FRAGILITY OF PROPERNESS

YASUO YOSHINOBU

Abstract. We prove that for any models $V \subsetneq W$ of ZFC with the same ordinals, there is a poset which is proper in V but not in W. This answers a question raised by Karagila.

In this short paper we prove the following theorem, which answers a question raised by Karagila [4, Problem 5]. The background of the question is explained in [5].

Theorem 1. Suppose $V \subsetneq W$ are models of ZFC with the same ordinals. Then there exists a poset $\mathbb{P}$ in V such that $\mathbb{P}$ is proper in V but not in W.

Proof. Let κ be the least ordinal such that ${}^{\kappa}\mathrm{Ord} \cap (W \setminus V) \neq \emptyset$. It is easy to see that κ is a regular infinite cardinal both in V and W. Let λ be the least ordinal such that ${}^{\kappa}\lambda \cap (W \setminus V) \neq \emptyset$. Then λ is a cardinal in V and satisfies $\lambda \geq 2$. Our proof of Theorem 1 is done in two cases.

Case 1 $\kappa > \omega$.

This case can be done with an argument similar to the one in [8, Section 2], which gave an example of a proper poset whose properness is destroyed by some κ-closed forcing. It was a variation of Shelah's example of a pair of proper posets whose product is improper (see [7, XVII Observation 2.12, p.826]).

Work in V first. Let T denote the tree ${}^{<\kappa}\lambda$ ordered by end-extension. Note that there are λ^{κ} branches through T. Let $\theta := \lambda^{\kappa}$, $\mathbb{P} := \mathrm{Add}(\omega, 1)$ and $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}} = \mathrm{Col}(\omega_1, \theta)$." Since $\dot{\mathbb{Q}}$ is σ-closed in $V^{\mathbb{P}}$, by Mitchell's theorem (see [6]) no branches through T are newly added by forcing over $\mathbb{P} * \dot{\mathbb{Q}}$, and thus there are exactly ω_1 branches through T in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$. Note that $\mathrm{cf}\kappa = \omega_1$ holds in $V^{\mathbb{P}*\dot{\mathbb{Q}}}$, and let $\dot{C}$ be a $(\mathbb{P} * \dot{\mathbb{Q}})$-name for a cofinal subset of κ of order type ω_1. In $V^{\mathbb{P}*\dot{\mathbb{Q}}}$ we let

$$T \restriction \dot{C} = \{t \in T \mid \mathrm{lh}(t) \in \dot{C}\}.$$

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Then $T \restriction \dot{C}$ forms a tree of height ω_1 with ω_1 cofinal branches. Now let $\dot{\mathbb{R}}$ denote a $(\mathbb{P} * \dot{\mathbb{Q}})$-name for a c.c.c. poset specializing $T \restriction \dot{C}$ (see [1, §7]). Then $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper in V.

Now let G be any $(\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}})$-generic filter over W (thus over V). While $T \restriction \dot{C}_G$ is specialized in $V[G]$, W (and thus $W[G]$) has branches through T which are not in V (and thus not in $V[G]$), and so $W[G]$ has branches through $T \restriction \dot{C}_G$ which are not in $V[G]$. Therefore ω_1 must be collapsed in $W[G]$, and thus $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is improper in W. □(Case 1)

Case 2 $\kappa = \omega$.

This case can be handled by generalizing the argument of Shelah (presented by Goldstern in [3]), showing that some σ-closed posets (for example $\mathrm{Col}(\omega_1, \omega_2)$) turns improper after adding a real in some ways (for example adding a Cohen real).

Lemma 2. There exists $\mu > \omega_1^W$, regular in W, such that $(\mathcal{P}_{\omega_1}\mu)^W \setminus V$ is stationary in W.

(Proof of Lemma 2)
Subcase (i) $\lambda = 2$ (namely there exists a real in $W \setminus V$).

In this subcase, the conclusion of Lemma 2 for $\mu = \omega_2^W$ directly follows from Gitik's theorem [2, Theorem 1.1].
Subcase (ii) Otherwise.

Pick an $f \in {}^{\omega}\lambda \cap (W \setminus V)$. Since no reals are in $W \setminus V$ in this subcase, it is easy to see that no $x \supseteq \mathrm{ran}(f)$ in V is countable in W. Pick a W-regular cardinal $\mu \geq \max\{\lambda, \omega_2^W\}$. Then in W, the set

$$X = \{x \in \mathcal{P}_{\omega_1}\mu \mid x \supseteq \mathrm{ran}(f)\}$$

does not intersect with V, and is stationary in $\mathcal{P}_{\omega_1}\mu$. □(Lemma 2)

Let μ be as in Lemma 2, and $\mathbb{P} = \mathrm{Col}(\omega_1, \mu)^V$. $\mathbb{P}$ is σ-closed and thus proper in V. Now work in W. Let θ be a sufficiently large cardinal. Then by Lemma 2,

$$Y = \{M \prec H_\theta \mid \mathbb{P} \in M, |M| = \omega, M \cap \mu \notin V\}$$

is stationary in $\mathcal{P}_{\omega_1}H_\theta$. Note that for each $M \in Y$, $M \cap \omega_1$ is an ordinal and so is $M \cap \omega_1^V$. We write $M \cap \omega_1^V$ as δ. For each $M \in Y$, if $p \in \mathbb{P}$ were $(M, \mathbb{P})$-generic, by a density argument we would have $\mathrm{ran}(p \restriction \delta) = M \cap \mu \notin V$, which is absurd since $p \in V$. Therefore $\mathbb{P}$ is not proper in W. □(Case 2)

□(Theorem 1)

Question In Theorem 1, can we always find $\mathbb{P}$ which is totally proper?

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Graduate School of Information Science
Nagoya University
Furo-cho, Chikusa-ku, Nagoya 464-8601
JAPAN
E-mail address: `yosinobu@is.nagoya-u.ac.jp`