On the Fourier coefficients of Siegel Eisenstein series and genus theta series

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0 Introduction

Siegel Eisenstein series are one of the most important and fundamental topics in the theory of Siegel modular forms. We define the Siegel Eisenstein series with level and characters as follows. Let l be a positive integer and ψ a Dirichlet character modulo l, that is not necessarily primitive. We write

$$\Gamma_0^g(l) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{Z}) \mid C \equiv 0 \bmod l\},$$

$$\Gamma_\infty = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp(g,\mathbb{Z})\}.$$

For an integer $k \geq 0$ such that $\psi(-1) = (-1)^k$, we define

$$E_{l,\psi}^{g,k}(Z) = \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0^g(l)} \psi(\det D) \det(CZ+D)^{-k},$$

here $Z \in \mathbb{H}_g$ the Siegel upper half space of degree g. The right hand side converges when $k > g+1$, then $E_{l,\psi}^{g,k}(Z) \in M_k(\Gamma_0^g(l), \overline{\psi})$.

The Fourier expansion of $E_{l,\psi}^{g,k}(Z)$ are written as

$$E_{l,\psi}^{g,k}(Z) = \sum_{S_g(\mathbb{Z})^* \ni T \geq 0} C(T) \exp(2\pi \mathbf{i} \operatorname{Tr}(TZ)),$$

here $S_g(\mathbb{Z})^*$ denotes the set of half integral symmetric matrices of size g. The aim of this article is to give an explicit formula of $C(T)$.

For that, first note that we may assume $T > 0$, since in the case of $\operatorname{rank} T = r < g$, $C(T)$ coincides with the Fourier coefficients of $E_{l,\psi}^{r,k}(z)$ ($z \in \mathbb{H}_r$). If $T > 0$, then $C(T)$ has an Euler product expression

$$C(T) = \xi_g(T,k) \prod_{p:\text{prime}} S_g^p(l,\psi,T,k). \tag{0.1}$$

Siegel gave an explicit formula of $\xi_g(T,k)$ as

$$\xi(T,k) = \frac{2^{-g(g-1)/2}(-2\pi\mathbf{i})^{gk}}{\Gamma_g(k)}(\det T)^{k-(g+1)/2}.$$

Here we put $\Gamma_g(s) = \pi^{g(g-1)/4}\prod_{j=0}^{g-1}\Gamma(s-j/2)$.

On the other hand the term $S_g^p(l,\psi,T,k)$ are called the Siegel series. We refer the precise definition in [Ta1, Definition 2.1]. We remark that the properties are quite different during two cases whether $p \mid l$ or $p \nmid l$.

In the case of $p \nmid l$, $S_g^p(l,\psi,T,k)$ are independent of l, thus we simply write $S_g^p(\psi,T,k)$. Moreover if $l=1$ they are denoted by $S_g^p(T,k)$. It is known that $S_g^p(T,k)$ are rational functions in p^{-k}. If we write $S_g^p(T,k) = R(p^{-k})$, then $S_g^p(\psi,T,k) = R(\psi(p)p^{-k})$, thus it suffices to consider the case $l=1$, i.e. the full modular case. An explicit formula of Siegel series are given in the famous paper of Katsurada ([Kat]), as a consequence the case of $p \mid l$ remains.

For $S_g^p(l,\psi,T,k)$ with $p \mid l$, only several examples of calculations were known in the case of small g (cf. [Mi], [Ta1], [Gu2], [Gu3], [Di]). For higher degree case, the computations seemed to be quite difficult. We note that in [Ta2], Takemori calculated the Fourier coefficients of $E_{l,\psi}^{g,k}(Z)$ for non-quadratic character ψ. Precisely to say, he treated the case that l is odd and for the decomposition $\psi = \prod_{p\mid l}\psi_p$, $\psi_p^2 \not\equiv 1$ for all $p \mid l$. In that case, the Fourier coefficients at $S_g(\mathbb{Z})^* \ni T > 0$ becomes quite simple: $S_g^p(l,\psi,T,k)$ are almost 1 except for the Euler factors of L-functions. In [Ta2], first Takemori constructed Siegel Eisenstein series with such simple Fourier coefficients, next he showed the coincidence of it with our $E_{l,\psi}^{g,k}(Z)$. However such a result does not hold for the case of quadratic or trivial character.

In this article we treat the Siegel Eisenstein series with trivial or quadratic characters, using the theory of genus theta series. In full modular case, it is known as Siegel's main theorem, that the genus theta series coincides with the Siegel Eisenstein series. In higher level case the situation becomes more complicated, but genus theta series belong to the space of Siegel Eisenstein series. Moreover Katsurada-Schulze Pirrot ([KS]) or Böcherer-Hironaka-Sato ([BHS]) showed that by taking suitable quadratic forms genus theta series form a basis of the space of Siegel Eisenstein series for square free level case. The Fourier coefficients of the genus theta series are known by Sato-Hironaka ([SH]), thus combining above results we can get the Fourier coefficients of Siegel Eisenstein series.

