{tr}\Big[(z - H_\cdot(h))^{-1}(H_\cdot(h) - z_0)^{-m}\Big]_1^2, \ \Im z > 0.$$

For $\theta \in \mathbb{R}$, we introduce

$$H_{j,\theta}(h) = -h^2 e^{-2\theta}\Delta + V_j(e^\theta x).$$

As in the proof of (3.6), analytic continuation argument shows that, there exists θ_0 small enough such that for any $\theta \in D(0, \theta_0)$ we have

$$\sigma_+(z, h) = (z - z_0)^m \mathrm{tr}\Big[(z - H_{\cdot,\theta}(h))^{-1}(H_{\cdot,\theta}(h) - z_0)^{-m}\Big]_1^2, \ \Im z > 0. \tag{4.5}$$

From now on, we fix $\theta = i\eta$ with $\eta > 0$, and we let Ω be a bounded complex neighborhood of J with $\overline{\Omega} \subset \Omega_\theta := \{z \in \mathbb{C}; \Re z > a, \ \Im z > -\eta\}$. Now, as in the proof of Theorem 1.5 of [4] (see also), we will reduce the study of the r.h.s. of (4.5) to the study of a finite rank operator.

Proposition 4.2. *There exist finite rank operators $K_j = K_j(z, h)$, $j = 1, 2$, in $L^2(\mathbb{R}^n)$ such that* $\mathrm{rank}\, K_j = \mathcal{O}(h^{-n})$, $\|K_j\| = \mathcal{O}(1)$ *and*

$$\sigma_+(z, h) = \big[\mathrm{tr}\,\big((\mathrm{Id} + K_\cdot)^{-1}\partial_z K_\cdot\big)\big]_1^2 + k, \tag{4.6}$$

where $k = k(z, h)$ is a holomorphic function in Ω satisfying the estimate $|k(z, h)| = \mathcal{O}(h^{-n})$.

Proof. Let M, R be two large constants, $\chi, \tilde{\chi} \in C_0^\infty(]-2R, 2R[; [0, 1]))$ equal to one on $[-R, R]$ with $\tilde{\chi} = 1$ near $\mathrm{supp}\chi$ and $f \in C_0^\infty(]-3R, 3R[; [0, 1])$ equal to 1 on $[-2R, 2R]$. We define

$$K(h) := iMf\left(-h^2\Delta + |x|^2\right)\tilde{\chi}(-h^2\Delta)\chi(|x|^2)\tilde{\chi}(-h^2\Delta)f\left(-h^2\Delta + |x|^2\right),$$

Clearly, $K(h)$ is a finite rank operator and

$$\mathrm{rank}\, K(h) \leq \mathrm{rank}\, f(-h^2\Delta + |x|^2) = \mathcal{O}(h^{-n}). \tag{4.7}$$

From the functional calculus for h-pseudodifferential operators, we know that the Weyl symbol of the operator $f(-h^2\Delta + |x|^2)$ has an asymptotic expansion of the form:

$$\sum_{k=0}^{N} h^k f^{(k)}(|\xi|^2 + |x|^2)a_k(x, \xi) + \mathcal{O}(h^N\langle(x, \xi)\rangle^{-\infty}), \ \forall N,$$

with symbols $a_k(x, \xi)$ depending on x^α, ξ^α with $|\alpha| \leq 2k$ (see the proof of Theorem 8.7 in [11]). Combining this with the fact that $f^{(k)}(|\xi|^2 + |x|^2)\tilde{\chi}(|\xi|^2)\chi(|x|^2) = 0$ for $k \geq 1$ as well as the fact that $\tilde{\chi}^{(l)}(|\xi|^2)\chi^{(m)}(|x|^2) = 0$ for $l, m \geq 1$, we deduce from the composition formula of h-pseudodifferential operators that

$$K(h) = iM\mathrm{Op}_h^w\left(f(|\xi|^2 + |x|^2)^2\tilde{\chi}(|\xi|^2)^2\chi(|x|^2)\right) + \mathrm{Op}_h^w(h^\infty\langle(x, \xi)\rangle^{-\infty}). \tag{4.8}$$

