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# CONGRUENCE PROPERTIES OF ENDO-CLASSES AND THE LOCAL JACQUET–LANGLANDS CORRESPONDENCE

by

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These are notes from a lecture I gave at RIMS, Kyoto, for the conference *Analytic, geometric and  $p$ -adic aspects of automorphic forms and  $L$ -functions*, from 20 to 24 January, 2020. I wish to thank the organizers of this conference. These notes are intended to be a survey of recent results that have been published in [31].

## Notation

Let us fix some notation:

- $F$  is a non-Archimedean locally compact field of residual characteristic  $p$ ,
- $\overline{F}$  is a separable closure of  $F$ ,
- $\mathcal{W}_F$  is the Weil group of  $\overline{F}/F$ ,
- $\mathcal{J}_F$  is the inertia subgroup of  $\mathcal{W}_F$ , that is its unique maximal compact subgroup,
- $\mathcal{P}_F$  is the wild inertia subgroup of  $\mathcal{W}_F$ , that is its unique maximal pro- $p$ -subgroup.

## 1. Irreducible representations of the Weil group

Given an irreducible smooth complex representation  $\sigma$  of  $\mathcal{W}_F$ , its restriction to  $\mathcal{P}_F$  decomposes into a direct sum of finitely many irreducible representations of  $\mathcal{P}_F$ . These irreducible representations form a single  $\mathcal{W}_F$ -orbit under the conjugacy action, denoted  $\mathcal{O}_F(\sigma)$ . All representations in this orbit occur in the restriction of  $\sigma$  to  $\mathcal{P}_F$  with the same multiplicity  $m = m(\sigma)$ .

Fix  $\alpha \in \mathcal{O}_F(\sigma)$ . Its stabilizer in  $\mathcal{W}_F$  is equal to  $\mathcal{W}_T$  for a uniquely determined tamely ramified, finite extension  $T$  of  $F$  contained in  $\overline{F}$ . The representation  $\alpha$  extends (non-canonically) to an irreducible representation  $\rho$  of  $\mathcal{W}_T$ . The representation  $\sigma$  can then be written:

$$\sigma \simeq \text{Ind}_{T/F}(\rho \otimes \tau)$$

where  $\tau$  is an irreducible representation of  $\mathcal{W}_T$  trivial on  $\mathcal{P}_T = \mathcal{P}_F$ , of dimension  $m$ , uniquely determined up to isomorphism, and where  $\text{Ind}_{T/F}$  denotes induction from  $\mathcal{W}_T$  to  $\mathcal{W}_F$ .

## 2. Cuspidal representations of $\mathrm{GL}_n(F)$

On the other hand, consider cuspidal irreducible smooth complex representations of the group  $G = \mathrm{GL}_n(F)$  for some  $n \geq 1$ . Bushnell and Kutzko have constructed in [8] an explicit family of pairs  $(\mathbf{J}, \boldsymbol{\lambda})$ , where  $\mathbf{J}$  is an open, compact mod centre subgroup of  $G$  and  $\boldsymbol{\lambda}$  is an irreducible representation of  $\mathbf{J}$ , such that:

- the compact induction of  $\boldsymbol{\lambda}$  from  $\mathbf{J}$  to  $G$  is irreducible and cuspidal for all pairs  $(\mathbf{J}, \boldsymbol{\lambda})$ ,
- any cuspidal irreducible representation of  $G$  occurs this way, for a pair  $(\mathbf{J}, \boldsymbol{\lambda})$ , uniquely determined up to  $G$ -conjugacy.

These pairs have the following properties:

(1) The group  $\mathbf{J}$  has a unique maximal compact subgroup  $\mathbf{J}^0$  and a unique maximal normal pro- $p$ -subgroup  $\mathbf{J}^1$ .

(2) The restriction of  $\boldsymbol{\lambda}$  to  $\mathbf{J}^1$  is a multiple of a single irreducible representation  $\eta$ .

