A CONSTRUCTION OF p-ADIC ASAI L-FUNCTIONS OVER CM FIELDS

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Abstract. This article is a survey on the author's preprint [Na], where the author constructs a p-adic Asai L-functions for irreducible cohomological cuspidal autmorphic representations of GL_2 over CM fields.

1. Introduction

This article is a report of the author's talk at the conference " Analytic, geometric and p-adic aspects of automorphic forms and L-functions " , which was held at RIMS, Kyoto university between 20th to 24th, January, 2020. The aim of this article is to summarize the main result in the talk, which can be found in the preprint [Na]. In [Na], the author discusses a construction of p-adic Asai L-functions for irreducible cohomological cuspidal autmorphic representations of GL_2 over CM fields. In this survey article, we write down the interpolation formula for p-adic Asai L-functions and we compare the formula with a conjecture on the existence of p-adic L-functions by Coates and Perrin-Riou ([CPR89], [Co89]). We also briefly discuss the strategy of the proof.

We introduce some notations to state the statement of the main theorem in [Na]. Fix an embedding $\overline{\mathbf{Q}} \to \mathbf{C}$ and fix an isomorphism $\mathbf{i}_p : \mathbf{C} \to \mathbf{C}_p$ for an odd prime p. Let F be a totally real field and E/F a CM extension. For each number field L, we denote by Σ_L (resp. I_L) the set of places of L (resp. the set of embeddings from L to $\mathbf{C}$). Define $\Sigma_{L,\infty}$ be the set of infinite places of L and put $r_L = \sharp\Sigma_{L,\infty}$.

Let $n = \sum_{\sigma\in I_E} n_\sigma \sigma$ be an element of $\mathbf{Z}_{\geq 0}[I_E]$ which satisfies the following two conditions:

- For all $\sigma, \tau \in I_E$, $n_\sigma \equiv n_\tau \bmod 2$;
- For each complex place $\sigma \in \Sigma_E$, $n_\sigma = n_{c\sigma}$, where c is the complex conjugate.

We choose an element $m = \sum_{\sigma\in I_E} m_\sigma \sigma$ of $\mathbf{Z}_{\geq 0}[I_E]$ which satisfies the following condition:

- For all $\sigma, \tau \in I_E$, $n_\sigma + 2m_\sigma = n_\tau + 2m_\tau$.

We denote $\Sigma_{\sigma\in I_E}\sigma \in \mathbf{Z}[I_E]$ by t.

Consider an irreducible unitary cohomological cuspidal automorphic representation π of $\mathrm{GL}_2(E_{\mathbf{A}})$ with the central character ω_π. Recall that, for each $\sigma \in \Sigma_{E,\infty}$, the Langlands parameter of π_σ is given by

$$\phi[\pi_\sigma] : W_{E_\sigma} = \mathbf{C}^\times \to \mathrm{GL}_2(\mathbf{C}); z \mapsto \mathrm{diag}(z^{\frac{n_\sigma+1}{2}}\overline{z}^{-\frac{n_\sigma+1}{2}}, z^{-\frac{n_\sigma+1}{2}}\overline{z}^{\frac{n_\sigma+1}{2}}),$$

where $\overline{*}$ is the complex conjugate of $*$. Assume that $p > \max\{n_\sigma \mid \sigma \in \Sigma_{E,\infty}\}$. For each ideal $\mathfrak{N} \subset \widehat{\mathcal{O}}_E$, define a subgroup $\mathcal{K}_1(\mathfrak{N})$ of $\mathrm{GL}_2(\widehat{\mathcal{O}}_E)$ to be

$$\mathcal{K}_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_E) \;\middle|\; c, d-1 \in \mathfrak{N} \right\}.$$

Suppose that π has the conductor $\mathfrak{N}$, that is, π has a $\mathcal{K}_1(\mathfrak{N})$-fixed vector and $\mathfrak{N}$ is minimal among such ideals. We also suppose that $\mathfrak{N}$ is prime to the discriminant of E/F.

Let $\mathrm{As}^+(\pi)$ be the Asai lift of π, which is an isobaric automorphic representation of $\mathrm{GL}_4(F_{\mathbf{A}})$. We denote by $\mathrm{As}^+_{\mathcal{M}}(\pi)$ the conjectural Asai motive attached to π, which satisfies

$$L(s, \mathrm{As}^+_{\mathcal{M}}(\pi)) = L(s+1, \mathrm{As}^+(\pi)).$$

We briefly recall the expected properties of $\mathrm{As}^+_{\mathcal{M}}(\pi)$ in Section 2.

Let $\alpha = \sum_{\sigma \in I_F} \alpha_\sigma \sigma \in \mathbf{Z}[I_F]$ satisfying the following conditions:

(Alp1) for each $\sigma \in I_F$, $0 \leq \alpha_\sigma \leq n_\sigma$;
(Alp2) for each $\sigma, \tau \in I_F$, we have $n_\sigma - \alpha_\sigma = n_\tau - \alpha_\tau$.

