

Nonisomorphic Smith equivalent modules over solvable groups

Toshio Sumi

Faculty of Arts and Science, Kyushu University

1 Introduction

Paul Smith posted the following question:

Question 1 ([17]) *Is it true that for any smooth action of a finite group G on a homotopy sphere with exactly two fixed points, the real G -modules determined on the tangent spaces at the two fixed points are isomorphic?*

If the action is semi-free, then the answer is yes [1]. Sanchez [16] showed that the answer is yes if the group G is of odd prime power order. However, Cappell and Shaneson [2] showed that the answer is no if G is a cyclic group of order $4n$ for $n \geq 2$. Many researchers studied this corresponding problem (eg. [15, 3, 7, 12, 10, 14]).

We say U and V are *Smith equivalent*, denoted by $U \simeq_S V$, if there exists a smooth G -action on a homotopy sphere S such that $S^G = \{x, y\}$ and the tangential G -modules $T_x(S)$ and $T_y(S)$ are isomorphic to U and V respectively. We put

$$\text{Sm}(G) := \{[U] - [V] \mid U \simeq_S V\} \subset \text{RO}(G)$$

which is called the *Smith set* of G . There are natural questions that for a group G , what the Smith set $\text{Sm}(G)$ is and in particular when the Smith set is not trivial.

Let $\mathcal{S}(G)$ be the set of all subgroups of G . For sets \mathcal{L} , \mathcal{P} of subgroups of G and a subset \mathcal{A} of $\text{RO}(G)$, we put

$$\begin{aligned} \mathcal{A}_{\mathcal{P}} &= \bigcap_{P \in \mathcal{P}} \ker(\text{Res}_P^G: \text{RO}(G) \rightarrow \text{RO}(P)) \cap \mathcal{A}, \\ \mathcal{A}^{\mathcal{L}} &= \bigcap_{L \in \mathcal{L}} \ker(\text{Fix}_L^G: \text{RO}(G) \rightarrow \text{RO}(N_GL/L)) \cap \mathcal{A}, \\ \mathcal{A}_{\mathcal{P}}^{\mathcal{L}} &= (\mathcal{A}_{\mathcal{P}})^{\mathcal{L}}. \end{aligned}$$

Let $\mathcal{P}_{\text{cyc}}(G)$ denote the set of cyclic subgroups of G of odd prime power order and let $\mathcal{P}_{\text{odd}}(G)$ denote the set of subgroups of G of odd prime power order. Clearly $\mathcal{P}_{\text{cyc}}(G) \subset \mathcal{P}_{\text{odd}}(G)$. By character theory, we have

$$\text{RO}(G)_{\mathcal{P}_{\text{odd}}(G)} = \text{RO}(G)_{\mathcal{P}_{\text{cyc}}(G)}.$$

Then by Sanchez's result, we have an implementation

$$\text{Sm}(G) \subset \text{RO}(G)_{\mathcal{P}_{\text{odd}}(G)}.$$

It is easy to see that for the set $\mathcal{Q}(G)$ of all subgroups of G of order 1, 2, 4, $\text{Sm}(G)$ is a subset of $\text{RO}(G)_{\mathcal{Q}(G)}$ and thus

$$\text{Sm}(G) \subset \text{RO}(G)_{\mathcal{P}_{\text{odd}}(G) \cup \mathcal{Q}(G)}.$$

However, $\text{Sm}(C_{4n})$ is not trivial for $n \geq 2$. Here C_{4n} denotes a cyclic group of order $4n$. In particular it is not satisfied that $\text{Sm}(G)$ is a subset of $\text{RO}(G)_{\mathcal{P}(G)}$.

We define a subset $\text{LSm}(G)$ called the Laitinen-Smith set of G , which is a subset of $\text{Sm}_{\mathcal{P}(G)}(G)$ as follows. We say U and V are *Laitinen-Smith equivalent*, denoted by $U \simeq_{LS} V$, if there exists a smooth G -action on a homotopy sphere S such that $S^G = \{x, y\}$, the tangential G -modules $T_x(S)$ and $T_y(S)$ are isomorphic to U and V respectively, and S^h is connected for any element h of G of 2-power order ≥ 8 . We put

$$\text{LSm}(G) := \{[U] - [V] \mid U \simeq_{LS} V\}.$$

The set $\text{LSm}(G)$ is a subset of $\text{Sm}(G)_{\mathcal{P}(G)}$, where $\mathcal{P}(G)$ denotes the set of subgroups of G of prime power order.