The above strategy is nothing new, however by the complicatedness of the explicit formula of local densities (or the Fourier coefficients of the genus theta series), taking liner combinations have not seemed to be an effective way (cf. [Boe]). Here we first show a simple description to write the Siegel Eisenstein series by the linear combinations of the genus theta series, next by using the properties of the coefficients, we simplify the formula for the Fourier coefficients.

1 genus theta series

We set $S_n(\mathbb{Z}) = \{Q \in M_n(\mathbb{Z}) \mid {}^tQ = Q\}$ the set of integral symmetric matrices, $S_n(\mathbb{Z})^* = \{Q = (q_{ij}) \in M_n(\mathbb{Q}) \mid {}^tQ = Q, q_{ii} \in \mathbb{Z},\ 2q_{ij} \in \mathbb{Z}\ (i \neq j)\}$ the set of half integral symmetric matrices. We put $S_n(\mathbb{Z})_e = \{Q = (q_{ij}) \in S_n(\mathbb{Z}) \mid q_{ii} \in 2\mathbb{Z}\}$ the set of even integral matrices, i.e. $Q \in S_n(\mathbb{Z})^*$ if and only if $2Q \in S_n(\mathbb{Z})_e$. Moreover $S_n^+(\mathbb{Z})$, $S_n^+(\mathbb{Z})^*$ and $S_n^+(\mathbb{Z})_e$ denote the subset of positive definite matrices in $S_n(\mathbb{Z})$, $S_n(\mathbb{Z})^*$ and $S_n(\mathbb{Z})_e$ respectively.

Definition 1.1 (1) We say $Q_1,\ Q_2 \in S_n(\mathbb{Z})^*$ are in the same *class* if there exists $U \in GL_n(\mathbb{Z})$ such that ${}^tUQ_1U = Q_2$.

(2) We say $Q_1,\ Q_2 \in S_n(\mathbb{Z})^*$ are in the same *genus* if for every prime p there exist $U_p \in GL_n(\mathbb{Z}_p)$ such that ${}^tU_pQ_1U_p = Q_2$, and Q_1, Q_2 have same signs.

We write $Q_1 \sim Q_2$ if Q_1 and Q_2 are in the same class. Then the set

$$\{S \in S_n(\mathbb{Z})^* \mid S \text{ is in the same genus as } Q\}/\sim$$

is finite, whose number $h(Q)$ is called the *class number* of Q.

For $Q \in S_{2k}^+(\mathbb{Z})^*$ we define the theta series

$$\vartheta^g(Q;Z) = \vartheta(Q;Z) = \sum_{N \in M_{2k,g}} \exp\big(2\pi\mathbf{i}\,\mathrm{Tr}({}^tNQNZ)\big) \quad Z \in \mathbb{H}_g,$$

that is a Siegel modular form of level l and character χ_Q. Here l is the level of $2Q$, i.e. the least integer $l > 0$ such that $l \cdot (2Q)^{-1} \in S_{2k}(\mathbb{Z})_e$. χ_Q is the quadratic or trivial character, that satisfies

$$\chi_Q(q) = \left(\frac{(-1)^k \det(2Q)}{q}\right)$$

for an odd prime q. Note that if k is even and $\det(2Q)$ is a square number, then χ_Q is the trivial character modulo l.

The Fourier coefficient of $\vartheta(Q;Z)$ at $T \in S_g(\mathbb{Z})^*$ is the representation number

$$r(Q,T) = \sharp\{N \in M_{2k,g}(\mathbb{Z}) \mid {}^tNQN = T\},$$

that is not easy to handle. Instead of that if we consider the weighted average of the theta series in the same genus, then its Fourier coefficients has an Euler product expression, thus we can treat it using local theories.

Let $Q_1, \ldots, Q_h$ be a representative set of the equivalence class in the same genus as in Q. Put

$$O(Q_i) = \{U \in GL_{2k}(\mathbb{Z}) \mid {}^tUQ_iU = Q_i\},$$

which is a finite group since each Q_i is positive definite. Then we define the genus theta series

$$\Theta(Q; Z) = \frac{1}{w} \sum_{i=1}^{h} \frac{\vartheta(Q_i; Z)}{\sharp O(Q_i)} \in M_k(\Gamma_0^g(l), \chi_Q), \quad w = \sum_{i=1}^{h} \frac{1}{\sharp O(Q_i)}.$$

Note that the constant term of the Fourier expansion of the genus theta series are 1.