By (Alp2), we sometimes identify $n - \alpha$ with an integer.

Let $\mu_{p^\infty} \subset \overline{\mathbf{Q}}$ be the group of p-power roots of 1. Fix a finite extension K_π of $\mathbf{Q}_p$ so that K_π contains all conjugates of E, all Hecke eigenvalues of π and the values of ω_π. Denote by $\mathcal{O}_\pi$ the ring of integers of K_π.

The following statement is the main theorem in [Na].

Theorem 1.1. ([Na, Theorem 8.8]) *Assume that*

- *π is nearly p-ordinary;*
- *$\omega_{\pi,p}$ is unramified;*
- *if π is conjugate self-dual, then $\alpha \neq n$;*
- *the conductor $\mathfrak{N}$ of π is square-free;*
- *for each $v \mid \mathfrak{N}_F := \mathfrak{N} \cap F$ with $v \nmid p$, suppose either one of the following conditions:*
  - *$\omega_{\pi,v}$ is ramified;*
  - *if $\omega_{\pi,v}$ is unramified, then $v = ww_c \mid \mathfrak{N}_F$ splits in E/F and one of π_w and π_{w_c} is an unramified principal representation and the other is a special representation.*

Then, there exists $\mathscr{L}_p^\alpha(\mathrm{As}^+_{\mathcal{M}}(\pi)) \in K_\pi \otimes_{\mathcal{O}_\pi} \mathcal{O}_\pi[[\mathrm{Gal}(F(\mu_{p^\infty})/F)]]$ for each finite-order Hecke character φ of a p-power conductor satisfying that $(-1)^{n-\alpha}\varphi(-1_\sigma) = 1$ for each $\sigma \in \Sigma_{F,\infty}$, we have

$$\widehat{\phi}(\mathscr{L}_p^\alpha(\mathrm{As}^+_{\mathcal{M}}(\pi))) = \mathcal{E}_\infty(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))\mathcal{E}_p(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))\frac{L(0, \mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))}{\Omega(\mathrm{As}^+_{\mathcal{M}}(\pi))}, \tag{1.1}$$

where $\phi = |\cdot|_{F_{\mathbf{A}}}^{n-\alpha}\varphi$ and $\widehat{\phi}$ is the p-adic avatar of ϕ, and

- *$\mathcal{E}_\infty(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ and $\mathcal{E}_p(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ are the modified Euler factor at ∞ and p respectively;*
- *$\Omega(\mathrm{As}^+_{\mathcal{M}}(\pi))$ is the period of $\mathrm{As}^+_{\mathcal{M}}(\pi)$ due to Coates* ([Co89, page 107])*, which is a product of Deligne's period $c^+(\mathrm{As}^+_{\mathcal{M}}(\pi))$ of $\mathrm{As}^+_{\mathcal{M}}(\pi)$ and a power of the circular constant.*

We will recall the definitions of the modified Euler factors $\mathcal{E}_\infty(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi)), \mathcal{E}_p(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ and the periods $\Omega(\mathrm{As}^+_{\mathcal{M}}(\pi))$ in Section 3.

Remark 1.2. (1) Let $\varphi : F^\times \backslash F_{\mathbf{A}}^\times \to \mathbf{C}^\times$ be a Hecke character satisfying the conditions in Theorem 1.1 and put $\phi = |\cdot|_{F_{\mathbf{A}}}^{n-\alpha}\varphi$. Then $L(0, \mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ is a critical value in the sense of Deligne ([De79]). The algebraicity of critical values of Asai L-functions is proved by Ghate ([Gh99a, page 635, Theorem 1], [Gh99b, page 106, Theorem 1]).

(2) If $F = \mathbf{Q}$, there is a work by Loeffler-Williams ([LW]). We can find similar results [Ba17] and [BGV]. Theorem 1.1 gives a generalization of [LW] in general CM fields case and this refines the results in [Ba17] and [BGV].

(3) Let $\mathrm{Tw}_p : K_\pi \otimes_{\mathcal{O}_\pi} \mathcal{O}_\pi[[\mathrm{Gal}(F(\mu_{p^\infty})/F)]] \to K_\pi \otimes_{\mathcal{O}_\pi} \mathcal{O}_\pi[[\mathrm{Gal}(F(\mu_{p^\infty})/F)]]$ be

$$\mathrm{Tw}_p(g) = \varepsilon_{\mathrm{cyc}}(g)g,$$

where $\varepsilon_{\mathrm{cyc}} : \mathrm{Gal}(F(\mu_{p^\infty})/F) \to \mathbf{C}_p^\times$ is the p-adic cyclotomic character. Then Coates and Perrin-Riou's conjecture predicts that

$$\mathrm{Tw}_p^{\alpha'-\alpha}(\mathscr{L}_p^{\alpha}(\mathrm{As}^+_{\mathcal{M}}(\pi))) = \mathscr{L}_p^{\alpha'}(\mathrm{As}^+_{\mathcal{M}}(\pi))$$

for each $0 \leq \alpha, \alpha' \leq n$. If the base field F is the rational number field, the above identity is established in [LW, Proposition 5.6]. So far, we can only interpolate the critical values twisted by only finite-order Hecke characters in Hilbert settings.