(3) The representation  $\eta$  extends (non-canonically) to a representation  $\boldsymbol{\kappa}$  of  $\mathbf{J}$ , and  $\boldsymbol{\lambda}$  is isomorphic to  $\boldsymbol{\kappa} \otimes \boldsymbol{\xi}$  for a uniquely determined irreducible representation  $\boldsymbol{\xi}$  of  $\mathbf{J}$  trivial on  $\mathbf{J}^1$ .

(4) The representation  $\eta$  is constructed as a Heisenberg representation from a character  $\theta$  of a smaller open subgroup  $H^1 \subseteq \mathbf{J}^1$  such that  $\mathbf{J}^1/H^1$  is a finite-dimensional  $\mathbb{F}_p$ -symplectic space.

We denote by  $\mathcal{C}(G)$  the set of all characters  $\theta$  obtained this way when one varies the pairs  $(\mathbf{J}, \boldsymbol{\lambda})$ , and by  $\mathcal{C}(F)$  the union of  $\mathcal{C}(\mathrm{GL}_n(F))$  for all  $n \geq 1$ . Bushnell and Henniart ([4]) equipped the set  $\mathcal{C}(F)$  with an equivalence relation called *endo-equivalence*. Equivalence classes for this relation are called *endo-classes*. The set  $\mathcal{E}(F)$  of endo-classes only depends on  $F$ .

**Remark 2.1.** — When one fixes  $n$ , two characters  $\theta, \theta' \in \mathcal{C}(\mathrm{GL}_n(F))$  defined on  $H^1, H'^1$  respectively are endo-equivalent if and only if there is a  $g \in \mathrm{GL}_n(F)$  such that  $\theta' = \theta^g$  on  $H'^1 \cap H^{1g}$ .

The endo-class of any character  $\theta$  as above occurring in a cuspidal irreducible representation  $\pi$  of  $G$  will be denoted  $\Theta(\pi)$ .

## 3. The Ramification Theorem

We now consider the local Langlands correspondence  $\pi \mapsto {}^L\pi$  (see [21, 17, 18]) between isomorphism classes of cuspidal irreducible representations of  $\mathrm{GL}_n(F)$  and isomorphism classes of irreducible  $n$ -dimensional representations of  $\mathcal{W}_F$ .

**Theorem 3.1** ([6]). — *Let  $\pi_1, \pi_2$  be cuspidal irreducible representations of  $\mathrm{GL}_{n_1}(F), \mathrm{GL}_{n_2}(F)$  respectively, and set  $\sigma_1 = {}^L\pi_1$  and  $\sigma_2 = {}^L\pi_2$ . Then:*

$$\mathcal{O}_F(\sigma_1) \cap \mathcal{O}_F(\sigma_2) \neq \emptyset \iff \Theta(\pi_1) = \Theta(\pi_2).$$

## 4. Inner forms and the local Jacquet-Langlands correspondence

More generally, the classification of cuspidal representations by compact induction, as well as the notion of endo-classes, also works for inner forms of general linear groups ([2, 16, 26, 27,

**28, 29, 3]**). It is natural to ask whether there exists, for inner forms, an analogue of Bushnell–Henniart’s Ramification Theorem 3.1.

Let  $H = \mathrm{GL}_m(D)$  be an inner form of  $G = \mathrm{GL}_n(F)$ , where  $D$  is a central division  $F$ -algebra of reduced degree  $d$  such that  $md = n$ . The local Jacquet–Langlands correspondence is a bijection  $\pi \mapsto {}^{JL}\pi$  between the discrete series of  $H$  and  $G$ , characterized by a character identity at elliptic regular conjugacy classes ([**19, 25, 13, 1**]).

