A summary on Zeta-functions of root systems and Poincaré polynomials of Weyl groups

立教大学理学部数学科　小森 靖 (Yasushi Komori)
Department of Mathematics, Rikkyo University
名古屋大学大学院 多元数理科学研究科　松本 耕二 (Kohji Matsumoto)
Graduate School of Mathematics, Nagoya University
首都大学東京大学院 理工学研究科　津村 博文 (Hirofumi Tsumura)
Department of Mathematics and Information Sciences
Tokyo Metropolitan University

1 Introduction

Witten zeta-functions were introduced as partition functions of quantum gauge theories and are expressed as

$$\zeta_W(s;G) = \sum_{\psi} \frac{1}{(\dim\psi)^s}, \tag{1.1}$$

where ψ runs over all finite dimensional irreducible representations of a connected compact semisimple Lie group G [20, 21]. Some of these zeta-functions are explicitly given as the following multiple Dirichlet series:

$$\sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s), \tag{1.2}$$

$$\sum_{m,n=1}^{\infty} \frac{2^s}{m^s n^s (m+n)^s}, \tag{1.3}$$

$$\sum_{m,n=1}^{\infty} \frac{6^s}{m^s n^s (m+n)^s (m+2n)^s}. \tag{1.4}$$

In [2–6, 8–10, 13] we consider multivariable analog of the above zeta-functions and call them zeta-functions of root systems and studied their special values at integers and established value relations among them. For example, (1.3) is generalized as

$$\zeta_2(s_{12}, s_{23}, s_{13}; A_2) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_{12}} n^{s_{23}} (m+n)^{s_{13}}}, \tag{1.5}$$

and a special value is given as

$$\zeta_2(2,2,2;A_2) = \frac{1}{6}(-1)^3 \frac{1}{3780} \frac{(2\pi i)^{2+2+2}}{2!2!2!} = \frac{\pi^6}{2835}, \tag{1.6}$$

where $\frac{1}{3780}$ is given by multiple analog of Bernoulli numbers. Then the next question arises naturally: What about functional relations? In the case of Euler-Zagier multiple zeta-functions, only harmonic products are known as functional relations on the whole space: For $s_1, s_2 \in \mathbb{C}$,

$$\zeta_{EZ,2}(s_1, s_2) + \zeta_{EZ,2}(s_2, s_1) = \zeta(s_1 + s_2) - \zeta(s_1)\zeta(s_2). \tag{1.7}$$

If we admit the restriction of the domain, we also have another type of functional relation [7,16]. As for the multiple zeta-functions of root systems, it is known that there are some functional relations. One of such relations is given in [5, 17, 19]. For $k_{12}, k_{13} \in \mathbb{N}$ and $s_{23} \in \mathbb{C}$,

$$\begin{aligned} &\zeta_2(k_{12}, s_{23}, k_{13}; A_2) + (-1)^{k_{12}} \zeta_2(k_{12}, k_{13}, s_{23}; A_2) + (-1)^{k_{12}+k_{13}} \zeta_2(s_{23}, k_{13}, k_{12}; A_2). \\ &= 2 \sum_{j_2=0}^{[k_{12}/2]} (-1)^{k_{12}} \binom{k_{12}+k_{13}-1-2j_2}{k_{13}-1} \zeta(2j_2)\zeta(k_{12}+k_{13}+s_{23}-2j_2) \\ &\quad + 2 \sum_{j_3=0}^{[k_{13}/2]} (-1)^{k_{13}} \binom{k_{12}+k_{13}-1-2j_3}{k_{12}-1} \zeta(2j_3)\zeta(k_{12}+k_{13}+s_{23}-2j_3). \end{aligned} \tag{1.8}$$

In particular, for $k_{12} = k_{13} = s_{23} = 3$, we have

$$(1-1+1)\zeta_2(3,3,3;A_2) = -40\zeta(0)\zeta(9) - 12\zeta(2)\zeta(7). \tag{1.9}$$