For an element g of G , the real conjugacy class of g , denoted by $(g)^\pm$ is the union of the conjugacy class of g and that of g^{-1} . A finite group G is called an Oliver group if there does not exist a sequence

$$P \triangleleft H \triangleleft G$$

such that P and G/H is of prime power order and H/P is cyclic.

Theorem 2 ([11, 5]) *The followings are equivalent.*

- (1) G is an Oliver group.
- (2) There exists a fixed-point-free G -action of a disk.
- (3) There exists a one fixed point G -action of a sphere.

Question 3 ([6]) *Let G be a finite Oliver group. Is it true that $\text{LSm}(G)$ is not trivial if G has at least two real conjugacy classes of elements of G not of prime power order?*

Morimoto [8] pointed out that the answer is no and moreover showed that $\text{Sm}(G)$ is a subset of $\text{RO}(G)^{\cap_2(G)}$, where $\cap_2(G)$ is the set of subgroups of G with index 1 or 2. Furthermore Morimoto and his students [4] showed that if a Sylow 2-subgroup of G is a normal subgroup of G then $\text{Sm}(G)$ is a subset of $\text{RO}(G)^{\cap(G)}$, where $\cap(G)$ is the set of normal subgroups of G with index 1 or prime.

In this note, we recall and study Oliver groups G satisfying that $\text{LSm}(G)^{\mathcal{L}(G)}$ is not trivial.

2 Some groups of which Smith sets are nontrivial

Let a_G denote the number of real conjugacy classes of elements not of prime power order. If $\text{LSm}(G)$ is not trivial then $a_G \geq 2$ holds. For $G = \text{Aut}(A_6)$, $a_G = 2$ and $\text{Sm}(G)$ is trivial. Let $b_{G,2}$ be the number of real conjugacy classes of $G/\cap_{L \in \cap_2(G)} L$ which is the image of the real conjugacy classes of elements of G not of prime power order.

Proposition 4 *If $a_G = b_{G,2}$ then $\text{LSm}(G)$ is trivial.*

Equivalently if $\text{LSm}(G)$ is not trivial then $a_G > b_{G,2}$ holds.

For a prime p , the Dress subgroup $O^p(G)$ is defined as the intersection of all subgroups of G with p -power index. Put

$$G^{\text{nil}} = \bigcap_p O^p(G).$$

The factor group G/G^{nil} is nilpotent. Let $\mathcal{L}(G)$ be the set of large subgroups, that is the set of subgroups of G which includes a Dress subgroup $O^p(G)$ for some prime p .

For a subset \mathcal{S} of $\mathcal{S}(G)$, we say that K is an \mathcal{S} -gap subgroup of G if there exists an $(\mathcal{S} \cap K)$ -free K -module V such that

$$\dim V^P > 2 \dim V^H$$

for any $P < H \leq K$ and $P \in \mathcal{P}(K)$, where

$$\mathcal{S} \cap K = \{S \cap K \mid S \in \mathcal{S}\}.$$

A finite group G is called a gap group if G is an $\mathcal{L}(G)$ -free gap subgroup of G . We say that a G -module V satisfies the *weak gap condition* if

$$\dim V^P \geq 2 \dim V^H$$

for any $P < H \leq G$ and $P \in \mathcal{P}(G)$. Also, we say that G satisfies the G^{nil} -coset condition¹ if there exists a G^{nil} -coset of G containing two elements x and y not of prime power order that are not real conjugate in G and, in addition,

- (1) the elements x and y are both in some $\mathcal{L}(G)$ -gap subgroup K of G , or
- (2) the orders of x and y are even and the elements of order 2 of $\langle x \rangle$ and $\langle y \rangle$ are conjugate in $G \setminus O^2(G)$, where $\langle x \rangle$ and $\langle y \rangle$ are the cyclic groups generated by x and y , respectively.