Given a discrete series representation  $\pi$  of  $H$ , it occurs as a subquotient of the parabolic induction of a cuspidal representation of the form:

$$\rho\chi_1 \otimes \cdots \otimes \rho\chi_r$$

where  $r$  is a divisor of  $m$ ,  $\rho$  is a cuspidal irreducible representation of  $\mathrm{GL}_{m/r}(D)$  and  $\chi_1, \dots, \chi_r$  are unramified characters of  $\mathrm{GL}_{m/r}(D)$ . Define  $\Theta(\pi) = \Theta(\rho)$ .

**Theorem 4.1** ([**32, 31, 14**]). — *The local Jacquet–Langlands correspondence preserves endo-classes.*

This result is one of the main steps in the explicit description of the local Jacquet–Langlands correspondence in terms of Bushnell–Kutzko’s simple types ([**31, 14**]).

In the sequel, I will explain a crucial step in the proof of this theorem. Surprisingly, it involves the *modular representation theory* of  $H$ .

## 5. Modular representations of $p$ -adic groups

Fix a prime number  $\ell$  different from  $p$  and an algebraic closure  $\overline{\mathbb{F}}_\ell$  of  $\mathbb{F}_\ell$ , and consider smooth  $\overline{\mathbb{F}}_\ell$ -representations of  $H$ . As in the complex case, one has Haar measures, normalized parabolic induction and restriction functors, cuspidal representations and uniqueness of the cuspidal support for irreducible representations. Here, as in the complex case, a representation is cuspidal if all its proper Jacquet modules are zero or, equivalently, if all its matrix coefficients are compactly supported mod centre. However, there is a crucial difference: an irreducible cuspidal representation may occur as a subquotient of a parabolically induced representation ([**33**] Corollaire 5).

Say an irreducible representation of  $H$  is supercuspidal when it does not occur as a subquotient of a parabolically induced representation. For irreducible representations of  $H$ , there is a well-behaved notion of supercuspidal support.

**Theorem 5.1** ([**23**]). — *Let  $\pi$  be an irreducible  $\overline{\mathbb{F}}_\ell$ -representation of  $H$ .*

(1) *There are a Levi subgroup  $M$  of  $H$  and an irreducible supercuspidal representation  $\rho$  of  $M$  such that  $\pi$  occurs as a subquotient of the parabolic induction of  $\rho$  to  $H$  with respect to any parabolic subgroup with Levi component  $M$ .*

(2) *The pair  $(M, \rho)$  is unique up to  $H$ -conjugacy.*

**Remark 5.2.** — Unlike uniqueness of the cuspidal support, Theorem 5.1 is non-trivial. Uniqueness of the supercuspidal support also holds for special linear  $p$ -adic groups [**9**] and small unramified unitary groups [**20**], but is false in general (see [**15, 12**] for a counterexample in the symplectic group  $\mathrm{Sp}_8$ ).

Based on the well-defined notion of supercuspidal support for the group  $H$ , we have the following decomposition theorem.

**Theorem 5.3** ([30]). — *The category  $\text{Rep}_{\overline{\mathbb{F}}_\ell}(H)$  of smooth  $\overline{\mathbb{F}}_\ell$ -representations of  $H$  decomposes into blocks, which correspond bijectively to inertial classes of supercuspidal pairs of  $H$ .*

## 6. Reduction mod $\ell$ of representations over $\ell$ -adic numbers

Now fix an algebraic closure  $\overline{\mathbb{Q}}_\ell$  of the field of  $\ell$ -adic numbers and consider smooth  $\overline{\mathbb{Q}}_\ell$ -representations of  $H$ . Let  $\overline{\mathbb{Z}}_\ell$  denote the ring of algebraic integers of  $\overline{\mathbb{Q}}_\ell$ . An irreducible representation  $\pi$  of  $H$  on a  $\overline{\mathbb{Q}}_\ell$ -vector space  $V$  is said to be *integral* if it has an  $H$ -stable  $\overline{\mathbb{Z}}_\ell$ -lattice  $L \subseteq V$ . We have the following  $p$ -adic Brauer-Nesbitt principle.