Set, for $j = 1, 2$,

$$\widehat{H}_{j,R}(h) := H_{j,\theta}(h) - K(h). \tag{4.9}$$

Then, modulo $\mathcal{O}(h^\infty\langle(x,\xi)\rangle^{-\infty})$, the symbol of $\widehat{H}_{j,R}(h)$ is given by

$$\widehat{H}_{j,R}(x,\xi)=e^{-2\theta}|\xi|^2+V_{j,\theta}(x)-iMf(|\xi|^2+|x|^2)^2\tilde{\chi}(|\xi|^2)^2\chi(|x|^2),$$

and for $z\in\Omega$, $|\widehat{H}_{j,R}(x,\xi)-z|^2=\mathcal{R}_1+\mathcal{R}_2$ with

$$\mathcal{R}_1=\left[|\xi|^2\cos(2\theta)\Re(V_{j,\theta}-z)\right]^2,$$
$$\mathcal{R}_2=\left[|\xi|^2\sin(2\theta)+Mf\left(|\xi|^2|x|^2\right)\tilde{\chi}^2(|\xi|^2)\chi(|x|^2)+\Im(z-V_{j,\theta})\right]^2.$$

Choose R large enough so that $R>2\sup_{x\in\mathbb{R}^n,z\in\Omega}|\Re(V_{j,\theta}(x)-z)|$. It follows that

$$\mathcal{R}_1\geq C(1+|\xi|^2)^2 \quad \text{for } |\xi|^2\geq R,\ x\in\mathbb{R}^n,\ z\in\Omega. \tag{4.10}$$

Next, we choose M large enough so that $M>\sup_{x\in\mathbb{R}^n,z\in\Omega}|\Im(z-V_{j,\theta}(x))|$. Then

$$\mathcal{R}_2\geq(M+\Im z-\Im V_{j,\theta}(x))^2\geq c>0 \quad \text{for } |\xi|^2\leq R,\ |x|^2\leq R,\ z\in\Omega. \tag{4.11}$$

It remains to esgtimate $|\widehat{H}_{jR}(x,\xi)-z|$ for $|\xi|^2\leq R$ and $|x|^2\geq R$. From (2.1) and the assumptions **(A2)**, **(A3)**, we have for $j=1,2$,

$$\Re V_{j,\theta}(x)=W\left(\frac{x}{|x|}\right)+o_R(1),\ \ \Im V_{j,\theta}(x)=o_R(1),$$

Since $\alpha:=\inf_{z\in\Omega}\Re z>\sup_{x\in S^{n-1}}W(x)=:\beta$, it follows that for θ small enough

$$\mathcal{R}_1\geq\left(\Re z-W\left(\frac{x}{|x|}\right)+o_R(1)-\cos(2\theta)|\xi|^2\right)^2\geq\tilde{c}>0 \quad \text{for } |\xi|^2\leq\frac{\alpha-\beta}{2\cos(2\theta)}. \tag{4.12}$$

On the other hand, for $|\xi|^2\geq\frac{\alpha-\beta}{2\cos(2\theta)}$ and $|x|^2\geq R$, we have

$$\mathcal{R}_2\geq\left(\sin(2\theta)|\xi|^2+Mf\left(|\xi|^2+|x|^2\right)\tilde{\chi}^2(|\xi|^2)\chi(|x|^2)+\Im z+O_R(1)\right)^2\geq c'>0, \tag{4.13}$$

uniformly for $z\in\Omega$ provided that $\Omega\subset\{z\in\mathbb{C},\Im z\geq-\eta\}$ with $0<\eta\ll 1$.