The structure of this article is as follows. In Section 2, we introduce the conjectural Asai motive $\mathrm{As}^+_{\mathcal{M}}(\pi)$ and basic notations on Asai L-functions $L(s, \mathrm{As}^+(\pi))$. In Section 3, the definitions of the modified Euler factor and periods are recalled according to [Co89]. The strategy of the proof of Theorem 1.1 is briefly explained in Section 4.

We fix some notations. For each number field L and $v \in \Sigma_L$, L_v is the completion of L at v. Let $\mathcal{O}_L$ is the ring of integers of L and $\widehat{\mathcal{O}}_L = \mathcal{O}_L \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$. For the completion $\mathcal{O}_{L,v}$ of $\mathcal{O}_L$ at $v \in \Sigma_L, v < \infty$, we denote by ϖ_v a uniformizer of $\mathcal{O}_{L,v}$ and put $q_v = \sharp\mathcal{O}_{L,v}/\varpi_v\mathcal{O}_{L,v}$. Put $L_\infty = \prod_{v\in\Sigma_{L,\infty}} L_v$. For each ideal $\mathfrak{M} \subset \widehat{\mathcal{O}}_L$, define compact subgroups of $\mathrm{GL}_2(\widehat{\mathcal{O}}_L)$ to be

$$\mathcal{K}_0^L(\mathfrak{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_L) \;\middle|\; c \in \mathfrak{M} \right\},$$
$$\mathcal{K}^L(\mathfrak{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_L) \;\middle|\; a-1, b, c, d-1 \in \mathfrak{M} \right\}.$$

Put

$$\mathcal{K}_\infty^F = F_\infty^\times \prod_{v\in\Sigma_{F,\infty}} \mathrm{SO}_2(\mathbf{R}), \quad \mathcal{K}_\infty^E = E_\infty^\times \prod_{v\in\Sigma_{E,\infty}} \mathrm{SU}_2(\mathbf{R}).$$

We frequently use the multi-index notation. We sometimes identify $a \in \mathbf{Z}$ with $at \in \mathbf{Z}[I_E]$.

2. Asai motives

In this section, we summarize notations on Asai L-functions, which are necessary to state the interpolation formulas of p-adic Asai L-functions. To begin with, we introduce conjectural motives attached to Asai L-functions, which we call Asai motives. The detail can be found in [Na, Section 3.2, 3.3].

The conjectural motive $\mathcal{M}[\pi]$ over E attached to π ([Cl90, Conjecture 4.5], [Hi94, Section 8]), which has a pure weight $n + 2m + 1$, satisfies

- $H_{\mathrm{B}}(\mathcal{M}[\pi]_\sigma) \otimes \mathbf{C} = H^{n_\sigma+m_\sigma+1,m_\sigma}(\mathcal{M}[\pi]) \oplus H^{m_\sigma,n_\sigma+m_\sigma+1}(\mathcal{M}[\pi]), \quad (\sigma \in I_E);$
- $L(s, \mathcal{M}[\pi]) = L(s + \frac{[\kappa]+1}{2}, \pi),$

where $H_{\mathrm{B}}(*)$ is the Betti realization of $(*)$ and we write $n + 2m = [\kappa]t$ for an integer $[\kappa] \in \mathbf{Z}$. Denote by $\mathrm{As}^+(\mathcal{M}[\pi])$ the Asai motive over F attached to π, that is, $\mathrm{As}^+(\mathcal{M}[\pi])$ is the descent of the motive $\mathcal{M}[\pi] \otimes \mathcal{M}[\pi]^c$ over E to F with the descent datum $v \otimes w \mapsto w \otimes v$ ([Gh99a, Section 4]). We put

$$\mathrm{As}^+_{\mathcal{M}}(\pi) = \mathrm{As}^+(\mathcal{M}[\pi])(n + 2m + 2),$$

which is a pure motive of the weight -2. Let $h(i,j) = \dim_{\mathbf{C}} H^{i,j}(\mathrm{As}^+_{\mathcal{M}}(\pi))$. Note that $\mathrm{As}^+_{\mathcal{M}}(\pi)$ has the following properties:

- $H_{\mathrm{B}}(\mathrm{As}^+_{\mathcal{M}}(\pi)_\sigma) \otimes \mathbf{C}$
  $= H^{-n_\sigma-2,n_\sigma}(\mathrm{As}^+_{\mathcal{M}}(\pi)_\sigma) \oplus H^{-1,-1}(\mathrm{As}^+_{\mathcal{M}}(\pi)_\sigma) \oplus H^{n_\sigma,-n_\sigma-2}(\mathrm{As}^+_{\mathcal{M}}(\pi)_\sigma);$
- $h(-n_\sigma - 2, n_\sigma) = h(n_\sigma, -n_\sigma - 2) = 1, \quad h(-1,-1) = 2;$
- the complex conjugate acts on $H^{-1,-1}(\mathrm{As}^+_{\mathcal{M}}(\pi))$ as $+1$.