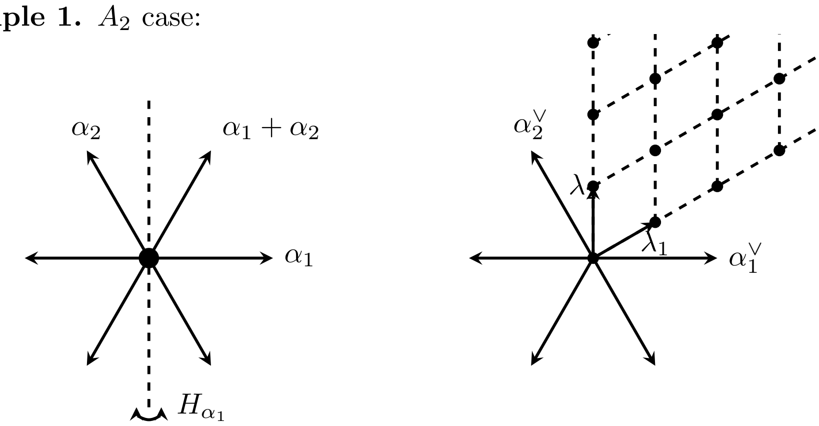
Our main purpose is to generalize this formula, that is, we understand the left-hand side by a group theoretic interpretation and the right-hand side by the Poincaré polynomials. For the details, see the forthcoming paper [14].

2 Zeta-Functions of Root Systems

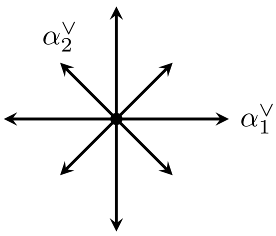
2.1 Root Systems

Let V be an r dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$ and $\Delta \subset V$ be a root system. Let σ_α be the reflection with respect to the hyperplane H_α orthogonal to $\alpha \in \Delta$ and W be the Weyl group, which is generated by all reflections σ_α. Let $\alpha^\vee$ be the coroot of α, which is equal to $2\alpha/\langle \alpha, \alpha \rangle$ and Δ_+ be the set of all positive roots. Let $\{\alpha_1, \dots, \alpha_r\}$ be the fundamental roots of Δ, which consists of a basis such that $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta_+$ with all $c_i \geq 0$. Let $P_{++} = \bigoplus \mathbb{Z}_{\geq 1}\lambda_i$ be the set of all strictly dominant weights, where $\{\lambda_1, \ldots, \lambda_r\}$ is a dual basis of $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$. For the geometric meaning of these symbols, see the following example [1].

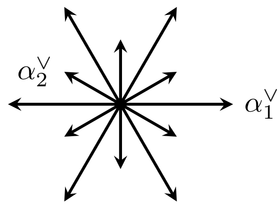
Example 1. A_2 case:



Example 2. C_2 case:



G_2:



2.2 Zeta-Functions of Root Systems

Definition 1 (Zeta-functions of root systems [3], multivariable Lerch analog)**.** For a root system Δ and for $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^{|\Delta_+|}$ and $\mathbf{y} \in V$, define

$$\zeta_r(\mathbf{s}, \mathbf{y}; \Delta) = \sum_{\lambda \in P_{++}} e^{2\pi i \langle \mathbf{y}, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}, \tag{2.1}$$

Example 3. We obtain the corresponding zeta-functions by formally replacing $\alpha_1^\vee$ and $\alpha_2^\vee$ by m and n appearing in positive coroots. For example, in the root systems of rank 2, we have

$$\zeta_2(\mathbf{s}, \mathbf{y}; A_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i (m y_1 + n y_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3}}, \tag{2.2}$$

$$\zeta_2(\mathbf{s}, \mathbf{y}; C_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i (m y_1 + n y_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}}, \tag{2.3}$$

$$\zeta_2(\mathbf{s}, \mathbf{y}; G_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i (m y_1 + n y_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4} (m+3n)^{s_5} (2m+3n)^{s_6}}. \tag{2.4}$$

Here and hereafter if the root system Δ is of type X_r, we write $\zeta_r(\mathbf{s},\mathbf{y};X_r)$ instead of $\zeta_r(\mathbf{s},\mathbf{y};\Delta)$ for short.