In order to write the Fourier coefficients of $\Theta(Q; Z)$, we define the local densities as follows. For a prime number p, put

$$r_{p^\nu}(Q, T) = \sharp\{N \in M_{m,g}(\mathbb{Z}/p^\nu) \mid {}^tNQN \equiv T \bmod p^\nu S_g(\mathbb{Z})^*\}$$

for $Q \in S_m(Z)^*$ and $T \in S_g(Z)^*$. We define

$$\alpha_p(Q, T) = \lim_{\nu \to \infty} p^{-\nu(mg - g(g+1)/2)} r_{p^\nu}(Q, T).$$

The right hand side is stable for sufficiently large ν. Then the following theorem is known as Siegel's main theorem.

Theorem 1.1 ([Si, Sats 1, (72)] or [Ki, Theorem 6.8.1])
Let $Q \in S_{2k}^+(\mathbb{Z})^*$, $T \in S_g^+(\mathbb{Z})^*$. For $2k > g + 1$, the Fourier coefficient of the genus theta series $\Theta(Q; Z)$ at T is given by

$$\alpha_\infty(Q, T) \prod_{p:\text{prime}} \alpha_p(Q, T),$$

with

$$\begin{aligned} \alpha_\infty(Q, T) &= (\det Q)^{-g/2} (\det T)^{(2k-g-1)/2} \frac{2^{-g(g-1)/2} \pi^{gk}}{\Gamma_g(k)} \\ &= \mathbf{i}^{gk} \det(2Q)^{-g/2}\, \xi_g(T, k). \end{aligned} \tag{1.1}$$

Remark In Siegel's original paper [Si] or [Ki], the value of $\alpha_\infty(Q, T)$ differs $2^{g(g-1)}$ from our result. However the definition of $\alpha_p(Q, T)$ is also different; they count the number of N such that ${}^tNQN \equiv p^\nu S_g(\mathbb{Z})$, thus the value of $\alpha_2(Q, T)$ is not equal to ours. By looking at [Ki, Lemma 5.6.5], we see that the final results are correct.

The relation between Siegel series and local densities are given by the following proposition.

Proposition 1.2 ([Sh, Lemma 3.5]) Let $Q \in S_{2k}^{+}(\mathbb{Z})^*$ and $T \in S_g(\mathbb{Z})^*$. For a prime number p (including $p = 2$) such that $(p, \det(2Q)) = 1$, we have

$$\alpha_p(Q, T) = S_g^p(\chi_Q, T, k)$$

The condition $(p, \det(2Q)) = 1$ is equivalent to $p \nmid \ell(2Q)$, here $\ell(2Q)$ denote the level of $2Q$. Combining the results above, we can compute the local densities $\alpha_p(Q, T)$ for $p \nmid \ell(2Q)$. In particular if $\det(2Q) = 1$ (it occurs only when $4 \mid k$), we have $E^{g,k}(Z) = \Theta(Q; Z)$. Here $E^{g,k}(Z)$ denote the Eisenstein series of weight k for the full modular group $Sp(g, \mathbb{Z})$.

In the case $\ell(2Q) > 1$ such a simple result does not hold, since we have several Siegel Eisenstein series corresponding to the 0-dimensional cusps. However it is known that the following properties hold. Let

$$\mathcal{E}_k(\Gamma_0^g(l), \chi_Q) = \langle E_{g,\psi}^{k,l}(Z)|_k\gamma \mid \gamma \in \Gamma^g, \psi\rangle_{\mathbb{C}} \cap M_k(\Gamma_0^g(l), \chi_Q)$$

be the space of Siegel Eisenstein series, where ψ runs through the set of Dirichlet characters modulo l such that $\psi(-1) = (-1)^k$.

Proposition 1.3 For $Q \in S_{2k}^{+}(\mathbb{Z})^*$ with $\det(2Q) = l$, $\Theta(Q; Z)$ are contained in the space of Siegel Eisenstein series $\mathcal{E}_k(\Gamma_0^g(l), \chi_Q)$.

Our strategy is as follows: we choose suitable Q's such that $\{\Theta(Q; Z)\}$ form a basis of the space of Siegel Eisenstein series. The Fourier coefficients of $\Theta(Q; Z)$ are given by local densities, and an explicit formula of local densities are known by [SH], for general degree case and any odd primes. Thus the Fourier coefficients of the Siegel Eisenstein series are written by a linear combinations of known values.

Our aim is to find the local p-factor of the Siegel series, thus we may assume $l = p$ is an odd prime.

From now on, we fix an odd prime p. Let χ_p denotes the quadratic character modulo p and χ_0 denote the trivial character modulo p.

Assume $k > g + 1$. For each j such that $1 \le j \le 2g + 2$, we fix quadratic forms $Q_{2k}^{(j)} \in S_{2k}^{+}(\mathbb{Z})^*$ such that

$$\ell(2Q_{2k}^{(j)}) = p, \quad \det(2Q_{2k}^{(j)}) = p^j.$$

The existence of such $Q_{2k}^{(j)}$ will be discussed in the next section. Then

$$\chi_{Q_{2k}^{(j)}} = \begin{cases} \chi_0 & j \text{ is even}, \\ \chi_p & j \text{ is odd}. \end{cases}$$

Proposition 1.4 The set $\{Q_{2k}^{(2j-1)}\}$ $(1 \le j \le g + 1)$ form a basis of $\mathcal{E}_k(\Gamma_0^g(p), \chi_p)$, and the set $\{Q_{2k}^{(2j)}\}$ $(1 \le j \le g+1)$ form a basis of $\mathcal{E}_k(\Gamma_0^g(l), \chi_0)$.

