

RESTRICTION OF EISENSTEIN SERIES AND STARK-HEEGNER POINTS

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This short note is written based on my talk on a joint work with Shunsuke Yamana [HY20] in the RIMS conference “Analytic, geometric and p -adic aspects of automorphic forms and L -functions” during January 20-24, 2020. ¹ The author thanks the organizers for their hospitality during the conference.

1. THE WORK OF DARMON, POZZI AND VONK

Let F be a real quadratic field and let \mathfrak{d} be the different of F/\mathbf{Q} . Let $x \mapsto \bar{x}$ denote the non-trivial automorphism of F and let $N : F \rightarrow \mathbf{Q}$, $N(x) = x\bar{x}$ be the norm map. Let Δ_F be the discriminant of F/\mathbf{Q} . Let $\text{Cl}^+(\mathcal{O}_F)$ be the narrow ideal class group of F . Let $\phi : \text{Cl}^+(\mathcal{O}_F) \rightarrow \overline{\mathbf{Q}}^\times$ be an odd narrow ideal class character, i.e. $\phi((\delta)) = -1$ for any $\delta \in \mathcal{O}_F$ with $\bar{\delta} = -\delta$. Let $L(s, \phi)$ be the Hecke L -function attached to ϕ . Fix an odd rational prime p unramified in F . Fix an embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ throughout.

To each odd character ϕ of the narrow ideal class group of a real quadratic field F , we associate a one-variable p -adic family $E_k^{(p)}(1, \phi)$ of Hilbert Eisenstein series on $\Gamma_0(p)$ over a real quadratic field F defined by the q -expansion:

$$E_k^{(p)}(1, \phi)(z_1, z_2) = L^{(p)}(1 - k, \phi) + \sum_{\beta \in \mathfrak{d}_F^{-1}, \beta > 0} \sigma_{k-1, \phi}^{(p)}(\beta \mathfrak{d}_F) q^\beta,$$

$$q^\beta = \exp(2\pi\sqrt{-1}(\beta z_1 + \bar{\beta} z_2)),$$

where $L^{(p)}(1 - k, \phi) = (1 - \phi(p)p^{k-1})L(1 - k, \phi)$ and

$$\sigma_{k-1, \phi}^{(p)}(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{a}, (\mathfrak{b}, p) = 1} N\mathfrak{b}^{k-1} \phi(\mathfrak{b}).$$

Consider the elliptic modular form $G_{2k}(\phi)$ of weight $2k$ defined by

$$G_{2k}(\phi) := e_{\text{ord}} \left(E_k^{(p)}(1, \phi)(z, z) \right),$$

where e_{ord} is Hida’s p -ordinary projector. Then one can interpolate the function $k \in \mathbf{Z}_{\geq 2} \mapsto G_{2k}(\phi)$ into a locally analytic functions on \mathbf{Z}_p valued in the space of p -adic elliptic modular forms. Suppose that

p is inert in F .

¹The paper [HY20] was finished later after the conference. The text of this note has some overlapping with the content in [HY20].

It can be verified that

$$e_{\text{ord}}G_k(\phi)|_{k=1} = 0.$$

In [DPV19], the authors investigate the spectral decomposition of the derivative

$$\frac{d}{dk} (e_{\text{ord}}G_k(\phi))|_{k=1} = \sum_f \lambda'_f \cdot f. \in M_2(\Gamma_0(p)),$$

where f runs over all the normalized Hecke eigenforms in $M_2(\Gamma_0(p))$. The main results of [DPV19] show that if f is an Eisenstein series, then λ'_f is essentially the p -adic logarithm of elliptic units over F in [DD06], while if f is a cusp form, then λ'_f can be expressed in terms of the product of special values of the L -function for f and the p -adic logarithms of Stark-Heegner points or introduced in [Dar01].