3 Special Zeta-Values (Review)

We extend $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}$ to $(s_\alpha)_{\alpha\in\Delta}$ by $s_\alpha=s_{-\alpha}$ and define $(w\mathbf{s})_\alpha=s_{w^{-1}\alpha}$. Then we have the following.

Theorem 1 (value relations [3, 5])**.** *For* $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

$$\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};\Delta)=(-1)^{|\Delta_+|}P(\mathbf{k},\mathbf{y};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr),\tag{3.1}$$

where $P(\mathbf{k},\mathbf{y};\Delta)$ *is a multiple periodic Bernoulli function, which will be defined below.*

Theorem 2 (special values [3, 5])**.** *For* $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in(2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k}=\mathbf{k}$ *for all* $w\in W$,

$$\zeta_r(\mathbf{k},\mathbf{0};\Delta)=\frac{(-1)^{|\Delta_+|}}{|W|}P(\mathbf{k},\mathbf{0};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr)\in\mathbb{Q}\pi^{\sum_{\alpha\in\Delta_+}k_\alpha}.\tag{3.2}$$

Example 4.

$$\begin{aligned}\zeta(2)&=\frac{-1}{2}\frac{1}{6}\frac{(2\pi i)^2}{2!}=\frac{\pi^2}{6}.\\ \zeta_2((2,4,4,2),\mathbf{0};C_2)&=\sum_{m,n=1}^{\infty}\frac{1}{m^2n^4(m+n)^4(m+2n)^2}\\ &=\frac{(-1)^4}{2^2 2!}\frac{53}{1513512000}\Bigl(\frac{(2\pi i)^2}{2!}\Bigr)^2\Bigl(\frac{(2\pi i)^4}{4!}\Bigr)^2=\frac{53}{6810804000}\pi^{12}.\end{aligned}\tag{3.3}$$

4 Multiple Periodic Bernoulli Functions (Review)

Let $\mathscr{V}$ be the set of all bases $\mathbf{V}\subset\Delta_+$ and $\mathbf{V}^*=\{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$ be the dual basis of $\mathbf{V}^\vee=\{\beta^\vee\}_{\beta\in\mathbf{V}}$. Let $Q^\vee=\bigoplus_{i=1}^r\mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $L(\mathbf{V}^\vee)=\bigoplus_{\beta\in\mathbf{V}}\mathbb{Z}\beta^\vee$. Note that $|Q^\vee/L(\mathbf{V}^\vee)|<\infty$. Fix a certain $\phi\in V$ and define a multiple generalization of the fractional part of real numbers as

$$\{\mathbf{y}\}_{\mathbf{V},\beta}=\begin{cases}\{\langle\mathbf{y},\mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi,\mu_\beta^{\mathbf{V}}\rangle>0),\\ 1-\{-\langle\mathbf{y},\mu_\beta^{\mathbf{V}}\rangle\} & (\langle\phi,\mu_\beta^{\mathbf{V}}\rangle<0).\end{cases}\tag{4.1}$$

Definition 2 (generating functions [3, 5])**.** For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$,

$$
\begin{aligned}
F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{V}\in\mathscr{V}} \Bigl(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle} \Bigr) \\
\times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \Bigl(\prod_{\beta\in\mathbf{V}} \frac{t_\beta \exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \Bigr).
\end{aligned}
\tag{4.2}
$$

Definition 3 (multiple periodic Bernoulli functions [3, 5])**.**

$$
F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} P(\mathbf{k},\mathbf{y};\Delta) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}. \tag{4.3}
$$

Remark. The A_1 case reduces to the classical generating function:

$$
F(t,y) = \frac{te^{t\{y\}}}{e^t-1} = \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!}. \tag{4.4}
$$

5 Functional Relations

Let I be a subset of $\{1,\ldots,r\}$. We will see that this determines which variables are complex. Let Δ_I be the subroot system of Δ with the fundamental roots $\{\alpha_i\}_{i\in I}$ and W^I be the minimal coset representatives of W/W_I with the Weyl group W_I of Δ_I, that is, $W = W_I W^I$.