Remark 5 (cf. [18, Theorem B]) *If an Oliver group G has an element x not of prime power order such that $\langle x \rangle \cap O^2(G)$ is a group of even order then $\langle x \rangle O^2(G)$ is an $\mathcal{L}(G)$ -gap subgroup of G .*

Theorem 6 (cf. [14, Theorem 5.6]) *If G satisfies (2), then there exists an element of $\text{RO}(G)_{\mathcal{P}(G)}^{\mathcal{L}(G)}$ can be described as the difference of nonisomorphic G -modules satisfying the weak gap condition and it belongs to $\text{LSm}(G)$.*

Theorem 7 ([8, 9, 14]) *Let G be a finite nonsolvable group. Suppose that there exist two or more real conjugacy classes of elements of G not of prime power order. Then G satisfies one of the following property.*

- (1) G is isomorphic to $\text{Aut}(A_6)$.
- (2) G is isomorphic to $P\Sigma L(2, 27)$.

¹It was defined in [14]. This version is a little bit extended so that [14, Theorem 5.6].

(3) G satisfies the G^{nil} -coset condition.

Furthermore, $\text{LSm}(G)$ is nontrivial for any group G satisfying (2) or (3), and if $\text{Sm}(G)$ is trivial then (1) holds.

Question 8 Determine nonsolvable groups G of which Smith set is trivial. Is it true that if $a_G \leq 1$, then $\text{Sm}(G) = 0$?

By the above theorem and [12, Classification Theorem], such groups are finitely many.

For an epimorphism $f: G \rightarrow F$, we easily see that $f^* \text{Sm}(F)$ isomorphic to $\text{Sm}(F)$, is a subset of $\text{Sm}(G)$. However it is not known whether the action on a sphere giving the Smith equivalence is effective. If G is an Oliver group and there is a pair (V, W) of $\mathcal{L}(F)$ -free F -modules such that $[V] - [W] \in \text{RO}(F)_{\mathcal{P}(F)}$ and both V and W satisfy the weak gap condition, then we construct an effective action of a sphere giving $f^*([V] - [W]) \in \text{Sm}(G)$.

Let $\text{WRO}(G)$ be the set of the difference of $\mathcal{L}(G)$ -free G -modules satisfying the weak gap condition and put $\text{WLO}(G) = \text{LO}(G) \cap \text{WRO}(G)$. The following proposition is known.

Proposition 9 Let G be an Oliver group.

$$\text{WLO}(G) \subset \text{LSm}(G)^{\mathcal{L}(G)} \subset \text{Sm}(G)_{\mathcal{P}(G)}.$$

Note that $\text{WLO}(N) = \text{LO}(N)$ for a nilpotent group N . We have $\text{LO}(N) \subset \text{LSm}(N)$. Then we have

Proposition 10 If $\text{LSm}(G) = 0$ then $\text{LSm}(G/G^{\text{nil}}) = 0$ holds.

We remark that nilpotent groups N with $\text{LO}(N) = 0$ are known:

Proposition 11 ([13]) Let N be a finite nilpotent group not of prime power order. If $\text{LO}(N) = 0$, then N is isomorphic to $C_2 \times Q$, Q a group of odd prime power order, or $P \times C_3$ for a 2-group P such that x and x^{-1} are conjugate for any element $x \in P$.

3 Case when $\text{LO}(G) \neq 0$ implies $\text{WLO}(G) \neq 0$

It is unknown that there exists a finite group G such that $\text{LO}(G) \neq \text{WLO}(G)$. There exist many groups satisfying the equality:

Proposition 12 If G is a gap group then $\text{LO}(G) = \text{WLO}(G)$.

Proposition 13 If an Oliver group G has an element whose order is divisible by at least three distinct primes, then $0 \neq \text{WLO}(G) \subset \text{LO}(G) \subset \text{LSm}(G)$.

Proof By the assumption, G is a gap group and satisfies (1) and (2) of the G^{nil} -condition. ■

Theorem 14 Let G be a finite Oliver group with $O^p(G) \neq G$ for some odd prime p . Then for any $x \in \text{LO}(G)$ there exists an $\mathcal{L}(G)$ -free G -module W satisfying the weak gap condition such that $x + [W]$ is represented by a $\mathcal{L}(G)$ -free G -module satisfying the weak gap condition, which implies $\text{WLO}(G) = \text{LO}(G) \subset \text{LSm}(G)$.