**Proposition 6.1** ([34] I.9.6). — *The  $\overline{\mathbb{F}}_\ell$ -representation  $L \otimes \overline{\mathbb{F}}_\ell$  has finite length. Its semi-simplification only depends on the isomorphism class of  $\pi$ , and not on the choice of  $L$ .*

The semi-simplification of  $L \otimes \overline{\mathbb{F}}_\ell$  will be denoted  $r_\ell(\pi)$  and called the reduction mod  $\ell$  of  $\pi$ . Two integral irreducible representations are said to be *congruent* if they have the same reduction mod  $\ell$ .

## 7. Congruence properties of the local Jacquet-Langlands correspondence

Now replace  $\mathbb{C}$  by  $\overline{\mathbb{Q}}_\ell$  in Paragraphs 3 and 4 by fixing a field isomorphism  $\iota : \mathbb{C} \rightarrow \overline{\mathbb{Q}}_\ell$ . Since cuspidal representations are characterized by the properties of the support of their matrix coefficients, base change from  $\mathbb{C}$  to  $\overline{\mathbb{Q}}_\ell$  preserves cuspidality. One thus gets a local Langlands correspondence between isomorphism classes of cuspidal irreducible  $\overline{\mathbb{Q}}_\ell$ -representations of  $\text{GL}_n(F)$  and isomorphism classes of irreducible  $n$ -dimensional  $\overline{\mathbb{Q}}_\ell$ -representations of  $W_F$ . The congruence properties of this local Langlands correspondence have been studied by Vignéras, Bushnell–Henniart and Dat ([35, 7, 10]).

For the local Jacquet–Langlands correspondence, one first has to check that the set of  $\overline{\mathbb{Q}}_\ell$ -representations of  $H$  coming from a discrete series complex representation by base change from  $\mathbb{C}$  to  $\overline{\mathbb{Q}}_\ell$  does not depend on the choice of  $\iota$ . This comes from the fact that discrete series complex representations can be characterized as Langlands quotients of certain induced representations.

**Theorem 7.1** ([11, 24, 31]). — *Let  $\pi, \pi'$  be integral discrete series  $\overline{\mathbb{Q}}_\ell$ -representations of  $H$ .*

- (1)  *${}^{JL}\pi$  and  ${}^{JL}\pi'$  are integral, and they are congruent if and only if  $\pi, \pi'$  are congruent.*
- (2)  *$r_\ell({}^{JL}\pi)$  and  $r_\ell({}^{JL}\pi')$  are in the same block of  $\text{Rep}_{\overline{\mathbb{F}}_\ell}(G)$  if and only if  $r_\ell(\pi)$  and  $r_\ell(\pi')$  are in the same block of  $\text{Rep}_{\overline{\mathbb{F}}_\ell}(H)$ .*

## 8. Linked representations

Now go back to complex representations.

**Definition 8.1.** — *Let  $\pi, \pi'$  be discrete series complex representations of  $H$ . They are:*

(1) *1-linked* if there are a prime number  $\ell \neq p$  and an isomorphism of fields  $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$  such that  $r_\ell(\iota^* \pi)$  and  $r_\ell(\iota^* \pi')$  are in the same block of  $\text{Rep}_{\overline{\mathbb{F}}_\ell}(H)$ .

(2) *linked* if there are discrete series representations  $\pi_0, \dots, \pi_r$  of  $H$  such that  $\pi_0 = \pi$ ,  $\pi_r = \pi'$  and  $\pi_i, \pi_{i-1}$  are 1-linked for all  $i \in \{1, \dots, r\}$ .

Any two linked discrete series representations of  $H$  have the same endo-class. The following theorem says that the converse is true.