We see that $\mathrm{As}^+_{\mathcal{M}}(\pi)$ is critical at $s=j$ if and only if j is an element in the following set:

$$\{j \mid \text{odd}, -n_{\min}-1 \le j \le -1\} \cup \{j \mid \text{even}, 0 \le j \le n_{\min}\}, \quad (n_{\min} := \min\{n_\sigma | \sigma \in I_E\}).$$

In particular, $\mathrm{As}^+_{\mathcal{M}}(\pi)$ is critical at $s=0$. Consider n as an element in $\mathbf{Z}[I_F]$, since $n_\sigma = n_{c\sigma}$ for each $\sigma \in I_E$. Let $\alpha = \sum_{\sigma \in I_F} \alpha_\sigma \sigma \in \mathbf{Z}[I_F]$ satisfying the conditions (Alp1), (Alp2) in Section 1. The following proposition immediately follows from the above descriptions.

Proposition 2.1. *Let $\varphi : F^\times \backslash F^\times_{\mathbf{A}} \to \mathbf{C}^\times$ be a Hecke character of finite-order and $\alpha \in \mathbf{Z}[I_F]$ as above. Put $\phi = |\cdot|^{n-\alpha}_{F_{\mathbf{A}}} \varphi$. Then $\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi)$ is critical at $s=0$ if and only if*

- *$n-\alpha$ is even and $\varphi_\sigma(-1) = 1$ for each $\sigma \in \Sigma_{F,\infty}$.*
- *$n-\alpha$ is odd and $\varphi_\sigma(-1) = -1$ for each $\sigma \in \Sigma_{F,\infty}$.*

For π, we have the Asai lift $\mathrm{As}^+(\pi)$ of π, which is an isobaric automorphic representation of $\mathrm{GL}_4(F_{\mathbf{A}})$ due to [Kr03, Theorem 6.3]. Recall that the Asai L-functions $L(s, \mathrm{As}^+(\pi)) = \prod_{v \in \Sigma_F} L(s, \mathrm{As}^+(\pi_v))$ is meromorphically continued to the whole $\mathbf{C}$-plane with possible pole at $s = 0$ or 1 satisfying

$$L(s, \mathrm{As}^+_{\mathcal{M}}(\pi)) = L(s+1, \mathrm{As}^+(\pi)).$$

Assuming the contragradient $\pi^\vee$ of π is not isomorphic to the complex conjugate π^c of π, it is known that $L(s, \mathrm{As}^+(\pi))$ is entire (see [GS15, Theorem 4.3]).

3. Modified Euler factors

In this section we summarize the modified Euler factors and periods which appear in the interpolation formula of Asai L-functions in Theorem 1.1. These constants are introduced in [CPR89] and [Co89] for general motives. We wirte down them in the case of Asai motives. The detail can be found in [Na, Section 4].

(Modifeid Euler factor $\mathcal{E}_v(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ at $v \mid p$)

Let $\psi_{\mathbf{Q}} = \otimes_{v \in \Sigma_{\mathbf{Q}}} \psi_{\mathbf{Q},v} : \mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}} \to \mathbf{C}^\times$ be the additive character such that $\psi_{\mathbf{Q},\infty}(x) = \exp(2\pi\sqrt{-1}x)$ $(x \in \mathbf{R})$. Let ψ_F be $\psi_{\mathbf{Q}} \circ \mathrm{Tr}_{F/\mathbf{Q}}$, where $\mathrm{Tr}_{F/\mathbf{Q}} : F_{\mathbf{A}} \to \mathbf{Q}_{\mathbf{A}}$ is the trace map, and $\psi_{F,v}$ the restriction of ψ_F to F_v.

Let $\chi_v : F_v^\times \to \mathbf{C}^\times$ be a continuous character. Define the γ-factor of $\mathrm{As}^+(\pi_v) \otimes \chi_v$ to be

$$\gamma(s, \mathrm{As}^+(\pi_v) \otimes \chi_v, \psi_{F,v}) = \epsilon(s, \mathrm{As}^+(\pi_v) \otimes \chi_v, \psi_{F,v}) \frac{L(1-s, \mathrm{As}^+(\pi_v)^\vee \otimes \chi_v^{-1})}{L(s, \mathrm{As}^+(\pi_v) \otimes \chi_v)}, \tag{3.1}$$

where $\epsilon(s, \mathrm{As}^+(\pi_v) \otimes \chi_v, \psi_{F,v})$ is the ϵ-factor and $\mathrm{As}^+(\pi_v)^\vee$ is the contragradient of $\mathrm{As}^+(\pi_v)$.