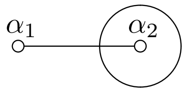
Theorem 3 (functional relations)**.** *For $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+}$ with $s_\alpha\in\mathbb{C}$ ($\alpha\in\Delta_{I+}$) and $s_\alpha = k_\alpha \in \mathbb{Z}_{\geq 2}$ ($\alpha\in\Delta_+\setminus\Delta_{I+}$), we have*

$$
\begin{aligned}
&\sum_{w\in W^I} \Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \Bigr) \zeta_r(w^{-1}\mathbf{s}, w^{-1}\mathbf{y};\Delta) \\
&\quad = (-1)^{|\Delta_+\setminus\Delta_{I+}|} \Bigl(\prod_{\alpha\in\Delta_+\setminus\Delta_{I+}} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!} \Bigr) \sum_{\lambda\in P_{I++}} \Bigl(\prod_{\alpha\in\Delta_{I+}} \frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}} \Bigr) P(\mathbf{k},\mathbf{y},\lambda;I;\Delta),
\end{aligned}
\tag{5.1}
$$

where $P(\mathbf{k},\mathbf{y},\lambda;I;\Delta)$ is a multiple periodic Bernoulli function associated with I, which will be defined below.

It should be noted that generally, the right-hand side consists of sum of several zeta-functions of lower rank.

Example 5. In the root system of type A_2, we choose $I = \{2\}$, which we express as the following diagram



(5.2)

where the circled node belongs to I. Then we have

$$\begin{aligned}
&\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2)\\
&=(-1)^2\left(\frac{(2\pi i)^{k_{12}}}{k_{12}!}\frac{(2\pi i)^{k_{13}}}{k_{13}!}\right)\\
&\qquad\times\sum_{m=1}^{\infty}\frac{1}{m^{s_{23}}}\left(\frac{b_0}{m^{k_{12}+k_{13}}}+\frac{b_2}{m^{k_{12}+k_{13}-2}}+\cdots+\frac{b_j}{m^{k_{12}+k_{13}-2j}}\right),
\end{aligned}\tag{5.3}$$

where $j=\max\{[k_{12}/2],[k_{13}/2]\}$ and $b_0,\ldots,b_j$ are certain real numbers. It should be noted that the right-hand side consists of sum of several Riemann zeta-functions.

To define a multiple periodic Bernoulli function associated with I, we need some definitions. Let $\mathscr{V}_I$ be the set of all bases of the form $\mathbf{V}=\mathbf{V}_I\cup\{\alpha_i\mid i\in I\}$ with $\mathbf{V}_I=\{\gamma_1,\ldots,\gamma_d\}\subset\Delta_+\setminus\Delta_{I+}$ and $p_{\mathbf{V}_I^\perp}$ be the projection defined by

$$p_{\mathbf{V}_I^\perp}(v)=v-\sum_{\gamma\in\mathbf{V}_I}\mu_\gamma^{\mathbf{V}}\langle\gamma^\vee,v\rangle\tag{5.4}$$

for $v\in V$.

