MATRIX FUNCTIONS AND MATRIX ORDER

MITSURU UCHIYAMA (SHIMANE UNIV.(EP), RITSUMEIKAN UNIV.(GUEST))

Abstract. This note is based on [8, 19]. The main purpose is to give a new method to construct an operator monotone function, and to give a characterization for an operator monotone function on a finite interval.

1. Introduction

Let $f(t)$ be a real continuous function defined on an interval J in the real axis. For a hermitian matrix A whose spectrum is in J, i.e. $A \in B_h(J)$, $f(A)$ is well-defined. f is called an *operator monotone function* on J and denoted by $f \in \mathbf{P}(J)$ if this map $A \mapsto f(A)$ preserves the matrix order, i.e.,

$$f(A) \leqq f(B) \text{ whenever } A \leqq B.$$

$\mathbf{P}_+(J)$ stands for $\{f | f \in \mathbf{P}(J), f(t) > 0\}$. f is said to be *operator decreasing* if $-f$ is operator.

g is called an *operator convex function*on J if it fulfills the operator inequality

$$g(sA + (1-s)B) \leqq sg(A) + (1-s)g(B)$$

for every $0 < s < 1$ and for every pair A, B with spectra in J.
An *operator concave function* is similarly defined. We here give some examples to help our comprehension. But the proves of some of them need subsequent results.

Example 1.1. (i) A power function t^λ is operator monotone and operator concave on $(0, \infty)$ for $0 \leq \lambda \leq 1$.
(ii) For $1 < \lambda \leq 2$, t^λ is operator convex but not operator monotone on $(0, \infty)$.
(iii) $1/t$ is operator decreasing and operator convex on $(0, \infty)$.
(iv) $1/t$ is operator decreasing and operator concave on $(-\infty, 0)$.
(v) $\tan t \in \mathbf{P}(-\pi/2, \pi/2)$.

We now refer to the excellent theorem:

Löwner (or Loewner)[12] Let J be open. Then $f \in \mathbf{P}(J)$ if and only if f has a holomorphic extension $f(z)$ to the open upper half plane

Π_+ which is a *Pick function.* In this case,

$$f(t) = \alpha + \beta t + \int_{-\infty}^{\infty} (-\frac{x}{x^2+1} + \frac{1}{x-t}) d\nu(x), \tag{1}$$

where α is real, $\beta \geqq 0$ and ν is a Borel measure so that

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} d\nu(x) < \infty, \quad \nu(J) = 0.$$

Refer to **Donoghue**[10] and **B. Simon**[16] for further study on this area.

It is well-known that $f(t)$ defined on a right half line is operator monotone if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$. This characterization does not hold if the domain of $f(t)$ is a finite interval; for instance, $\tan t \in \mathbf{P}(-\pi/2, \pi/2)$ is neither operator concave nor numerically concave.

In this note we will give a characterization of an operator monotone function on a finite interval.

A relationship between the operator monotone function and the operator convex function has been investigated:

Bendat-Sherman [4]. $g(t)$ is operator convex on an open interval J if and only if

$$K_g(t, t_0) := \frac{g(t)-(t_0)}{t-t_0} \quad (t \neq t_0), \quad K_g(t_0, t_0) = g'(t_0)$$

is in $\mathbf{P}(J)$ for every $t_0 \in J$.

M. Uchiyama [18]. Let $g(t)$ be a C^1-function on an open interval J. Then $g(t)$ is operator convex if $K_g(t, t_0)$ is operator monotone for one point $t_0 \in J$.

B. Simon(2017) showed us that $\frac{\tan t}{t}$ is operator convex since

$$\frac{\frac{\tan t}{t} - 1}{t} \in \mathbf{P}(-\pi/2, \pi/2).$$

The following characterization for an operator convex function is fundamental for subsequent study on operator inequality.

C. Davis[9]. g is operator convex on J if and only if

$$Pg(A_P)P \leq Pg(A)P$$

for every A with spectrum in J and for every orthogonal projection P, where A_P is the compression of A to the range of P.

Definition 1.1 (L. Brown [6, 7]). g is called a *strongly operator convex function* and denoted by $g \in \mathbf{SOC}(J)$ if

$$Pg(A_P)P \leq g(A)$$

for every A and for every orthogonal projection P.