From (4.10), (4.11), (4.12) and (4.13) we deduce that, uniformly for $(x,\xi)\in\mathbb{R}^{2n}$ and $z\in\Omega$,

$$|\widehat{H}_{j,R}(x,\xi)-z|\geq C(1+|\xi|^2), \tag{4.14}$$

modulo $\mathcal{O}(h^\infty\langle x,\xi\rangle^{-\infty})$. Hence, for h small enough, the operator $\widehat{H}_{j,R}(h)-z$ is elliptic for $z\in\Omega$. Therefore, $z-\widehat{H}_{jR}(h)$ is invertible for h small enough, and

$$\|(z-\widehat{H}_{j,R}(h))^{-1}\|=\mathcal{O}(1), \tag{4.15}$$

uniformly for $z\in\Omega$. Moreover, for h small enough and $z\in\Omega$, $(z-\widehat{H}_{j,R}(h))^{-1}$ is an h-pseudo-differential operator. By construction, we have

$$z-H_{j,\theta}(h)=\left(\mathrm{Id}-K(h)(z-\widehat{H}_{j,R}(h))^{-1}\right)(z-\widehat{H}_{j,R}(h)),$$

which yields

$$(z-H_{j,\theta}(h))^{-1}=(z-\widehat{H}_{j,R}(h))^{-1}\left(\mathrm{Id}-K(h)(z-\widehat{H}_{j,R}(h))^{-1}\right)^{-1}, \tag{4.16}$$

uniformly for $z\in\Omega$ and h small enough.

We can now decompose the right hand side of (4.5) as $\sigma_+(z,h) = I_1 + I_2 + I$, where

$$I_j = (-1)^j (z-z_0)^m \mathrm{tr}\Big[\Big((z-H_{j,\theta}(h))^{-1} - (z-\widehat{H}_{j,R}(h))^{-1}\Big)(H_{j,\theta}(h)-z_0)^{-m}\Big]$$
$$= (-1)^j (z-z_0)^m \mathrm{tr}\Big[(z-\widehat{H}_{j,R}(h))^{-1} K(h)(z-H_{j,\theta}(h))^{-1}(H_{j,\theta}(h)-z_0)^{-m}\Big],$$
$$I = (z-z_0)^m \mathrm{tr}\Big[(z-\widehat{H}_{\cdot,R}(h))^{-1}(z_0-H_{\cdot,\theta}(h))^{-m}\Big]_1^2.$$

Clearly , I is analytic on Ω. On the other hand, the h-pseudo-differential calculus shows that $I = \mathcal{O}(h^{-n})$. Now we treat I_j. From the resolvent equation we have

$$(z-z_0)^m (z-H_{j,\theta}(h))^{-1}(z_0-H_{j,\theta}(h))^{-m} = (z-H_{j,\theta}(h))^{-1} - \sum_{k=0}^{m-1}(z_0-z)^k (z_0-H_{j,\theta}(h))^{-k-1}.$$

Using the above equality and the cyclicity of the trace we deduce that

$$I_j = (-1)^j \mathrm{tr}\left((z-H_{j,\theta}(h))^{-1} K(h)(z-\widehat{H}_{j,R}(h))^{-1}\right) + g_j(z,h),$$

wher $z \to g_j(z,h)$ is analytic on Ω and $|g_j(z,h)| = \mathcal{O}(h^{-n})$ uniformly for $z \in \Omega$. Inserting the right hand side of (4.16) in the above equality and using the cyclicity of the trace, we obtain

$$I_j = (-1)^j \mathrm{tr}\left(\left(\mathrm{Id} - K(h)(z-\widehat{H}_{j,R}(h))^{-1}\right)^{-1} K(h)(z-\widehat{H}_{j,R}(h))^{-2}\right) + g_j(z,h)$$
$$= (-1)^j \mathrm{tr}\left(\mathrm{Id} + K_j(z,h))^{-1}\partial_z K_j(z,h)\right) + g_j(z,h),$$

where

$$K_j(z,h) = -K(h)(z-\widehat{H}_{j,R}(h))^{-1}. \tag{4.17}$$

It follows from (4.9) and (4.15) that

$$\mathrm{rank}\, K_j(z,h) = \mathcal{O}(h^{-n}). \tag{4.18}$$

This concludes the proof of Proposition 4.2. □

Proof. of Theorem 4.1. This follows from a routine application of Proposition 4.2. For the reader's convenience we give the main steps of the proof. Set