Then we obtain the following:

Theorem and Definition 4 (generating function)**.** *For* $\mathbf{t}_I=(t_\alpha)_{\alpha\in\Delta_+\setminus\Delta_{I+}}$ *and* $\lambda\in P_I$,

$$\begin{aligned}
F(\mathbf{t}_I,\mathbf{y},\lambda;I;\Delta)&=\sum_{\mathbf{V}\in\mathscr{V}_I}\Bigg(\prod_{\gamma\in\Delta_+\setminus\Delta_{I+}\cup\mathbf{V}_I}\frac{t_\gamma}{t_\gamma-\sum_{\beta\in\mathbf{V}_I}t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle-2\pi\sqrt{-1}\langle\gamma^\vee,p_{\mathbf{V}_I^\perp}(\lambda)\rangle}\Bigg)\\
&\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}\exp(2\pi\sqrt{-1}\langle\mathbf{y}+q,p_{\mathbf{V}_I^\perp}(\lambda)\rangle)\Bigg(\prod_{\beta\in\mathbf{V}_I}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Bigg)\\
&=\sum_{\mathbf{k}\in\mathbb{N}_0^{|\Delta_+\setminus\Delta_{I+}|}}P(\mathbf{k},\mathbf{y},\lambda;I;\Delta)\prod_{\alpha\in\Delta_+\setminus\Delta_{I+}}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}.
\end{aligned}\tag{5.5}$$

In particular, if $I=\emptyset$, $F(\mathbf{t}_I,\mathbf{y},\lambda;I;\Delta)$ reduces to the generating function for value relations:

$$\begin{aligned}
F(\mathbf{t}_\emptyset,\mathbf{y},\lambda;\emptyset;\Delta)=F(\mathbf{t},\mathbf{y};\Delta)&=\sum_{\mathbf{V}\in\mathscr{V}}\Bigg(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}}\frac{t_\gamma}{t_\gamma-\sum_{\beta\in\mathbf{V}}t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle}\Bigg)\\
&\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)}\Bigg(\prod_{\beta\in\mathbf{V}}\frac{t_\beta\exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1}\Bigg).
\end{aligned}\tag{5.6}$$

Remark. In the proof of this theorem, we use the results in [12].

6 Examples

6.1 A_r Case

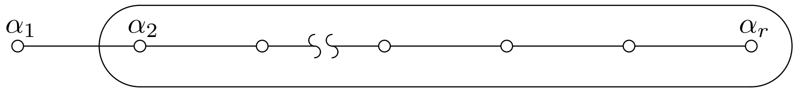
We use the following realization of the root system of type A_r:

$$\Delta_+ = \{e_i - e_j \mid 1 \leq i < j \leq r+1\} \subset \mathbb{R}^{r+1}, \qquad (\langle e_i, e_j \rangle = \delta_{ij}). \tag{6.1}$$

Then the zeta-function of type A_r is expressed as

$$\zeta_r((s_{ij})_{1\leq i<j\leq r}, (y_i)_{1\leq i\leq r}; A_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\exp(2\pi\sqrt{-1}\sum_{1\leq i\leq r} m_i y_i)}{\prod_{1\leq i<j\leq r+1}(m_i + \cdots + m_{j-1})^{s_{ij}}}. \tag{6.2}$$

We choose $I = \{2, \ldots, r\}$ and $I^c = \{1\}$ as in the following Dynkin diagram.



(6.3)

Then we have the following theorem:

Theorem 5 (generating function)**.** *Put* $t_{e_1 - e_i} = t_i$ *for* $2 \leq i \leq r+1$.

$$\begin{aligned} &F((t_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \\ &= \sum_{j=2}^{r+1} \prod_{i=2}^{j-1} \frac{t_i}{t_i - t_j + 2\pi\sqrt{-1}(m_i + \cdots + m_{j-1})} \prod_{i=j+1}^{r+1} \frac{t_i}{t_i - t_j - 2\pi\sqrt{-1}(m_j + \cdots + m_{i-1})} \\ &\quad \times \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_i y_i\Big)\Big) \frac{t_j \exp(t_j\{y_1\})}{e^{t_j} - 1}. \end{aligned} \tag{6.4}$$