Stark-Heegner points are local points defined by theory of p -adic double integration on the product of p -adic upper half planes and conjectured to be rational over Hilbert class fields of F . The work [DPV19] may shed some light on the global nature of Stark-Heegner points in the future provided one has some K -theoretic construction of Hilbert Eisenstein series.

2. STATEMENT OF THE MAIN RESULT

Now we introduce our recent work on partial generalizations of [DPV19] to the two-variable setting by introducing the cyclotomic variable. For $x \in \mathbf{Z}_p^\times$, let $\omega(x)$ be the Teichmüller lift of $x \pmod{p}$ and let $\langle x \rangle := x\omega^{-1}(x) \in 1 + p\mathbf{Z}_p$. Let $\mathcal{X} := \{x \in \mathbf{C}_p \mid |x|_p \leq 1\}$ be the p -adic closed unit disk and let $A(\mathcal{X})$ be the ring of rigid analytic functions on \mathcal{X} . For each ideal $\mathfrak{m} \triangleleft \mathcal{O}_F$ coprime to p , define $\sigma_\phi(\mathfrak{m}) \in A(\mathcal{X} \times \mathcal{X})$ by

$$\sigma_\phi(\mathfrak{m})(k, s) = \sum_{\mathfrak{a} \triangleleft \mathcal{O}_F, \mathfrak{a} | \mathfrak{m}} \phi(\mathfrak{a}) \langle N(\mathfrak{a}) \rangle^{\frac{k-s}{2}} \langle N(\mathfrak{m}\mathfrak{a}^{-1}) \rangle^{\frac{s-2}{2}}.$$

Let $\mathcal{X}^{\text{cl}} := \{k \in \mathbf{Z}^{\geq 2} \mid k \equiv 2 \pmod{2(p-1)}\}$ be the set of classical points in \mathcal{X} . Let $h = \#\text{Cl}^+(\mathcal{O}_F)$. Fix a set $\{\mathfrak{t}_\lambda\}_{\lambda=1, \dots, h}$ of representatives of the narrow ideal class group $\text{Cl}^+(\mathcal{O}_F)$ with $(\mathfrak{t}_\lambda, p\mathcal{O}_F) = 1$. For each classical point $k \in \mathcal{X}^{\text{cl}}$, the classical Hilbert-Eisenstein series $E_{\frac{k}{2}}(1, \phi)$ on $\text{SL}_2(\mathcal{O}_F)$ of parallel weight $\frac{k}{2}$ is determined by the normalized Fourier coefficients

$$c(\mathfrak{m}, E_{\frac{k}{2}}(1, \phi)) = \sigma_\phi(\mathfrak{m})(k, s), \quad c_\lambda(0, E_{\frac{k}{2}}(1, \phi)) = 4^{-1}L(1 - k/2, \phi).$$

Let I_F be the set of integral ideals of F . Let $\mathfrak{n} \in I_F$ and p be coprime. Let $\mathcal{M}^{(2)}(\mathfrak{n})$ be the space of two-variable p -adic families of Hilbert *semi-cusp* forms² of tame level \mathfrak{n} , which consists of functions

$$f: I_F \rightarrow \mathcal{A}(\mathcal{X} \times \mathcal{X}), \quad \mathfrak{m} \mapsto c(\mathfrak{m}, f)$$

²Recall that a Hilbert semi-cusp form is a Hilbert modular form having no constant in the Fourier expansion around the cusps at the infinity.

such that the specialization $f(k, s) = \{c(\mathfrak{m}, f)(k, s)\}$ is the set of normalized Fourier coefficients of a p -adic Hilbert semi-cusp forms of parallel weight k on $\Gamma_0(p\mathfrak{n})$ for (k, s) in a p -adically dense subset $U \subset \mathbf{Z}_p \times \mathbf{Z}_p$. Define $\widehat{E}_\phi^{\{p\}} : I_F \rightarrow A(\mathcal{X} \times \mathcal{X})$ by the data

$$\begin{aligned} c(\mathfrak{m}, \widehat{E}_\phi^{\{p\}}) &= \sigma_\phi(\mathfrak{m}) \text{ if } (\mathfrak{m}, p\mathcal{O}_F) = 1, \\ c(\mathfrak{m}, \widehat{E}_\phi^{\{p\}}) &= 0 \text{ otherwise.} \end{aligned}$$