$$D_j(z,h) = \det\left(\mathrm{Id} + K_j(z,h)\right).$$

Notice that

$$\partial_z \ln D_j(z,h) = \mathrm{tr}\left((\mathrm{Id} + K_j(z,h))^{-1}\partial_z K_j(z,h)\right), \tag{4.19}$$

and recall that the resonances of $H_j(h)$ in Ω lie in the lower half plane, and are the eigenvalues of $H_{j,\theta}(h)$. Combining this with (4.16) and (4.18) we deduce that the zeros of $D_j(z,h)$ in Ω are the resonances of $H_j(h)$ in Ω, and that the multiplicity agree. Hence, one has

$$D_j(z,h) = G_j(z,h)\Pi_{\omega\in \mathrm{Res}(H_j(h)),\Im\omega\le 0}\left(z-\omega\right), \tag{4.20}$$

where $G_j(z,h)$ are non-vanishing holomorphic functions in Ω. On the other hand, using (4.16) and (4.20) we deduce by a standard arguments of complex analysis that $G_j(z,h) = \mathcal{O}\left(e^{\mathcal{O}(1)h^n}\right)$ and $|G_j(z,h)| \ge C_1 e^{-C_1 h^{-n}}$ on Ω and $\Omega \cap \{|\Im z| \ge \epsilon\}$ respectively. Combining this with the Harnack inequality we get

$$|\partial_z \ln\left(G_j(z,h)\right)| = \mathcal{O}(h^{-n}),$$

which together with (4.20) yields Theorem 4.1. □

Remark 4.3. *The above arguments also show that the number of resonances of $H_j(h)$ in Ω is $\mathcal{O}(h^{-n})$. For the details we refer to [4].*

As in [4] (Theorem 2) and [9] (Theorem 2-3), the following result is a consequence of Theorem 4.1.

Theorem 4.4. *Assume* **(A1)-(A3)** *and that $\nabla_x V_j(x) \neq 0$ if $V_j(x) \in J$ for $j = 1, 2$. Then we have*

$$\xi(\lambda, h) = (2\pi h)^{-n} c_0(\lambda) + \mathcal{O}(h^{-n+1}), \tag{4.21}$$

uniformly for $\lambda \in J$, where

$$c_0(\lambda) = \kappa_0 \int \left[\Big(\lambda - V_\cdot(x)\Big)_+^{\frac{n}{2}} \right]_1^2 dx.$$

Moreover, if there exists $\delta > 0$ such that

$$\mathrm{Res}(H_j(h)) \cap \Big(J - i[0, h^\delta]\Big) = \emptyset, \ j = 1, 2, \tag{4.22}$$

then $\xi'(\lambda, h)$ has a complete asymptotic expansion with smooth coefficients

$$\xi'(\lambda, h) \sim \sum_{k=0}^{\infty} b_k(\lambda) h^{k-n}, \tag{4.23}$$

as $h \searrow 0$ uniformly for $\lambda \in J$. In particular $(2\pi)^n b_0(\lambda) = c_0'(\lambda)$.

Proof. Let g and Ψ be smooth functions with supports in small neighborhoods of J and zero respectively, with $\Psi = 1$ near zero. According to Theorem 12.2 in [11] (see also [13, 18, 19, 30, 32]), the following full asymptotic expansion holds uniformly for $\lambda \in J$ as $h \searrow 0$:

$$\mathcal{F}_h \Psi * (g\xi')(\lambda, h) \sim \sum_{k=0}^{\infty} b_k(\lambda) h^{k-n}. \tag{4.24}$$

When ξ were a monotone function as in the case of eigenvalue counting function, the Weyl asymptotis (4.21) would follow simply from this formula by a Tauberian argument. However, it is not the case for the SSF. To overcome this difficulty, we use (4.4).