This proposition is first proved by Katusrada-Shulze Pillot ([KS, Theorem 5.1]) in the case of $\chi = \chi_p$, after that Böcherer-Hironaka-Sato extended the result to the case including trivial character or the case of square-free level ([BHS, Corollary 5.2]).

In both of the above papers they showed the linearly independence of the local densities $\alpha_p(Q_{2k}^{(j)}, T)$, as an function of $T \in S_g(\mathbb{Z})^*$. Here we give an another proof of this proposition by using the theta transforms, i.e. not local theory but global theory. From our proof we can easily find the way to write the Siegel Eisenstein series as a linear combinations of genus theta series.

First we recall the 0-dimensional cusps of $\Gamma_0^g(p)\backslash\mathbb{H}_g$. A representative set of $\Gamma_0^g(p)\backslash\Gamma^g/\Gamma_\infty^g$ is given by

$$\Gamma_0^g(p)\backslash\Gamma^g/\Gamma_\infty^g = \bigcup_{r=0}^{g} \mathcal{M}_r, \quad \mathcal{M}_r = \begin{pmatrix} E_r & -I_r \\ I_r & E_r \end{pmatrix},$$

with

$$E_r = \mathrm{diag}(\underbrace{0,\ldots,0}_{r}, \underbrace{1,\ldots,1}_{g-r}), \quad I_r = \mathrm{diag}(\underbrace{1,\ldots,1}_{r}, \underbrace{0,\ldots,0}_{g-r}).$$

In particular $\mathcal{M}_0 = 1_{2g}$ and $\mathcal{M}_g = J_g = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}$. For $F \in M_k(\Gamma_0^g(p), \psi)$, $A_r(F)$ denote the constant term of the Fourier expansion of $F|_k\mathcal{M}_r$. For $k > g+1$ and $\chi = \chi_p$ or χ_0, the linear map

$$\mathcal{E}_k(\Gamma_0^g(p), \chi) \to \mathbb{C}^{g+1}, \quad F \mapsto \big(A_r(F)\big)_r$$

is isomorphic. In particular $\dim \mathcal{E}_k(\Gamma_0^g(p), \chi_p) = \dim \mathcal{E}_k(\Gamma_0^g(p), \chi_0) = g+1$.

Now we can show that the genus theta series satisfies

$$A_r\big(\Theta(Q_{2k}^{(j)}; Z)\big) = \big((-\mathbf{i})^k p^{-j/2}\big)^r. \tag{1.2}$$

Proof (Proof of Proposition 1.4) Change $j \mapsto 2j-1$ or $j \mapsto 2j$ in (1.2), we consider the matrices

$$\Big(\big((-\mathbf{i})^k p^{1/2-j}\big)^r\Big)_{\substack{0\le r\le g \\ 1\le j\le g+1}} \quad \text{or} \quad \Big(\big((-\mathbf{i})^k p^{-j}\big)^r\Big)_{\substack{0\le r\le g \\ 1\le j\le g+1}}, \tag{1.3}$$

whose determinants are non-zero since those are Vandermonde matrices. This shows our assertion. □

Our Siegel Eisenstein series $E_{p,\chi}^{g,k}(Z)$ ($\chi = \chi_p$ or χ_0) are characterized by the condition

$$A_r(E_{p,\chi}^{g,k}(Z)) = \begin{cases} 1 & r = 0 \\ 0 & 1 \le r \le g. \end{cases} \tag{1.4}$$

Let $(c_i)_{1\le i\le g+1}$ be the first column vector of the inverse matrices of (1.3) (it is common for both matrices). Then c_i satisfies

$$\sum_{i=1}^{g+1} p^{-ai} c_i = \begin{cases} 1 & a=0, \\ 0 & 1\le a\le g. \end{cases} \tag{1.5}$$

Explicitly we can write $c_j = \prod_{\substack{m=1\\ m\neq j}}^{g+1} (p^{m-j}-1)^{-1}$, but we use only the properties of (c_i).

Now we can write the Siegel Eisenstein series by the linear combination of the genus theta series.

Theorem 1.5 Assume $k > g+1$, then we have

$$E^{g,k}_{p,\chi_p}(Z) = \sum_{j=1}^{g+1} c_j \Theta(Z; Q^{(2j-1)}_{2k}), \quad E^{g,k}_{p,\chi_0}(Z) = \sum_{j=1}^{g+1} c_j \Theta(Z; Q^{(2j)}_{2k}).$$

In terms of Siegel series, the above theorem becomes as follows. For $\chi = \chi_p$ or χ_0, $S^p_g(p, \chi, T, k)$ is simply denoted by $S^p_g(\chi, T, k)$.