By definition, for $(k, s) \in \mathcal{X}^{\text{cl}} \times \mathcal{X}^{\text{cl}}$ with $k \geq 2s$, we have

$$\widehat{E}_\phi^{\{p\}}(k, s) = \langle \Delta_F \rangle^{\frac{s-2}{2}} \cdot \theta^{\frac{s-2}{2}} E_{\frac{k+4-2s}{2}}^{\{p\}}(1, \phi),$$

where $E_k^{\{p\}}(1, \phi)$ is the p -depletion of $E_k(1, \phi)$ and θ is Serre's differential operator defined by $c(\mathfrak{m}, \theta f) = \Delta_F^{-1} N_{F/\mathbf{Q}}(\mathfrak{m}) c(\mathfrak{m}, f)$. Therefore, $\widehat{E}_\phi^{\{p\}}(k, s)$ is a p -adic Hilbert modular form of parallel weight k for all $(k, s) \in \mathbf{Z}_p^2$, and $\widehat{E}_\phi^{\{p\}} \in \mathcal{M}^{(2)}(\mathcal{O}_F)$. For each prime ideal \mathfrak{q} , define $\mathbf{U}_\mathfrak{q} : \mathcal{M}^{(2)}(\mathfrak{n}) \rightarrow \mathcal{M}^{(2)}(\mathfrak{n}\mathfrak{q})$ by $c(\mathfrak{m}, \mathbf{U}_\mathfrak{q} f) := c(\mathfrak{m}\mathfrak{q}, f)$. Let N be a positive integer such that $p \nmid N$ and

$$(\text{Splt}) \quad N\mathcal{O}_F = \mathfrak{N}\overline{\mathfrak{N}}, \quad (\mathfrak{N}, \overline{\mathfrak{N}}) = 1.$$

Define $\mathbf{E}_\phi \in \mathcal{M}^{(2)}(\mathfrak{N})$ by

$$\mathbf{E}_\phi := \prod_{\mathfrak{q}|\mathfrak{N}} (1 - \phi(\mathfrak{q})^{-1} \langle N(\mathfrak{q}) \rangle^{\frac{2s-2-k}{2}} \mathbf{U}_\mathfrak{q}) \cdot \widehat{E}_\phi^{\{p\}}$$

and the *diagonal restriction* $\mathbf{G}_\phi \in A(\mathcal{X} \times \mathcal{X})[[q]]$ of \mathbf{E}_ϕ by

$$\mathbf{G}_\phi := \sum_{n>0} \left(\sum_{\beta \in \mathfrak{d}_+^{-1}, \text{Tr}(\beta)=n} c(\beta\mathfrak{d}, \mathbf{E}_\phi) \right) q^n,$$

where \mathfrak{d}_+^{-1} is the additive semigroup of totally positive elements in \mathfrak{d}^{-1} .