In fact, let ξ_j' be the sum over resonances and eigenvalues of H_j in the RHS of (4.4). Then they are positive in the sense of distribution, and Tauberian arguments work for ξ_1 and ξ_2. To treat the term involving $\Im r(\lambda, h)$, notice that, by Cauchy's inequalities and (4.3), we have $|\partial_z^k r(z, h)| \leq C_k h^{-n}$, which together with Lemma 3.5 yields

$$\mathcal{F}_h \Psi * (g \Im r)(\lambda, h) = g(\lambda) \Im r(\lambda, h) + \mathcal{O}(h^\infty). \tag{4.25}$$

This completes the proof of (4.21). For more details we refer to Theorem 2 in [4].

Let us now sketch the proof of (4.23). By hypothesis (4.22), the RHD of (4.4) equals

$$\xi'(\lambda, h) = \Im r(\lambda, h) + \left[\sum_{\omega \in \mathrm{Res} H_\cdot(h) \cap \Omega} \frac{-\Im \omega}{|\lambda - \omega|^2} \right]_1^2. \tag{4.26}$$

For fixed ω with $\Im \omega \leq -h^\delta$, we apply Lemma 3.5 to $f_h(\lambda) = \frac{-\Im \omega g(\lambda)}{|\lambda - \omega|^2}$ to get

$$\mathcal{F}_h \Psi * f_h(\lambda, h) = \frac{-\Im \omega g(\lambda)}{|\lambda - \omega|^2} + \mathcal{O}(h^\infty).$$

Combining this with Remark 4.3, (4.25) and (4.26), we obtain

$$\mathcal{F}_h\Psi * (g\xi')(\lambda, h) = g(\lambda)\xi'(\lambda, h) + \mathcal{O}(h^\infty).$$

Therefore, (4.23) follows from the weak asymptotics (4.24). $\square$

Remark 4.5. *Notice that if $V_1 = W$ is a pure homogeneous potential of degree zero then the operator $H_1(h) = -h^2\Delta + W(x)$ has no eigenvalues (see [14]). On the other hand, the arguments in the proof of Theorem 3.2 and Proposition 4.2 show that $H_1(h) = -h^2\Delta + W(x)$ has no resonances near λ for $\lambda > \sup W$. In this case we may write (4.4) as follows*

$$\xi'(\lambda, h) = \Im r(\lambda, h) + \sum_{\omega \in \mathrm{Res} H_2(h) \cap \Omega} \frac{-\Im\omega}{|\lambda - \omega|^2} + \sum_{\omega \in \sigma_{pp}(H_2(h)) \cap \Omega} \delta(\lambda - \omega).$$

In general one cannot exclude the existence of embedded eigenvalue of a perturbation of pure homogeneous potential (see [1, 15]). Nevertheless, as will be shown in the next section the only possible threshhold energies of $H_j(h)$ are the critical eigenvalue of W.

5. Semi-classical Mourre estimate

In this section, we prove a semi-classical Mourre estimate away from critical values in $W(S^{n-1})$. This shows that the only possible threshhold energies of $H_j(h)$ are those in

$$\mathcal{C}_{\mathrm{cr}} = \{\lambda \in \mathbb{R}; \exists \omega \in S^{n-1} \text{ such that } W(\omega) = \lambda \text{ and } \nabla W(\omega) = 0\}.$$

From now on, we denote $V = V_1$ and $H = -h^2\Delta + V(x)$. We assume

(A4) $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$, and there exists a homogeneous function $W \in C^4(\mathbb{R}^n; \mathbb{R})$ of degree zero such that

$$\lim_{|x|\to\infty} \Big(x^\alpha \partial_x^\beta V(x) - x^\alpha \partial_x^\beta W(x) \Big) = 0 \text{ for all } |\alpha| + |\beta| \le 4. \tag{5.1}$$

We introduce a function $F(x)$ with a positive parameter β and a differential operator A_F by

$$F(x) = \frac{1}{2}(1 - 2\beta V_1(x))|x|^2, \quad 2A_F = \nabla F \cdot hD_x + hD_x \cdot \nabla F. \tag{5.2}$$

Theorem 5.1. *For $\lambda \notin \mathcal{C}_{\mathrm{cr}}$, there exist small positive constants ϵ, h_0 and compact operators $\mathbf{K}_j$, $j = 1, 2$, such that, for $\beta > 0$ small enough and $f \in C_0^\infty(]\lambda - \epsilon, \lambda + \epsilon[; \mathbb{R})$ we have*

$$f(H)[H, A_F]f(H) \ge Chf(H)^2 + \mathbf{K}, \tag{5.3}$$

uniformly for $h \in]0, h_0]$.