Theorem 1.6 For $k > g+1$ and $T \in S^+_g(\mathbb{Z})^*$, we have

$$S^p_g(\chi_p, T, k) = \mathbf{i}^{gk} \sum_{j=1}^{g+1} p^{g(1/2-j)} c_j\, \alpha_p\big(Q^{(2j-1)}_{2k}, T\big),$$

$$S^p_g(\chi_0, T, k) = \mathbf{i}^{gk} \sum_{j=1}^{g+1} p^{-gj} c_j\, \alpha_p\big(Q^{(2j)}_{2k}, T\big).$$

The term $\mathbf{i}^{gk}$ and $p^{g(1/2-j)}$ (resp. p^{-gj}) comes from (1.1).

2 Jordan decomposition of the quadratic forms

In this section, we show the existence of the $Q^{(j)}_{2k}$, that we considered in the previou section. It is well-known that an integral quadratic form is diagonalized over $\mathbb{Z}_p$ for each odd prime p. We also find the diagonal form in the same $\mathbb{Z}_p$ equivalent class of $Q^{(j)}_{2k}$.

First we recall the Haase invariant of the quadratic forms. Let q be a prime number. For any $Q \in S_g(\mathbb{Z})^*$, there exist $g \in GL_g(\mathbb{Q}_q)$ such that ${}^t gQg = \mathrm{diag}(a_1, \dots, a_g)$. Then we define the *Hasse invariant* $\mathrm{inv}_q(Q)$ by

$$\mathrm{inv}_q(Q) = \prod_{i<j} (a_i, a_j)_q \in \{\pm 1\},$$

here $(\cdot,\cdot)_q$ denote the Hilbert symbol. This value is independent of the choice of g.

For $S_1, S_2 \in S_g(\mathbb{Z})^*$, we write $S_1 \sim_q S_2$ if there exist $U_q \in GL_g(\mathbb{Z})$ such that ${}^tU_q S_1 U_q = S_2$.

The existence of $Q_{2k}^{(j)}$ comes from the following proposition.

Proposition 2.1 Let d be a positive integer. Assume that for each prime q, there exist $S_q \in S_g(\mathbb{Z}_q)$ such that $\det S_q = d$. Then there exist $S \in S_g^+(\mathbb{Z})$ with $S \sim_q S_q$ for each q if and only if $\prod_q \mathrm{inv}(S_q) = 1$.

Note that $\mathrm{inv}_q(S) = 1$ if $(q, 2d) = 1$, thus the infinite product is well-defined. For the proof we refer [Ca, Chapter 6, Theorem 1.3 and Chapter 9, Theorem 1.2].

Since our $Q_{2k}^{(j)} \in S_g^+(\mathbb{Z})^*$ not in $S_g^+(\mathbb{Z})$, we show the existence of $2Q_{2k}^{(j)} \in S_{2k}^+(\mathbb{Z})_e$. Our conditions are

$$\ell\big(2Q_{2k}^{(j)}\big) = p, \quad \det\big(2Q_{2k}^{(j)}\big) = p^j, \quad 2Q_{2k}^{(j)} \text{ is even.}$$

First we consider the Jordan decomposition in $\mathbb{Z}_2$. Since $2Q_{2k}^{(j)}$ is even, we have

$$2Q_{2k}^{(j)} \sim_2 H_k \quad \text{or} \quad 2Q_{2k}^{(j)} \sim_2 (H_{k-1} \perp W),$$

with

$$H_k = \underbrace{H \perp \cdots \perp H}_{k}, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The Hasse invariants are computed as

$$\mathrm{inv}_2(H_k) = (-1)^{k(k-1)/2}, \quad \mathrm{inv}_2(H_{k-1} \perp W) = -(-1)^{k(k-1)}.$$

By comparing the determinants we have that if j is even then $2Q_{2k}^{(j)} \sim_2 H_k$, and if j is odd then

$$2Q_{2k}^{(j)} \sim_2 H_k \iff p \equiv \pm 1 \bmod 8$$
$$2Q_{2k}^{(j)} \sim_2 (H_{k-1} \perp W) \iff p \equiv \pm 3 \bmod 8.$$