By definition $\mathbf{G}_\phi(k, s)$ is the q -expansion of a p -adic elliptic modular on $\Gamma_0(pN)$ of weight k obtained from the diagonal restriction of $\widehat{E}_\phi^{\{p\}}(k, s)$ for $(k, s) \in \mathcal{X}^{\text{cl}} \times \mathcal{X}^{\text{cl}}$ with $k \geq 2s$. Let \mathcal{U} be an appropriate neighborhood around $2 \in \mathcal{X}$. Let $\mathbf{S}^{\text{ord}}(N)$ be the space of ordinary $A(\mathcal{U})$ -adic elliptic cusp forms on $\Gamma_0(Np)$, consisting of q -expansion $\mathbf{f} = \sum_{n>0} c(n, \mathbf{f}) q^n \in A(\mathcal{U})[[q]]$ such that the weight k specialization \mathbf{f}_k is a p -ordinary cusp forms of weight k on $\Gamma_0(pN)$ for $k \in \mathcal{X}^{\text{cl}}$. By Hida theory, we know $\mathbf{S}^{\text{ord}}(N)$ is a free $A(\mathcal{U})$ -module of finite rank. It can be shown that the image $e\mathbf{G}_\phi$ under Hida's ordinary projector actually belongs to $\mathbf{S}^{\text{ord}}(N) \widehat{\otimes}_{A(\mathcal{U})} A(\mathcal{U} \times \mathcal{X})$, where $A(\mathcal{U})$ is regarded as a subring of $A(\mathcal{U} \times \mathcal{X})$ via the pull-back of the first projection $\mathcal{U} \times \mathcal{X} \rightarrow \mathcal{U}$. We can thus decompose

$$e\mathbf{G}_\phi = \sum_{\mathbf{f}} \mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}} \cdot \mathbf{f} + (\text{old forms}), \quad \mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}} \in A(\mathcal{U} \times \mathcal{X}),$$

where \mathbf{f} runs over the set of primitive Hida families of tame conductor N . We shall call $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}} \in A(\mathcal{U} \times \mathcal{X})$ the twisted triple product p -adic L -function attached to the p -adic Hilbert Eisenstein series \mathbf{E}_ϕ and a primitive Hida family \mathbf{f} . We provide the following derivative formula for $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}$, which partially generalizes [DPV19, Theorem C(2)] to elliptic newforms of split tame conductor.

Theorem (H.- and S. Yamana). *Let E be an elliptic curve over \mathbf{Q} of conductor pN with N satisfying (Splt). Let $\mathbf{f} \in A(\mathcal{U})[[q]]$ be a primitive Hida family of tame level N such that the weight two specialization \mathbf{f}_2 is the elliptic newform associated with E . Suppose that p is inert in F . Then $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(2, s) = 0$ and*

$$\begin{aligned} \frac{d}{dk}(\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(k, s+1))|_{k=2} &= \frac{1}{2}(1 + \phi(\mathfrak{N})^{-1}w_N) \cdot \log_E P_\phi \cdot L_p(E, s) \\ &\quad \times \frac{c_f}{m_E^2 2^{\alpha(E)}} \langle \Delta_F \rangle^{\frac{s-1}{2}}, \end{aligned}$$

where

- $\log_E P_\phi$ is the p -adic logarithm of the twisted Stark-Heegner point $P_\phi \in E(F_p) \otimes \mathbf{Q}(\phi)$ introduced in [Dar01, (182)],
- $L_p(E, s)$ is the Mazur-Tate-Teitelbaum cyclotomic p -adic L -function for E ,
- $c_f \in \mathbf{Z}^{>0}$ is the congruence number for f , $m_E \in \mathbf{Q}^\times$ is the Mainn constant for E and $2^{\alpha(E)} = [\mathbf{H}_1(E(\mathbf{C}), \mathbf{Z}) : \mathbf{H}_1(E(\mathbf{C}), \mathbf{Z})^+ \oplus \mathbf{H}_1(E(\mathbf{C}), \mathbf{Z})]$.

Remark 2.1.

- The definition of Stark-Heegner points P_ϕ for odd ϕ in [Dar01] depends on a choice of the purely imaginary period $\Omega_{\bar{E}}$. In the above theorem, we require $(\sqrt{-1})^{-1}\Omega_{\bar{E}}$ to be positive.
- Our main motivation for this two-variable generalization is that we have the non-vanishing of the p -adic L -function $L_p(E, s)$ thanks to Rohrlich's theorem [Roh84], so $\log_E P_\phi$ can be computed from the twisted triple product p -adic L -function even when the central value $L(E, 1)$ vanishes.
- The Eisenstein contribution in the spectral decomposition in Part (2) of [DPV19, Theorem C] is connected with the p -adic logarithms of elliptic units over F , while in our two-variable setting, $e\mathbf{G}_\phi$ is a p -adic family of cusp forms, so we do not get any information for elliptic units.