Proof. A straightforward calculation shows that

$$i[H, A_F] = 2h(hD, (\nabla \otimes \nabla\, F)hD) + \beta h|x|^2|\nabla V|^2 + r(x, h)$$

where $\nabla \otimes \nabla\, F = \mathrm{Id} - \beta\nabla \otimes \nabla\, (|x|^2 V(x))$ is the Hessian matrix and

$$r(x, h) = -\frac{h^3}{2}\Delta^2 F(x) - h(1 - 2\beta V(x))x \cdot \nabla V(x). \tag{5.4}$$

It follows from the assumption (5.1) that $\nabla \otimes \nabla\, (|x|^2 V(x))$ is a bounded symmetric matrix. Hence, for β small enough, we have

$$2h(hD, (\nabla \otimes \nabla\, F)hD) \ge -h^2\Delta.$$

The condition (5.1) also implies that $r(x,h)$ is a continuous function decaying at infinity. By Rellich's theorem, $r(x,h)(H+i)^{-1}$ is compact, and $r(x,h)f(H) = r(x,h)(H+i)^{-1}(H+i)f(H)$ is also compact for all $f \in C_0^\infty(\mathbb{R})$, since $(H+i)f(H)$ is bounded. Therefore, there exits a compact operator K_1 such that for β small enough, we have

$$f(H)i[H,A_F]f(H) \geq hf(H)H_\beta f(H) + K_1, \quad H_\beta := -h^2\Delta + \beta|x|^2|\nabla V|^2. \tag{5.5}$$

Now we fix $\lambda \notin \mathcal{C}_r$. Then there exist $\epsilon, \kappa > 0$ such that

$$|\nabla W(\omega)| \geq \kappa, \text{ for all } \omega \in S^{n-1} \text{ with } W(\omega) \in]\lambda - 3\epsilon, \lambda + 3\epsilon[. \tag{5.6}$$

We divide the unit sphere S^{n-1} into three open subsets $S^{n-1} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$, where

$$\mathcal{O}_1 = \{\lambda - 3\epsilon < W(\omega) < \lambda + 3\epsilon\}, \quad \mathcal{O}_2 = \{W(\omega) < \lambda - 2\epsilon\}, \quad \mathcal{O}_3 = \{\lambda + 2\epsilon < W(\omega)\}.$$

Let $\{\chi_0, \chi_1, \chi_2, \chi_3\}$ be a smooth partition of unity of $\mathbb{R}^n$, $\sum_{k=1}^3 \chi_k^2(x) = 1$, satisfying

$$\text{supp}\chi_0 \subset \{x \in \mathbb{R}^n; |x| \leq R\}, \text{ and } \chi_0(x) = 1 \text{ for } |x| < R/2,$$
$$\text{supp}\chi_k \subset \{x \in \mathbb{R}^n; |x| > R/2, \frac{x}{|x|} \in \mathcal{O}_k\}, \text{ and } x \cdot \nabla\chi_k(x) = 0 \text{ for } |x| > 3R/4.$$

We choose R large enough such that for $|x| > \frac{R}{2}$ one has

$$||x|^2|\nabla V(x)|^2 - |x|^2|\nabla W(x)|^2| \leq \frac{\kappa}{2}, \quad |V(x) - W(x)| \leq \frac{\epsilon}{2}. \tag{5.7}$$