Let $v \in \Sigma_F, v \mid p$, which is unramifed in E/F. Let $w, w_c \in \Sigma_E$ so that $v = ww_c$ if v is split in E/F. If v is inert in E/F, write $w = v \in \Sigma_E$. Suppose that π is nearly p-ordinary. For each $v \mid p$, we write $\pi_v = (\pi_w, \pi_{w_c})$ (resp. π_w) if v is split (resp. inert). Let $\alpha_{\pi_w}, \beta_{\pi_w}$ (resp.$\alpha_{\pi_{w_c}}, \beta_{\pi_{w_c}}$if v is split) be the Satake parameter of π_w (resp. π_{w_c}). Write $\alpha_{\pi_v} = \alpha_{\pi_w}\alpha_{\pi_{w_c}}, \beta_v = \beta_{\pi_w}\beta_{\pi_{w_c}}$ if v is split, and $\alpha_{\pi_v} = \alpha_{\pi_w}, \beta_{\pi_v} = \beta_{\pi_w}$ if v is inert. Since π is nearly p-ordinary, we may assume that $\{\varpi_w^m\}^{-1}\beta_{\pi_w} q_w^{\frac{[\kappa]+1}{2}}$ is a p-adic unit, where $\{\varpi_w^m\} \in \mathcal{O}_\pi$ is defined as in [Hi94, Section 4] (see also [Na, Section 4]). Note that $\{\varpi_w^m\}^{-1}\beta_{\pi_w} q_w^{\frac{[\kappa]+1}{2}}$ is an eigenvalue of the modified Hecke operator $U_0(\varpi_w)(w \mid p)$.

For $\gamma \in \mathbf{C}^\times$, define an unramified character $\chi_\gamma : F_v^\times \to \mathbf{C}^\times$ so that $\chi_\gamma(\varpi_v) = \gamma$. Let $\gamma(s, \chi_{\alpha_{\pi_v}}\varphi_v, \psi_{F,v})$ is the γ-factor of $\chi_{\alpha_{\pi_v}}\varphi_v$ associated with the fixed additive character $\psi_{F,v}$. Define the modified Euler factor $\mathcal{E}_v(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ to be

$$\mathcal{E}_v(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi)) L_v(0, \mathrm{As}^+_{\mathcal{M}}(\pi)(\phi)) = \frac{\gamma(n-\alpha+1, \chi_{\alpha_{\pi_v}}\varphi_v, \psi_{F,v})}{\gamma(n-\alpha+1, \mathrm{As}^+(\pi_v) \otimes \varphi_v, \psi_{F,v})},$$

where $\gamma(s, \mathrm{As}^+(\pi_v) \otimes \varphi_v, \psi_{F,v})$ is the γ-factor in (3.1). Suppose that π_v is unramified and the base field F is the rational number field. Then this definition is compatible with the definition of the modified Euler factor in [Co89, Section 2, (18)].

(Modifeid Euler factor $\mathcal{E}_v(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ at $v \mid \infty$) Let $v \in \Sigma_{F,\infty}$. Put $\Gamma_{\mathbf{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. Define the modified Euler factor $\mathcal{E}_v(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi))$ to be

$$\mathcal{E}_v(\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi)) = \begin{cases} \dfrac{\sqrt{-1}^{-(2n_v-\alpha_v+2)}}{(-1)\times\Gamma_{\mathbf{R}}(1-(n_v-\alpha_v))^2}, & (n_v-\alpha_v : \text{even}, \varphi_v(-1)=1), \\ \dfrac{\sqrt{-1}^{-(2n_v-\alpha_v+2)}}{\Gamma_{\mathbf{R}}(-(n_v-\alpha_v))^2}, & (n_v-\alpha_v : \text{odd}, \varphi_v(-1)=-1)). \end{cases}$$

This definition coincides with the definition of the modified Euler factor in [Co89, Section 1, (4)].

(Period $\Omega(\mathrm{As}^+_{\mathcal{M}}(\pi))$) Let $c^+(\mathrm{As}^+_{\mathcal{M}}(\pi))$ be the Deligne's period attached to the Asai motive $\mathrm{As}^+_{\mathcal{M}}(\pi)$. Coates introduced a constant $\tau(\mathrm{As}^+_{\mathcal{M}}(\pi))$ in [Co89, (12)] and a period $\Omega(\mathrm{As}^+_{\mathcal{M}}(\pi))$ to describe Deligne's conjecture on critical values ([De79, Conjecture 1.8]) in terms of the complete L-functions. We write down it explicitly in our setting as follows:

$$\tau(\mathrm{As}^+_{\mathcal{M}}(\pi)) = -n - 4t, \quad \Omega(\mathrm{As}^+_{\mathcal{M}}(\pi)) = (2\pi\sqrt{-1})^{\tau(\mathrm{As}^+_{\mathcal{M}}(\pi))} \times c^+(\mathrm{As}^+_{\mathcal{M}}(\pi)).$$

[Gh99a, page 613, Proposition 3, page 637, Remark 3] shows that $\Omega(\mathrm{As}^+_{\mathcal{M}}(\pi))$ can be taken as Hida's canonical period $\Omega_{\pi,p}$. (See also the argument in [Na, Section 3.2, 4] for the detail.) Assume that $\mathrm{As}^+_{\mathcal{M}}(\pi)(\phi)$ is critical at $s = 0$. Then [Co89, page 107, Period Conjecture] predicts the right-hand side of (1.1) gives an element in $\overline{\mathbf{Q}}$. The algebraicity of the right-hand side of (1.1) is proved by Ghate ([Gh99a, page 635, Theorem 1], [Gh99b, page 106, Theorem 1]), if ϕ is of the form $|\cdot|^{n-\alpha}_{F_{\mathbf{A}}}$ for even $n-\alpha$ $(0 \leq \alpha \leq n)$.