Proof It suffice to show that

$$\text{WRO}(G)_{\mathcal{P}(G)}^{\{O^2(G)\}} = \text{RO}(G)_{\mathcal{P}(G)}^{\{O^2(G)\}}$$

for an Oliver group G . Since if G is a gap group then it is clear, we assume that G is not a gap group. First we assume that $[G : O^2(G)] = 2$. Then by [19, Theorem 7.7], the group G^{nil} is of odd order. By Sylow's theorem, we see the claim. In the case when G is an Oliver nongap group with $[G : O^2(G)] > 2$. By [18, Theorem A], we see the claim. ■

Suppose that G/G^{nil} has 2-power order. Let x and y be elements of G not of prime power order such that $xO^2(G) = yO^2(G)$ and $|xO^2(G)| > 2$. Then we can construct a nontrivial element of $\text{WLO}(G)$ by using elements of $\text{RO}(\langle x \rangle)_{\mathcal{P}(G)}$ and $\text{RO}(\langle y \rangle)_{\mathcal{P}(G)}$ by using the G^{nil} -condition. Thus we see

Theorem 15 *Let G be an Oliver group. Suppose that any subgroup K of G with $[K : O^2(G)] = 2$ is a gap group. Then $\text{LO}(G) = \text{WLO}(G) \subset \text{LSm}(G)$.*

Therefore, the case when $[G : G^{\text{nil}}] = 2$ is important:

Theorem 16 *Let G be a finite nongap Oliver group such that G/G^{nil} is cyclic. $\text{LO}(G) \otimes \mathbb{Q} \neq \text{WLO}(G) \otimes \mathbb{Q}$ if and only if there exist two elements x and y of G not of prime power order, $|x|$ and $|y|$ is even and not divisible by 4, and elements of order 2 of $\langle x \rangle$ and $\langle y \rangle$ are elements of $G \setminus O^2(G)$ and are not conjugate in G .*

Remark 17 *There exist finite Oliver groups G such that*

- (1) G is not a gap group,
- (2) $[G : G^{\text{nil}}] = 2$, and
- (3) there exist two elements of $G \setminus G^{\text{nil}}$ of order 2 which are not conjugate in G .

4 Oliver solvable groups

Nonsolvable groups are Oliver groups. In this section we consider finite solvable Oliver nongap groups. Ronald Solomon pointed out the structure of a nongap group G with $[G : G^{\text{nil}}] = 2$ as follows. Suppose that there exists an element of order 2 of $G \setminus G^{\text{nil}}$. The group G/G_2 is a solvable group for some normal subgroup G_2 of G of odd order: There exist normal subgroups G_1, G_2 of G such that

- (1) $G^{\text{nil}} \geq G_1 \geq G_2$,
- (2) G_2 is of odd order,
- (3) G_1/G_2 is a 2-group,
- (4) G^{nil}/G_1 is an abelian group of odd order, and every element of $(G/G_1) \setminus (G^{\text{nil}}/G_1)$ is an element of order 2.

Note that $(G \setminus G^{\text{nil}})/G_2$ does not have an element not of prime power order.
 For $K \leq G$, we put

$$c_{K,G} := \#\{(k)^\pm \mid k \in \text{NPP}(K)\},$$

where $(k)^\pm$ is the real conjugacy class of k . Recall the following proposition.

Proposition 18 ([12, Second Rank Lemma and Subgroup Lemma]) $\text{LSm}(G) \neq 0$ for an Oliver group G with $c_{G^{\text{nil}},G} \geq 2$.

Thus suppose that $c_{G^{\text{nil}},G} \leq 1$. Under this condition, we see

Theorem 19 Let G be a nongap Oliver solvable group with $[G : G^{\text{nil}}] = 2$ and $c_{G^{\text{nil}},G} \leq 1$. Then $\text{LSm}(G) = \text{Sm}(G)_{\mathcal{P}(G)} = \text{WRO}(G)_{\mathcal{P}(G)}^{\{G^{\text{nil}}\}} = \text{RO}(G)_{\mathcal{P}(G)}^{\{G^{\text{nil}}\}}$ and $\text{Ind}_{G^{\text{nil}}}^G \text{RO}(G^{\text{nil}})_{\mathcal{P}(G)}^{\{G^{\text{nil}}\}} = 0$.