3. THE IDEA OF THE PROOF

We briefly explain the idea of the proof. Let $\mathcal{L}_p(\mathbf{f}/F, \phi, k)$ be the (odd) square-root p -adic L -function associated with the primitive Hida family \mathbf{f} and the character ϕ constructed in [BD09, Definition 3.4] with $w_\infty = -1$ and let $L_p(\mathbf{f}, k, s)$ be the Mazur-Kitagawa two-variable p -adic L -function so

that $L_p(\mathbf{f}, 2, s)$ is the cyclotomic p -adic L -function for \mathbf{f}_2 . We prove the following factorization formula of $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}$:

$$(3.1) \quad C^*(k) \cdot \mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(k, s+1) = \mathcal{L}_p(\mathbf{f}/F, \phi, k) \cdot L_p(\mathbf{f}, k, s),$$

where $C^*(k)$ is a meromorphic function on \mathcal{X} holomorphic at all classical points $k \in \mathcal{X}^{\text{cl}}$ with $C^*(2) = 1$. By the very construction, the square root p -adic L -function $\mathcal{L}_p(\mathbf{f}/F, \phi, k)$ interpolates the toric period integrals $B_{\mathbf{f}_k}^\phi$. Thus we get $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(2, s) = \mathcal{L}_p(\mathbf{f}/F, \phi, 2) = 0$ by a classical theorem of Saito and Tunnell. Moreover, from the formula [BD09, Corollary 2.6], it is not difficult to deduce that the first derivative of $\mathcal{L}_p(\mathbf{f}/F, \phi, k)$ at $k = 2$ is $2^{-1}(1 + w_N\phi(\mathfrak{N})^{-1})\log_E P_\phi$, and hence we obtain Theorem from (3.1). The factorization formula (3.1) is established by the explicit interpolation formulae on both sides. In particular, the interpolation formula for $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(k, s)$ is the most technical part of this paper. Roughly speaking, for $(k, s) \in \mathcal{X}^{\text{cl}} \times \mathcal{X}^{\text{cl}}$ with $k \geq 2s$, Hida's p -adic Rankin-Selberg method shows that $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(k, s)$ is interpolated by the inner product between the diagonal restriction of a nearly holomorphic Hilbert Eisenstein series $\mathbf{E}_\phi(k, s)$ and \mathbf{f}_k . Therefore, a result of Keaton and Pitale [KP19, Proposition 2.3] tells us that $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(k, s)$ is a product of (i) the Waldspurger toric period integral $B_{\mathbf{f}_k}^\phi$ of \mathbf{f}_k over F twisted by ϕ , (ii) the special value $L(\mathbf{f}_k, s)$ of the L -function for \mathbf{f}_k and (iii) local zeta integrals $Z_{\mathcal{D}}(s, B_{W_v})$ for every place of \mathbf{Q} . Now items (i) and (ii) are basically interpolated by $\mathcal{L}(\mathbf{f}/K, \phi, k)$ and $L_p(\mathbf{f}, k, s)$, so our task is to evaluate explicitly these local zeta integrals, which occupy the main body of Section 4. By the explicit interpolation formulae of these p -adic L -functions, we find immediately that the ratio C^* between $\mathcal{L}_p(\mathbf{f}/F, \phi, k) \cdot L_p(\mathbf{f}, k, s)$ and $\mathcal{L}_{\mathbf{E}_\phi, \mathbf{f}}(k, s+1)$ is independent of s , and hence C^* is a meromorphic function in k only. Finally, by a standard argument using Rohrlich's result on the non-vanishing of the cyclotomic p -adic L -functions for elliptic modular forms, we can conclude that $C^*(k)$ is holomorphic at all $k \in \mathcal{X}^{\text{cl}}$ and $C^*(2)$ is essentially the congruence number.

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