4. Strategy of proof

The algebraicity of the critical values of Asai L-functions is proved by studying the cup product of the differential forms associated with cusp forms and Eisenstein cohomology classes. This is done by Ghate ([Gh99a], [Gh99b]). The key point of the construction of p-adic L-functions is to make the critical values of Asai L-functions twisted by Hecke character of conductor p-power a good family satisfying distribution properties. For this purpose, we have to consider a cohomological interpretation of partial Asai L-functions generalizing the work of Ghate, which is called Birch lemma. The integral expression of partial Asai L-functions is given by a cup product of cusp forms and Eisenstein series. Hence the choices of cusp forms and Eisenstein series are key ingredients of the construction.

In this section, we briefly describe this strategy. In particular, the choice of Eisenstein series (resp. cusp forms) is introduced in Section 4.1 (resp. 4.2). The Birch lemma, which gives a cohomological interpretation of partial Asai L-functions, is given in Section 4.3. The sketch of the construction is given in Section 4.4.

In the following argument, we assume that $\mathfrak{N}$ is not trivial and that the central character ω_π of π is trivial for the sake of the simplicity. (If $\mathfrak{N}$ is trivial, then we have to introduce the choice of an " auxiliary prime " for the convergence of Eisenstein series in certain cases.)

4.1. A good family of Eisenstein series.

We firstly choose a good family of Eisenstein series $E_{r,s}$ on $\mathrm{GL}_2(F_{\mathbf{A}})$ for $r \in \mathbf{Z}, r \geq 0$ and $s \in \mathbf{C}$. The choice of Eisenstein series depends on the choice of the Schwartz function $\Phi^{(r)} = \prod_{v\in\Sigma_F} \Phi^{(r)}_v$ on $F^{\oplus 2}_{\mathbf{A}}$. Note that $\Phi^{(r)}_v$ depends on r only for places $v \in \Sigma_F$ above p. Since the key point of the construction of the Eisenstein series is the choice of $\Phi^{(r)}_v$ for each $v \mid p$, we explicitly describe it as follows.

Define a Bruhat-Schwartz function $\mathbb{I}_{\mathcal{O}_{F,v}^{\oplus 2}}$ on $F_v^{\oplus 2}$ to be the chracteristic function of $\mathcal{O}_{F,v}^{\oplus 2}$. By using the above notations, we define a Bruhat-Schwartz function $\Phi_v^{(r)}$ on $F_v^{\oplus 2}$ to be

$$\Phi_v^{(r)}(x,y)=\psi_{F,v}\left(\frac{x}{p^r}\right)\mathbb{I}_{\mathcal{O}_{F,v}^{\oplus 2}}(x,y)\quad ((x,y)\in F_v^{\oplus 2}).$$

Note that $\Phi_v^{(r)}$ is invariant under the right-translation by $\mathcal{K}^F(p^r)$.

We also choose $\Phi_\sigma^{(r)}$ for each infinite place $\sigma\in\Sigma_F$ in an explicit way. The choice of $\Phi_\sigma^{(r)}$ makes $E_{r,s}$ an automorphic form on $\mathrm{GL}_2(F_{\mathbf{A}})$ of the parallel weight $2n-2\alpha+2t$. Let $\mathcal{L}(2n-2\alpha;\mathbf{C})$ be the local system on $Y_{\mathcal{K}}(p^r):=\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(F_{\mathbf{A}})/\mathcal{K}_\infty^F\mathcal{K}_0^E(\mathfrak{N})\cap\mathcal{K}^F(p^r)$, which is defined by the symmetric tensor product of order $2n-2\alpha$ of the standard representation of $\mathrm{GL}_2(F_\infty)$. Then the Eisenstein series $E_{r,0}$ defines a cohomology class $\delta(E_{r,0})$ in $H^{r_F}(Y_{\mathcal{K}}(p^r),\mathcal{L}(2n-2\alpha;\mathbf{C}))$, which is called an Eisenstein cohomology class. By studying the constant term of $E_{r,s}$, we find that $\delta(E_{r,0})$ is a cohomology class with a rational coefficient ([Ha87, Section IV]). See [Na, Section 5, 6.3] for the detail.

4.2. **Restriction of cusp forms.** We introduce the choice of cusp forms on $\mathrm{GL}_2(E_{\mathbf{A}})$ and we describe its restriction to $\mathrm{GL}_2(F_{\mathbf{A}})$ in terms of cohomology classes.