By the so-called IMS localization formula, i.e.,

$$H_\beta = \sum_{k=0}^3 \chi_k H_\beta \chi_k - h^2 \sum_{k=0}^3 (\nabla\chi_k)^2,$$

it follows from (5.5) that

$$f(H)i[H,A_F]f(H_1) \geq h\sum_{k=0}^3 I_k + K_2, \quad I_k = f(H)\chi_k H_\beta \chi_k f(H), \tag{5.8}$$

where

$$K_2 = K_1 + f(H)\chi_0 H_\beta \chi_0 f(H) - h^2 \sum_{k=0}^3 f(H)(\nabla\chi_k)^2 f(H)$$

is a compact operator for the same reason as K_1, since χ_0 has a compact support and $\sum_{k=0}^3 (\nabla\chi_k)^2$ tends to zero as $|x| \to \infty$ by the homogeneity of χ_k.

First, we prove (5.3) for $j = 1$. Let us investigate I_1, I_2 and I_3. We begin with I_1. On the support of χ_1, we have by the homogeneity of W,

$$|x|^2|\nabla W(x)|^2 \geq \kappa.$$

Combining this with (5.7), we obtain $\chi_1 H_\beta \chi_1 \geq \frac{\kappa\beta}{2}\chi_1^2$ and hence

$$I_1 \geq \frac{h\kappa\beta}{2} f(H)\chi_1^2 f(H). \tag{5.9}$$

Next, we study I_2. From now on we restrict the support of f to $]\lambda - \epsilon, \lambda + \epsilon[$. This implies $f(t)(t - \lambda) \geq -\epsilon f(t)$, and hence, by the spectral theorem,

$$f(H)(H - \lambda) \geq -\epsilon f(H). \tag{5.10}$$

On the support of χ_2, we have

$$I_2 \geq hf(H)\chi_2(-h^2\Delta)\chi_2 f(H) = h(I_{2,1} + I_{2,2}),$$
$$I_{2,1} = f(H)\chi_2(H-\lambda)\chi_2 f(H), \quad I_{2,2} = f(H)\chi_2(\lambda - V)\chi_2 f(H).$$

Since $I_{2,1} = f(H)(H-\lambda)\chi_2^2 f(H) + h^2 f(H)(\Delta\chi_2)\chi_2 f(H) + 2hf(H)\nabla\chi_2 \cdot h\nabla f(H)$, the estimate (5.10) and the fact that $f(H)\Delta\chi_2, f(H)\nabla\chi_2$ are compact operators as well as that $h\nabla f(H)$ is bounded lead us to the estimate

$$I_{2,1} \geq -\epsilon f(H)\chi_2^2 f(H) + K_3,$$

where K_3 is a compact operator.

On the other hand, on the support of χ_2 we have $\lambda - V \geq \frac{3}{2}\epsilon$. Therefore

$$I_{2,2} \geq \frac{3}{2}\epsilon f(H)\chi_2^2 f(H).$$

Summing these estimates about $I_{2,1}$ and $I_{2,2}$, we get

$$I_2 \geq \frac{1}{2}\epsilon f(H)\chi_2^2 f(H) + K_3. \tag{5.11}$$

Finally, we show that I_3 is a compact operator. On the support of χ_3, one has $V(x) \geq \lambda + \frac{3\epsilon}{2}$. Thus, the support of χ_3 is contained in the classically forbidden region of the operator $f(H)$. In particular, by the semiclassical Weyl calculus, one has $f(H)\chi_3 = \mathcal{O}\left(h^\infty \langle\xi\rangle^{-\infty}\langle x\rangle^{-\infty}\right)$ on the symbolic level. Therefore $f(H)\chi_3$, and hence I_3, are compact operators.

Combining this with (5.8), (5.9), (5.11) and using the fact that $f(H)\chi_0^2 f(H)$ is also a compact operator, we get, with another compact operator K_4,

$$f(H)i[H, A_F]f(H) \geq \frac{h}{2}\min\left(\kappa\beta, \epsilon\right) f(H)^2 + K_4 \tag{5.12}$$

This ends the proof of the theorem. □

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