One can see the following elementary facts.
⋆ A strongly operator convex function is operator convex.
⋆ A positive constant function is strongly operator convex.
⋆ The identity function $f(t) = t$ is not strongly operator convex on any interval.

2. Strongly operator convex functions (Brown-U)

Theorem 2.1 ([8]). Let $g(t)$ be a continuous function on J such that $g(t) > 0$. Then the following are mutually equivalent.
(i) $g \in \mathbf{SOC}(J)$.
(ii)

$$\begin{aligned}&\frac{1}{2}g(A) + \frac{1}{2}g(B) - g(\frac{A+B}{2})\\ \geq&\frac{1}{2}\left(g(A) - g(B)\right)\{g(A) + g(B)\}^{-1}\left(g(A) - g(B)\right).\end{aligned}$$

(iii) $1/g(t)$ is operator concave.
(iv) $g(t) > 0$ and

$$\begin{aligned}&S^*g(A)S + \sqrt{I - S^*S}g(B)\sqrt{I - S^*S}\\ &- g(S^*AS + \sqrt{I - S^*S}B\sqrt{I - S^*S})\\ &\geq X\{\sqrt{I - SS^*}g(A)\sqrt{I - SS^*} + Sg(B)S^*\}^{-1}X^*\end{aligned}$$

for every contraction S and for every pair of bounded self-adjoint operators A, B with spectra in J, where

$$X = S^*g(A)\sqrt{I - SS^*} - \sqrt{I - S^*S}g(B)S^*.$$

Theorem 2.2 ([8]). Let $f(t)$ be a continuous function on J and $t_0 \in J$. Then

$$f(t) \in \mathbf{P}(J) \iff K_f(t, t_0) \in \mathbf{SOC}(J).$$

⋆ This gives a new method to construct an operator monotone function.

Example 2.1. Since $\tan t \in \mathbf{P}(-\frac{\pi}{2}, \frac{\pi}{2})$, $\frac{\tan t}{t} \in \mathbf{SOC}(J)$ and hence

$$\frac{\tan t - t}{t^2} = \frac{\frac{\tan t}{t} - 1}{t - 0} \in \mathbf{P}(J).$$

Example 2.2. Since $\frac{\tan t}{t} \in \mathbf{SOC}(J)$ for $J = (-\pi/2, \pi/2)$, $\frac{t}{\tan t}$ is operator concave, i.e., $-\frac{t}{\tan t}$ is operator convex. Thus

$$\frac{1}{t} - \cot t = \frac{-\frac{t}{\tan t} + 1}{t - 0} \in \mathbf{P}(J).$$

Example 2.3. *Since* $t^\alpha \in \mathbf{P}(J)$, *where* $0 < \alpha < 1$ *and* $J = (0, \infty)$, $\frac{t^\alpha - 1}{t-1} \in \mathbf{SOC}(J)$, *and hence*

$$\frac{t^{\alpha-1} - 1}{t^\alpha - 1} = \frac{-\frac{t-1}{t^\alpha - 1} + 1}{t} \in \mathbf{P}(0, \infty)$$

.

Theorem 2.3 ([8]). (i)$0 \neq g \in \mathbf{SOC}(-\infty, \infty)$ if and only if $g(t) > 0$ and $g(t)$ is constant.
(ii) $0 \neq g \in \mathbf{SOC}(a, \infty)$ if and only if $g(t) > 0$ and $g(t)$ is operator decreasing.
(iii) $0 \neq g \in \mathbf{SOC}(-\infty, b)$ if and only if $g(t) > 0$ and $g(t) \in \mathbf{P}(-\infty, b)$.

Proposition 2.4 ([8], cf. Ju. L. Šmul'jan[17]). Let $f(t)$ be a function on a finite interval (a, b). Then

(i) If f is operator concave and operator monotone on (a, b), then f has an extension $\tilde{f}$ to (a, ∞) such that $\tilde{f}$ is operator concave and operator monotone on (a, ∞).
(ii) If f is operator convex and operator decreasing on (a, b), then f has an extension $\tilde{f}$ to (a, ∞) such that $\tilde{f}$ is operator convex and operator decreasing on (a, ∞).
(iii) If f is operator convex and operator monotone on (a, b), then f has an extension $\tilde{f}$ to $(-\infty, b)$ such that $\tilde{f}$ is operator convex and operator monotone on $(-\infty, b)$.
(iv) If f is operator concave and operator decreasing on (a, b), then f has an extension $\tilde{f}$ to $(-\infty, b)$ such that $\tilde{f}$ is operator concave and operator decreasing on $(-\infty, b)$.