Next we consider the Jordan decomposition in $\mathbb{Z}_p$. Let

$$X_j = \mathrm{diag}(\underbrace{1,\ldots,1}_{2k-j}, \underbrace{p,\ldots,p}_{j}), \quad Y_j = \mathrm{diag}(\underbrace{1,\ldots,1}_{2k-j-1}, \gamma, \underbrace{p,\ldots,p}_{j-1}, p\gamma),$$

here γ is a fixed element in $\mathbb{Z}_p^\times$, that does not contained in $\mathbb{Z}_p^{\times 2}$. Then it is known that $Q_{2k}^{(j)} \sim_p X_j$ or Y_j. The Hasse invariants are $\mathrm{inv}_p(X_j) = \chi_p(-1)^{j(j-1)/2}$, $\mathrm{inv}_p(Y_j) = -\chi_p(-1)^{j(j-1)/2}$. Thus we can show the existence and the Jordan decomposition of $2Q_{2k}^{(j)}$ by choosing X_j or Y_j so that the product of the Hasse invariant becomes 1. Finally in order to find the Jordan decomposition of $Q_{2k}^{(j)}$, we consider the condition that $2X_j$ and X_j are equivalent in $\mathbb{Z}_q$. If j is even then $2X_j \sim_p X_j$, on the other hand if j is odd then $2X_j \sim_p X_j$ if and only if $2 \in \mathbb{Z}_p^{\times 2}$, i.e. $p \equiv 1, 7 \bmod 8$. As a consequence we have the following result.

Lemma 2.2 (1) Assume that j is odd. If $j \equiv 1 \bmod 4$, then

$$Q_{2k}^{(j)} \sim_p \begin{cases} X_j & k \equiv 0, 1 \bmod 4, \\ Y_j & k \equiv 2, 3 \bmod 4, \end{cases}$$

if $j \equiv 3 \bmod 4$ then

$$Q_{2k}^{(j)} \sim_p \begin{cases} X_j & k \equiv 0, 3 \bmod 4, \\ Y_j & k \equiv 1, 2 \bmod 4. \end{cases}$$

(2) Assume that j is even. If $j \equiv 0 \bmod 4$,

$$Q_{2k}^{(j)} \sim_p \begin{cases} X_j & k \equiv 0 \bmod 4, \\ Y_j & k \equiv 2 \bmod 4, \end{cases}$$

If $j \equiv 2 \bmod 4$ then

$$Q_{2k}^{(j)} \sim_p \begin{cases} X_j & k \equiv 0,\ p \equiv 1 \bmod 4 \quad \text{or} \quad k \equiv 2,\ p \equiv -1 \bmod 4, \\ Y_j & k \equiv 0,\ p \equiv -1 \bmod 4 \quad \text{or} \quad k \equiv 2,\ p \equiv 1 \bmod 4. \end{cases}$$

3 An explicit formula of the Siegel series

Now we give an explicit formula of the Fourier coefficients of Siegel Eisenstein series. Let l be an odd integer and ψ be a Dirichlet character modulo l such that $\psi^2 = 1$. Then the Siegel series are given by

$$\prod_{p \nmid l} S_g^p(\psi, T, k) \prod_{p \mid l} S_g^p(p^{e_p}, \psi, T, k),$$

here $e_p = \operatorname{ord}_p l$. For a prime $p \nmid l$, $S_g^p(\psi, T, k)$ is calculated by Katsurada [Kat] as explained above. The case of $p \mid l$, it suffices to compute $S_g^p(p^{e_p}, \chi, T, k)$ with $\chi = \chi_p$ or χ_0 (see [Ta1, Proposition 2.3]). The case of $e_p \geq 2$ we use the following lemma.

Lemma 3.1 Let ψ be a Dirichlet character modulo p, that is also regarded as the Dirichlet character modulo p^e $(e \geq 2)$. Then

$$E_{p^e,\psi}^{g,k}(Z) = E_{p,\psi}^{g,k}(p^{e-1}Z).$$

Thus we have

$$S_g^p(p^e, \chi, T, k) = \begin{cases} p^{(1-e)(gk-g(g+1)/2)} S_g^p(\chi, p^{-e+1}T, k) & T \equiv 0 \bmod p^{e-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore it suffices to consider the case of level p. The following lemma is essential for our method.

Lemma 3.2 Let p be an odd prime, $T \in S_g^+(\mathbb{Z})^*$. Then we have

$$\alpha_p(Q_{2k}^{(j)}, T) = \sum_{m=0}^{g} p^{mj/2} R_m(T,k),$$

here $R_m(T,k)$ depends only on T, k and j mod 2.

Remark Similar results are proved in [KS] or [BHS], where they treat the quadratic forms of the type $H_{2k-j} \perp pH_j$. Our assertion is that, if we take $Q_{2k}^{(j)}$ instead of them, then $R_m(T,k)$ depends only on j mod 2.