**Theorem 8.2** ([31]). — *Two complex discrete series representations of  $H$  are linked if and only if they have the same endo-class.*

Putting all these results together, we get:

**Corollary 8.3.** — *Let  $\pi, \pi'$  be discrete series complex representations of  $H$ . Then  $\pi, \pi'$  have the same endo-class if and only if  ${}^{JL}\pi, {}^{JL}\pi'$  have the same endo-class.*

It follows that, given a discrete series complex representation  $\pi$  of  $H$ , the endo-class  $\Theta({}^{JL}\pi)$  only depends on  $\Theta(\pi)$ . This defines a map from the set of endo-classes of discrete series representation of  $H$  to that of endo-classes of discrete series representation of  $G$ , and one verifies that this map is bijective. Moreover, varying  $n$  and the inner form, one gets a bijective map:

$$j_F : \Theta \mapsto {}^{JL}\Theta$$

from  $\mathcal{E}(F)$  to itself. Theorem 4.1 is then equivalent to saying that this map is the identity.

## 9. The restriction map

Let  $K$  be a finite, tamely ramified extension of  $F$  contained in  $\overline{F}$ . Associated with it, there is the set  $\mathcal{E}(K)$  of  $K$ -endo-classes. It corresponds, via the Ramification Theorem 3.1, to the set of  $\mathcal{W}_K$ -conjugacy classes of irreducible representations of  $\mathcal{P}_K = \mathcal{P}_F$ . Bushnell-Henniart ([4]) have defined a surjective map:

$$\rho_{K/F} : \mathcal{E}(K) \rightarrow \mathcal{E}(F)$$

called the *restriction map*. It corresponds, through the local Langlands correspondence, to passing from  $\mathcal{W}_K$ -conjugacy classes to  $\mathcal{W}_F$ -conjugacy classes of irreducible representations of  $\mathcal{P}_F$ .

## 10. The last step

Given an endo-class  $\Theta \in \mathcal{E}(F)$ , there exists an  $m \geq 1$  such that the group  $H = \text{GL}_m(D)$  has a cuspidal representation  $\pi$  with the following properties:

- (1) the endo-class of  $\pi$  is equal to  $\Theta$ ,
- (2) the transfer  ${}^{JL}\pi$  is a cuspidal representation of  $G = \text{GL}_n(F)$ , with  $n = md$ .

Let  $K$  be the unique finite unramified extension of  $F$  contained in  $\overline{F}$  of degree the torsion number of  $\pi$ , that is the number of unramified characters  $\chi$  of  $H$  such that  $\pi\chi \simeq \pi$ . This number only depends on  $n$  and  $\Theta$ . Choose an endo-class  $\Psi \in \mathcal{E}(K)$  whose restriction to  $\mathcal{E}(F)$  is equal to  $\Theta$ .

Following Bushnell-Henniart's analysis of the traces of  $\pi$  and  ${}^{JL}\pi$  at certain well chosen elliptic regular conjugacy elements in terms of types ([5]), one get the following result:

**Proposition 10.1** ([5] 6, [31] 7, [14] 4). — *One has  $\mathbf{j}_F\Theta = \rho_{K/F}(\mathbf{j}_K\Psi)$ .*

Fix an  $F$ -embedding of  $K$  in  $M_m(D)$ , and let  $H_K$  be the centralizer of  $K$  in  $H$ . Define  $G_K$  similarly. Let  $\rho$  be a cuspidal representation of  $H_K$  of endo-class  $\Psi$ . The endo-class of its transfer to  $G_K$  is thus  $\mathbf{j}_K\Psi$ .

Dotto's idea ([14]) is to choose the integer  $m \geq 1$  so that  $G_K$  and  $H_K$  are  $K$ -isomorphic. (For instance, choose  $m$  large enough so that  $G_K$  and  $H_K$  are both split over  $K$ .) We thus may identify  $G_K$  and  $H_K$  so that  $\rho$  is isomorphic to its transfer to  $G_K$ . This gives us  $\mathbf{j}_K\Psi = \Psi$ . Applying the restriction map and Proposition 10.1, we get the desired identity  $\mathbf{j}_F\Theta = \rho_{K/F}(\Psi) = \Theta$ .

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