Remark 2.1. We did not know whether this result had been known or not. However **B. Simon** referred us to [17] about (i).

3. Matrix means

Let us quickly remember essential results about matrix means.
Andersen-Duffin [1]. For $A, B \geq 0$, the (matrix) harmonic mean $A \,!\, B$ is defined by

$$A \,!\, B = 2A(A+B)^+B (= 2B(A+B)^+A),$$

where X^+ denotes the generalized inverse of X.
$A\,!\,B = (\frac{A^{-1}+B^{-1}}{2})^{-1}$ if A, B are invertible.

Pusz-Woronowicz [15]**, Ando**[2]. For $A, B \geq 0$ the matrix geometric mean $A\#B$ is defined by

$$A\#B = \max\{X \geq 0 : \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\}.$$

Pedersen-Takesaki [14]**, Ando**[2].

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

if A is invertible.

Remark 3.1. $0 \leq C \leq A\#B \not\Rightarrow \begin{pmatrix} A & C \\ C & B \end{pmatrix} \geq 0.$

4. Main Results

In this section we investigate matrix functions by making use of matrix means, especially harmonic mean. See [19] for detail. We start with new elementary result on matrix mean.

Lemma 4.1. If $A, B \geqq 0$ and $A + B > 0$, then

$$\frac{A+B}{2} - A\,!\,B = \frac{1}{2}(A-B)(A+B)^{-1}(A-B).$$

Lemma 4.2. If $B \geqq A \geqq 0$ and $B > 0$, then

$$B - A = \big(B + (A\#B)\big)\,!\,\big(B - (A\#B)\big).$$

Lemma 4.3. Let $0 < A \leqq B$. Then the operator equation

$$0 < X \leqq Y, \quad A = X\,!\,Y, \quad B = \frac{X+Y}{2}$$

has a unique solution

$$X = B - (B-A)\#B, \quad Y = B + (B-A)\#B.$$

Moreover, we have $A\#B = X\#Y$.

Lemma 4.4. For $A, B, C \geqq 0$

$$C \leqq A\,!\,B \Longleftrightarrow \begin{pmatrix} A & C \\ C & B \end{pmatrix} \geqq \tfrac{1}{2}\begin{pmatrix} C & C \\ C & C \end{pmatrix} \geq 0$$

We are ready to proceed to matrix functions.

Theorem 4.1. *Let $g(t) > 0$ be a continuous function defined on J. Then*

$$g \in \mathbf{SOC}(J) \Leftrightarrow \begin{cases} g(\frac{A+B}{2}) \leqq g(A)\,!\,g(B) \leqq \\ \leqq \frac{g(A)+g(B)}{2} \quad (\forall A, B). \end{cases}$$

Theorem 4.2. (i) if $f(t) > 0$ on $(0, b)$, where $0 < b \leqq \infty$, then

$$f \in \mathbf{P}_+(0, b) \Leftrightarrow f(A \,!\, B) \leqq f(A) \,!\, f(B) \ \ (\forall A, B).$$

(ii) $f \in \mathbf{P}(0, b) \Leftrightarrow \begin{cases} f(A \,!\, B) \leqq \frac{f(A)+f(B)}{2} \ \ (\forall A, B), \\ f(0+) < \infty. \end{cases}$

(iii) if $f(t) > 0$ on (a, ∞), where $-\infty < a$, then

$$f \in \mathbf{P}_+(a, \infty) \Leftrightarrow \begin{cases} f(A) \,!\, f(B) \leqq f(\frac{A+B}{2}) \ \ (\forall A, B), \\ f(\infty) > 0. \end{cases}$$

(iv) $f(t) \in \mathbf{P}(0, \infty) \Leftrightarrow f(A \,!\, B) \leqq f(\frac{A+B}{2}) \ \ (\forall A, B)$.