We write $R_m(T,k)$ as $R_m(T,k)_e$ or $R_m(T,k)_o$ according as j is even or odd. Then for example the case $\chi = \chi_p$,

$$\begin{aligned} S_p^g(\chi_p, T, k) &= \mathbf{i}^{gk} \sum_{j=1}^{g+1} c_j p^{g(1/2-j)} \alpha_p(Q_{2k}^{(2j-1)}, T) \\ &= \mathbf{i}^{gk} \sum_{j=1}^{g+1} \sum_{m=0}^{g} p^{(g-m)/2} p^{-(g-m)j} c_j R_m(T,k)_o \\ &= \mathbf{i}^{gk} R_g(T,k)_o \end{aligned}$$

by (1.5). As a consequence we have

$$S_g^p(\chi_p, T, k) = \mathbf{i}^{gk} R_g(T,k)_o, \quad S_g^p(\chi_0, T, k) = \mathbf{i}^{gk} R_g(T,k)_e.$$

It means that the Siegel series are nothing but the partial sum of the local densities.

For the proof of the lemma, we use the explicit formula by [SH]. We use the same notation as [SH], but in order to avoid the confusion of letters, we change a few notations. The size m of the matrix S is changed to $2k$, the matrix size n of T is changed to g, also the notations n_i, $n(i)$ are change to g_i, $g(i)$. The letters k in [SH] are changed to m in our notations.

Many invariants are contained in the explicit formula of $\alpha(Q,T)$, but there are a few invariants associated with Q. We write

$$Q_{2k}^{(j)} \sim_p \mathrm{diag}(u_1 p^{\alpha_1}, \ldots, u_{2k} p^{\alpha_{2k}}).$$

Then the term of p-power that depends on j is of the form $\prod_l p^{jg_l/2}$. Here g_l is the order of I_l for the partition of $I = \{1, \ldots, g\}$ into $I = I_0 \cup \cdots \cup I_r$.

The other contributions of$j$ are in the terms

$$\chi_p(-1)^{\sharp A(\lambda) + [\sharp A(\lambda)/2]} \prod_{m \in A(\lambda)} \chi_p(u_m). \qquad (*)$$

Here λ runs through the finite set of negative integers,

$$A(\lambda)=\begin{cases}\{1,2,\ldots,2k-j\} & \lambda \text{ is odd}\\ \{2k-j+1,\ldots,2k\} & \lambda \text{ is even.}\end{cases}$$

By using Lemma 2.2, we can compute that

$$(*)=\begin{cases}\mathbf{i}^k=(-1)^{k/2} & j \text{ is even,}\\ \mathbf{i}^k\varepsilon_p & j \text{ is odd,}\end{cases}\quad \text{here}\quad \varepsilon_p=\begin{cases}1 & p\equiv 1 \bmod 4\\ \mathbf{i} & p\equiv 3 \bmod 4.\end{cases}$$

In particular $(*)$ only depends on j mod 2, which proved our lemma □

Our main result is as follows. In addition to the notation in [SH], we define

$$\xi_{i,\lambda}(T,k)_e=2\cdot\mathbf{i}^k\prod_{m\in B_i(\lambda)}\chi_p(v_m)$$
$$\times\begin{cases}0 & \beta_i+\lambda\geq 0,\ \sharp B_i(\lambda):\text{odd}\\ (1-p^{-1})\chi_p(-1)^{[\sharp B_i(\lambda)/2]} & \beta_i+\lambda\geq 0,\ \sharp B_i(\lambda):\text{even}\\ \chi_p(-v_i)\chi_p(-1)^{[\sharp B_i(\lambda)/2]} & \beta_i+\lambda=-1,\ \sharp B_i(\lambda):\text{odd}\\ -p^{-1/2}\chi_p(-1)^{[\sharp B_i(\lambda)/2]} & \beta_i+\lambda=-1,\ \sharp B_i(\lambda):\text{even,}\end{cases}$$

$$\xi_{i,\lambda}(T,k)_o=2(-\mathbf{i})^k\varepsilon_p\prod_{m\in B_i(\lambda)}\chi_p(v_m)$$
$$\times\begin{cases}0 & \beta_i+\lambda\geq 0,\ \sharp B_i(\lambda):\text{even}\\ (1-p^{-1})\chi_p(-1)^{[\sharp B_i(\lambda)/2]} & \beta_i+\lambda\geq 0,\ \sharp B_i(\lambda):\text{odd}\\ \chi_p(v_i)\chi_p(-1)^{[\sharp B_i(\lambda)/2]} & \beta_i+\lambda=-1,\ \sharp B_i(\lambda):\text{even}\\ -p^{-1/2}\chi_p(-1)^{[\sharp B_i(\lambda)/2]} & \beta_i+\lambda=-1,\ \sharp B_i(\lambda):\text{odd.}\end{cases}$$

Then we have the following.