Let $f:\mathrm{GL}_2(E_{\mathbf{A}})\to\otimes_{v\in\Sigma_{E,\infty}}\mathrm{Sym}^{2n_v+2}(\mathbf{C}^{\oplus 2})$ be a cusp form such that the unitarization of the automorphic representation associated with f is π. Let $\mathcal{L}(n;\mathbf{C})$ be a local system on $Y_{\mathcal{K}}^E:=\mathrm{GL}_2(E)\backslash\mathrm{GL}_2(E_{\mathbf{A}})/\mathcal{K}_\infty^E\mathcal{K}_0^E(\mathfrak{N})$, which is defined by the symmetric tensor product of order n of the standard representation of $\mathrm{GL}_2(E_\infty)$. Write $\delta(f)$ be the image in the cuspidal cohomology group $H_{\mathrm{cusp}}^{r_E}(Y_{\mathcal{K}}^E,\mathcal{L}(n;\mathbf{C}))$ via the Eichler-Shimura-Harder map.

Suppose that f is the p-stabilized newform. Let $\xi\in E$ such that

- $\mathrm{Tr}_{E/F}(\xi)=0$;
- $\mathcal{O}_{E,p}=\langle 1,\xi\rangle_{\mathcal{O}_{F,p}}$.

Put $h_\xi^{(r)}=\begin{pmatrix}p^r&\xi\\0&1\end{pmatrix}\in\mathrm{GL}_2(E_p)$. Let us denote by $\varrho(h_\xi^{(r)})$ the right-translation by $h_\xi^{(r)}$. By using the natural inclusion $\mathrm{GL}_2(F_{\mathbf{A}})\to\mathrm{GL}_2(E_{\mathbf{A}})$, the pull-back of the local system $\mathcal{L}(n;\mathbf{C})$ decomposes into $\oplus_\alpha\mathcal{L}(2n-2\alpha;\mathbf{C})$, and the projector to each α-th component is explicitly described as a differential operator. Hence we can define an element $\varrho(h_\xi^{(r)})\delta^\alpha(f)$ in $H_{\mathrm{c}}^{r_F}(Y_{\mathcal{K}}(p^r),\mathcal{L}(2n-2\alpha;\mathbf{C}))$, where H_{c}^* denotes the compactly supported cohomology, as the α-th component of the pull-back of $\varrho(h_\xi^{(r)})\delta(f)$. See [Na, Section 6] for the detail.

4.3. **Birch lemma.** We give a cohomological interpretation of partial Asai L-functions by using cohomology classes $\delta(E_{r,s})$ and $\varrho(h_\xi^{(r)})\delta^\alpha(f)$ which are introduced in the previous subsections.

The determinant map induces the following map:

$$\det{}_r:Y_{\mathcal{K}}(p^r)\to F^\times F_{\infty,+}^\times\backslash F_{\mathbf{A}}^\times/(\widehat{\mathcal{O}}_F^{(p)})^\times(1+p^r\mathcal{O}_{F,p})=:\mathrm{Cl}_F^+(p^r).$$

For each $x\in\mathrm{Cl}_F^+(p^r)$, define $Y_{\mathcal{K}}(p^r)_x$ to be the inverse image $\det_r^{-1}(x)$ of x. We denote by $\varrho(h_\xi^{(r)})\delta^\alpha(f)_x$ (resp. $\delta(E_{r,0})_x$) the pull-back of $\varrho(h_\xi^{(r)})\delta^\alpha(f)$ (resp. $\delta(E_{r,0})$) via $Y_{\mathcal{K}}(p^r)_x\to Y_{\mathcal{K}}(p^r)$. Then we define $I_{r,x,s}^\alpha$ to be the cup product of $\varrho(h_\xi^{(r)})\delta^\alpha(f)_x$ and $\delta(E_{r,0})_x$, which is described by a certain integral on $Y_{\mathcal{K}}(p^r)_x$.

Identifying $\mathrm{Sym}^{2n_v+2}(\mathbf{C}^{\oplus 2})$ with the space $\mathbf{C}[S_v,T_v]_{2n_v+2}$ of the homogeneous polynomials of the degree $2n_v+2$, we denote by $f^i(g)$ the coefficient of $(-1)^{n+1-i}\binom{2n+2}{n+1-i}S^{n+1+i}T^{n+1-i}$

in $f(g) \in \otimes_{v\in\Sigma_{F,\infty}} \mathbf{C}[S_v, T_v]_{2n_v+2}$ for each $-n-1 \leq i \leq n+1$. Let

$$C(\alpha, i) =(-1)^n(-1)^{\frac{i-\alpha}{2}}\binom{n}{\alpha}^2\binom{2n+2}{n+1-i}^{-1} \times \sum_t (-1)^t\binom{\alpha}{t}\binom{2n-2\alpha+2}{n-2t+i+1},$$