Remark 4.1. $\star$ We can get a characterization for $f \in \mathbf{P}(a, b)$ from (ii) by translation.
$\star$ The constraints $f(0+) < \infty$ in (ii) and $f(\infty) > 0$ in (iii) are both indispensable; for instance, $f(t) = \frac{1}{t}$ satisfies $f(A \,!\, B) = \frac{1}{2}(f(A) + f(B))$ and $f(0+) = \infty$, but it is operator decreasing.
$\star$ $\Rightarrow$ in (iii) and $\Rightarrow$ in (iv) are well-known.

Example 4.1. (i) $\tan(A \,!\, B) \leqq \tan A \,!\, \tan B$ for $0 < A, B < \pi/2$.

(ii) $(A \,!\, B)^\alpha \leqq A^\alpha \,!B^\alpha \leqq \frac{1}{2}(A^\alpha + B^\alpha) \leqq (\frac{A+B}{2})^\alpha$ for $A, B > 0$.

(iii) $\log(A \,!\, B) \leqq \frac{1}{2}(\log A + \log B) \leqq \log \frac{A+B}{2}$ for $A, B > 0$.

Corollary 4.3. Let σ be a symmetric operator mean defined in [11] and $f \in \mathbf{P}_+(0, \infty)$, then

$$f(A \,!\, B) \leqq f(A)\sigma f(B) \leqq f(\frac{A+B}{2}) \ (\forall A, B > 0).$$

Conversely, each of the following implies $f \in \mathbf{P}_+(0, \infty)$:
(i) $f(A \,!\, B) \leqq f(A)\sigma f(B) \ (\forall A, B), \ f(0+) < \infty$,
(ii) $f(A)\sigma f(B) \leqq f(\frac{A+B}{2}) \ (\forall A, B), \ f(\infty) > 0$,
(iii) $f(A \,!\, B) \leqq f(\frac{A+B}{2}) \ (\forall A, B)$.

We remark that **Ando-Hiai**[3] has shown that
if $f > 0$, $f(A)\sigma f(B) \leqq f(\frac{A+B}{2})$ and σ is not the harmonic mean, then $f \in \mathbf{P}_+(0, \infty)$.

Proposition 4.4. *Let $g(t) > 0$ on J. TFAE*
(i) $g \in \mathbf{SOC}(J)$,
(ii) $f(g(t)) \in \mathbf{SOC}(J)$ *for every* $f \in \mathbf{P}_+(0, \infty)$,
(iii)

$$\begin{pmatrix} f(g(A)) & f(g(\frac{A+B}{2})) \\ f(g(\frac{A+B}{2})) & f(g(B)) \end{pmatrix} \geqq 0$$

for every $f \in \mathbf{P}_+(0,\infty)$ and for every $A, B \in B_h(J)$,

(iv) *$f(g(t))$ is operator convex on J for every $f \in \mathbf{P}_+(0,\infty)$.*

(v) *$f(g(t))$ is operator convex on J for every $f \in \mathbf{P}(0,\infty)$.*

Corollary 4.5. ([19], cf.[13]) For $0 < A, B, C < b$, the following are mutually equivalent:

(i) $C \leqq A \,!\, B$.

(ii) $f(C) \leqq f(A) \,!\, f(B)$ for every $f \in \mathbf{P}_+(0,b)$.

(iii) $\begin{pmatrix} f(A) & f(C) \\ f(C) & f(B) \end{pmatrix} \geqq \frac{1}{2}\begin{pmatrix} f(C) & f(C) \\ f(C) & f(C) \end{pmatrix} \geqq 0$ for every $f \in \mathbf{P}_+(0,b)$.

(iv) $\begin{pmatrix} f(A) & f(C) \\ f(C) & f(B) \end{pmatrix} \geqq 0$ for every $f \in \mathbf{P}_+(0,b)$.

(v) $f(C) \leqq \frac{1}{2}(f(A) + f(B))$ for every $f \in \mathbf{P}(0,b)$.

Remark 4.2.

$$0 \leq C \leq A \,!\, B \Leftrightarrow \begin{pmatrix} f(A) & f(C) \\ f(C) & f(B) \end{pmatrix} \geqq 0 \ (\forall f \in \mathbf{P}_+(0,\infty))$$

was given in [13] , which is an interesting paper, but there is an essential mistake in the proof.

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