The proof depends on the structure of a nongap group G with $[G : G^{\text{nil}}] = 2$ by Ronald Solomon. Note that if $\text{LO}(G) \neq 0$ then $0 \neq \text{WLO}(G) \subset \text{LSm}(G)$ for any nonsolvable group G .

Finally remark that there exist a few Oliver groups G such that $\text{LO}(G) = 0$ and $\text{LSm}(G) \neq 0$. The author expects that this is a rare case.

References

- [1] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic differential operators*, Bull. Amer. Math. Soc. **72** (1966), 245–250.
- [2] Sylvain E. Cappell and Julius L. Shaneson, *Fixed points of periodic differentiable maps*, Invent. Math. **68** (1982), no. 1, 1–19.
- [3] Karl Heinz Dovermann and Ted Petrie, *Smith equivalence of representations for odd order cyclic groups*, Topology **24** (1985), no. 3, 283–305.
- [4] Akihiro Koto, Masaharu Morimoto, and Yan Qi, *The Smith sets of finite groups with normal Sylow 2-subgroups and small nilquotients*, J. Math. Kyoto Univ. **48** (2008), no. 1, 219–227.
- [5] Erkki Laitinen and Masaharu Morimoto, *Finite groups with smooth one fixed point actions on spheres*, Forum Math. **10** (1998), no. 4, 479–520.
- [6] Erkki Laitinen and Krzysztof Pawałowski, *Smith equivalence of representations for finite perfect groups*, Proc. Amer. Math. Soc. **127** (1999), no. 1, 297–307.
- [7] Mikiya Masuda and Ted Petrie, *Lectures on transformation groups and Smith equivalence*, Group actions on manifolds (Boulder, Colo., 1983), Contemp. Math., vol. 36, Amer. Math. Soc., Providence, RI, 1985, pp. 191–242.
- [8] Masaharu Morimoto, *Smith equivalent $\text{Aut}(A_6)$ -representations are isomorphic*, Proc. Amer. Math. Soc. **136** (2008), no. 10, 3683–3688.

- [9] ———, *Nontrivial $\mathcal{P}(G)$ -matched \mathcal{S} -related pairs for finite gap Oliver groups*, J. Math. Soc. Japan **62** (2010), no. 2, 623–647.
- [10] Masaharu Morimoto and Krzysztof Pawałowski, *Smooth actions of finite Oliver groups on spheres*, Topology **42** (2003), no. 2, 395–421.
- [11] Bob Oliver, *Fixed point sets and tangent bundles of actions on disks and Euclidean spaces*, Topology **35** (1996), no. 3, 583–615.
- [12] Krzysztof Pawałowski and Ronald Solomon, *Smith equivalence and finite Oliver groups with Laitinen number 0 or 1*, Algebr. Geom. Topol. **2** (2002), 843–895 (electronic).
- [13] Krzysztof Pawałowski and Toshio Sumi, *Finite groups with Smith equivalent, nonisomorphic representations*, Proceedings of 34th Symposium on Transformation Groups, Wing Co., Wakayama, 2007, pp. 68–76.
- [14] ———, *The Laitinen conjecture for finite non-solvable groups*, Proc. Edinb. Math. Soc. (2) **56** (2013), no. 1, 303–336.
- [15] Ted Petrie, *Smith equivalence of representations*, Math. Proc. Cambridge Philos. Soc. **94** (1983), no. 1, 61–99.
- [16] Cristian U. Sanchez, *Actions of groups of odd order on compact, orientable manifolds*, Proc. Amer. Math. Soc. **54** (1976), 445–448.
- [17] P. A. Smith, *New results and old problems in finite transformation groups*, Bull. Amer. Math. Soc. **66** (1960), 401–415.
- [18] Toshio Sumi, *Gap modules for semidirect product groups*, Kyushu J. Math. **58** (2004), no. 1, 33–58.
- [19] ———, *Smith sets of non-solvable groups whose nilquotients are cyclic groups of order 1, 2, or 3*, Geometry of transformation groups and combinatorics, RIMS Kôkyûroku Bessatsu, B39, Res. Inst. Math. Sci. (RIMS), Kyoto, 2013, pp. 149–165.

Faculty of Arts and Science
 Kyushu University
 Fukuoka, 819-0395
 JAPAN
 E-mail address: sumi@artsci.kyushu-u.ac.jp