$$f^\alpha(g) = \sum_{-n-1\leq i\leq n+1} C(\alpha, i)f^i(g).$$

Recall that $n + 2m = [\kappa]t$ for an integer $[\kappa] \in \mathbf{Z}$. For each integer $a \in \mathbf{Z}$ and each Hecke character $\varphi : F^\times\backslash F^\times_{\mathbf{A}} \to \mathbf{C}^\times$, put $\varphi_a = |\cdot|^a_{F_{\mathbf{A}}}\varphi$. Suppose that the conductor of φ is divisible by p^r. Then we have

$$\begin{aligned}\sum_{x\in \mathrm{Cl}^+_F(p^r)} \widehat{\varphi_{2[\kappa]}}(x)I^\alpha_{r,x,s} =&[\mathrm{GL}_2(\widehat{\mathcal{O}}_F) : \mathcal{K}^F_0(\mathfrak{N}_F) \cap \mathcal{K}^F(p^r)]p^{-mr} \\ &\times \int_{\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(F_{\mathbf{A}})/F^\times_\infty} f^\alpha(gh^{(r)}_\xi)E_{r,s}(g)|\cdot|^{2[\kappa]}_{\mathbf{A}}\varphi(\det g)\mathrm{d}g.\end{aligned} \tag{4.1}$$

We will find that the right-hand side of (4.1) coincides with the right-hand side of (1.1) up to a simple factor. Hence the identity (4.1) shows that $I^\alpha_{r,x,s}$ gives a cohomological interpretation of partial Asai L-functions. See [Na, Section 6, 7] for the detail.

4.4. **Construction.** For each place $w \in \Sigma_E, w \mid p$, let $U_0(w)$ be the Hecke operator at w normalized as in [Hi94, Section 4]. Let $\lambda_{w,0}$ be the eigenvalue of $U_0(w)$ with respect to f. Recall that π is defined to be nearly p-ordinary if $\lambda_{w,0}$ is a p-adic unit. Put $\lambda_{p,0} = \prod_{w|p} \lambda_{w,0}$. Let

$$\widetilde{I}^\alpha_{r,x,s} = \frac{1}{\Omega_{\pi,p}} \times \lambda^{-r}_{p,0} \prod_{v\in\Sigma_F, v|p} q_v^{2(s+n-\alpha)r} \times I^\alpha_{r,x,s}.$$

We denote by pr_r the natural projection $\mathrm{Cl}^+_F(p^{r+1}) \to \mathrm{Cl}^+_F(p^r)$. Then we have the following proposition:

Proposition 4.1. (Distribution property, [Na, Corollary 8.7]) *For each $r \geq 1$, we have*

$$\sum_{y\in \mathrm{pr}_r^{-1}(x)} \widetilde{I}^\alpha_{r+1,y,s} = \widetilde{I}^\alpha_{r,x,s}.$$

Let $\mathrm{rec} : \mathrm{Cl}^+_F(p^r) \xrightarrow{\sim} \mathrm{Gal}(F(p^r)/F)$ be the geometrically normalized reciprocity map. Put $\sigma_x = \mathrm{rec}(x)$ for $x \in \mathrm{Cl}^+_F(p^r)$. Define

$$\mathscr{L}^\alpha_{p,r}(\mathrm{As}^+_{\mathcal{M}}(\pi)) =(*) \sum_{x\in\mathrm{Cl}^+_F(p^r)} \widetilde{I}^\alpha_{r,x,0}\sigma_x \in \mathcal{O}_\pi[\mathrm{Gal}(F(p^r)/F)],$$

where $(*)$ is a certain easy constant. The distribution property (Proposition 4.1) shows that the system $\left\{\mathscr{L}^\alpha_{p,r}(\mathrm{As}^+_{\mathcal{M}}(\pi))\right\}_{r\geq1}$ is a projective system. We define

$$\mathscr{L}^\alpha_p(\mathrm{As}^+_{\mathcal{M}}(\pi)) =(*') \times \mathrm{Tw}_p^{[\kappa]+(\alpha+2m)} \varprojlim_r \mathscr{L}^\alpha_{p,r}(\mathrm{As}^+_{\mathcal{M}}(\pi)) \in K_\pi \otimes_{\mathcal{O}_\pi} \mathcal{O}_\pi[[\mathrm{Gal}(F(\mu_{p^\infty})/F)]],$$

where $(*')$ is a certain easy constant. See [Na, Section 8.3] for the detail.

By unfolding, the right-hand side of (4.1) is described by a product of certain local integrals. Hence the proof of the interpolation formula for $\mathscr{L}^\alpha_p(\mathrm{As}^+_{\mathcal{M}}(\pi))$ is reduce to the calculation of these local integrals. The proof of the explicit formula of local integrals at p-adic places is done by using the local functional equation of Asai L-functions ([CCI20]). See [Na, Section 10] for the detail. The explicit formula of local integrals at infinite places is given by modifying the argument [Gh99a, page 629, Conjecture 1] and [LSO14, Theorem 4.1], where they use the classical language, into the adelic language. See [Na, Section 11] for the detail. The calculations for places $v \in \Sigma_F, v \nmid p\infty$ are easy ([Na, Section 9]).

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