Theorem 3.3 (Main theorem) We have

$$S_g^p(\chi,T,k)=\mathbf{i}^{gk}\sum_{\substack{\sigma\in\mathfrak{S}_g\\ \sigma^2=1}}2^{-c_1(\sigma)}(1-p^{-1})^{c_2(\sigma)}p^{-c_2(\sigma)}\sum_{I=I_0\cup\cdots\cup I_r}p^{-\tau(\{I_i\})-t(\sigma,\{I_i\})}$$
$$\times\sum_{\{\nu\}}p^{-\sum_{l=0}^{r}\nu_l g(l)}\prod_{l=0}^{r}\widetilde{\Xi}_{l,\nu_0+\cdots+\nu_l}(\sigma;T,k)_*,$$

here $*$ represents e or o according as $\chi=\chi_0$ or χ_p. We put

$$\widetilde{\Xi}_{l,\lambda}(\sigma,T,k)_*=p^{\widetilde{\rho}_{l,\lambda}(\sigma;T,k)}\prod_{\substack{i\in I_l\\ \sigma(i)=i}}\xi_{i,\lambda}(T,k)_*,$$

with

$$\widetilde{\rho}_{l,\lambda}(\sigma;T,k) = kg_l\lambda + \frac{1}{2}\sum_{i\in I_l}\sum_{m=1}^{g}\min\{\beta_m + e_{\sigma,i,m} + \lambda, 0\},$$

and $\xi_{i,\lambda}(T,k)_e$ or $\xi_{i,\lambda}(T,k)_o$ are given as above. Finally the index $\{\nu\}$ runs through the set

$$\{(\nu_0,\nu_1,\ldots,\nu_r) \in \mathbb{Z}\times\mathbb{N}^r \mid -b_l(\sigma,T) \le \nu_0+\nu_1+\cdots+\nu_l \le -1,\ (0 \le \forall l \le r)\}.$$

Thus we can compute the Fourier coefficients of Siegel Eisenstein series for odd level.

References

[An] A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundl. math. Wiss. 286, Springer-Verlag, 1987.

[Boe] S. Böcherer, On the space of Eisenstein series for $\Gamma_0(p)$: Fourier expansions. With an appendix by H. Katsurada. Comment. Math. Univ. St. Pauli **63** (2014), no. 1-2, 3–22

[BHS] S. Böcherer and Y. Hironaka and F. Sato, Lineare independence of local densities of quadratic forms and its application to the theory of Siegel modular forms, Contemporary Mathematica, **493**, 2009, 51–82.

[Ca] J. W. S. Cassels, *Rational Quadratic forms*, Academic Press, New York, 1978.

[Di] M. J. Dickson, Fourier coefficients of degree two Siegel-Eisenstein series with trivial character at squarefree level, Ramanujan J. **37** (2015), no. 3, 541–562.

[Fr] E. Freitag, Siegel Eisenstein series of arbitraly level and theta series, Abh. Math. Sem. Univ. Hamburg **66** (1996), 229–247.

[Gu1] K. Gunji, The dimension of the space of Siegel-Eisenstein series of weight one, Math. Z. **260** (2008), 187–201.

[Gu2] K. Gunji, On the Siegel Eisenstein series of degree two for low weights, J. Math. Soc. Japan **67** (2015), no. 3, 1043–1067.

[Gu3] K. Gunji, On the Fourier coefficients of Siegel Eisenstein series of degree 2 for prime level, preprint.

[IS] T. Ibukiyama and H. Saito, On the zeta functions associated to symmetric matrices, I: an explicit form of zeta functions, Amer. J. Math. **117** (1995), 1097–1155.

[Kat] H. Katsurada, An explicit formula for Siegel series, Amer. J. Math. **121** (1999), 415–452.

[Ki] Y. Kiaoka, *Arithmetic of quadratic forms*, Cambridge Tracts in Mathematics, **106**. Cambridge University Press, Cambridge, 1993.

[KS] H. Katsurada and R. Schulze-Pillot, Genus theta series, Hecke operators and the basis problem for Eisenstein series, "Automorphic forms and zeta functions - Proceedings of the conference in memory of Tsuneo Arakawa", Word Scientific, 2006, 234–261.

[Mi] Y. Mizuno, An explicit arithmetic formula for the Fourier coefficients of Siegel-Eisenstein series of degree two and square-free odd levels, Math. Z. **263** (2009), no. 4, 837–860.

[SH] F. Sato and Y. Hironaka, Local densities of representations of quadratic forms over p-adic integers (the non dyadic case). J. Number Theory **83** (2000), 106–136.

[Sh] G. Shimura, Euler products and Fourier coefficients of automorphic forms on symplectic groups, Invent. Math. **116** (1994), 531–576.

[Si] C.L. Siegel, Über die analytische Theorie der quadratischen Formen I, Ann. Math. **36** (1935), 527-606

[Ta1] S. Takemori, p-adic Siegel Eisenstein series of degree two, J. Number Theory **132** (2012), 1203–1264.

[Ta2] S. Takemori, Siegel Eisenstein series of degree n and Λ-adic Eisenstein series, J. Number Theory **149** (2015), 105–138.

[Ya] T. Yang, An explicit formula for local densities of quadratic forms, J. Number Theory **72** (1998), 309–356.

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