Theorem 6 (multiple periodic Bernoulli function)**.**

$$\begin{aligned} &F((t_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \\ &= \sum_{k_2, \ldots, k_{r+1} \geq 0} P((k_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \frac{t_2^{k_2} \cdots t_{r+1}^{k_{r+1}}}{k_2! \cdots k_{r+1}!}, \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} &P((k_i)_{2\leq i\leq r+1}, (y_j)_{1\leq j\leq r}, (m_i)_{2\leq i\leq r}; \{2, \ldots, r\}; A_r) \\ &= k_2! \cdots k_{r+1}! \sum_{j=2}^{r+1} \Big(\prod_{\substack{i=2 \\ i\neq j}}^{r+1} \delta_{k_i \neq 0}\Big) \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i - y_1) + \sum_{i=j}^{r} m_i y_i\Big)\Big) \\ &\quad \times \Big(\sum_{\substack{l_2, \ldots, l_{r+1} \geq 0 \\ l_2 + \cdots + l_{r+1} = k_j}} \frac{B_{l_j}(\{y_1\})}{l_j!} \prod_{\substack{2\leq i\leq r+1 \\ i\neq j}} (-1)^{k_i - 1} \binom{k_i + l_i - 1}{l_i} \Big(\frac{1}{2\pi\sqrt{-1}m_{ij}}\Big)^{k_i + l_i} \Big), \end{aligned} \tag{6.6}$$

with

$$m_{ij}=\begin{cases} m_i+\cdots+m_{j-1} & (i<j)\\ -(m_j+\cdots+m_{i-1}) & (i>j). \end{cases} \tag{6.7}$$

Theorem 7. *For $(s_{ij})_{1\le i<j\le r+1}$ with $s_{1j}=k_{1j}$ $(2\le j\le r+1)$, we have*

$$\begin{aligned}
&\sum_{j=0}^{r}\Bigl(\prod_{i=1}^{j}(-1)^{k_{1,i+1}}\Bigr)\zeta_r((s_{(1\cdots j+1)pq})_{1\le p<q\le r+1},(y_2-y_1,\ldots,y_{j+1}-y_1,y_{j+1},\ldots,y_r);A_r)\\
&=-\sum_{j=2}^{r+1}\sum_{\substack{l_2,\ldots,l_{r+1}\ge 0\\ l_2+\cdots+l_{r+1}=k_{1,j}}}(-1)^{k_{1,2}+\cdots+k_{1,j-1}+l_{j+1}+\cdots+l_{r+1}}(2\pi\sqrt{-1})^{l_j}\frac{B_{l_j}(\{y_1\})}{l_j!}\\
&\quad\times\prod_{\substack{2\le i\le r+1\\ i\ne j}}\binom{k_{1,i}+l_i-1}{l_i}\zeta_{r-1}((s_{pq}+\delta_{p<j}\delta_{q=j}(k_{1,p}+l_p)+\delta_{p=j}\delta_{q>j}(k_{1,q}+l_q))_{2\le p<q\le r+1},\\
&\qquad\qquad(y_2-y_1,\ldots,y_{j-1}-y_1,y_j,\ldots,y_r);A_{r-1}).
\end{aligned} \tag{6.8}$$

Remark. It should be noted that this is a special case. Generally, $\zeta_r(\mathbf{s},\mathbf{y};X_r)$'s are not necessarily described in terms of $\zeta_{r-1}(\mathbf{s},\mathbf{y};X_{r-1})$. It depends on the pair (X_r,I). We need more general multiple zeta-functions, which may not be classified as zeta-functions of root systems.

Remark. Other special cases are $(B_r,\{2,\ldots,r\})$, $(C_r,\{2,\ldots,r\})$.

Example 6. Set $r=2$, $(y_1,y_2)=(0,0)$. For $s_{23}\in\mathbb{C}$,

$$\begin{aligned}
&\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2)\\
&=2\sum_{j_2=0}^{[k_{12}/2]}(-1)^{k_{12}}\binom{k_{12}+k_{13}-1-2j_2}{k_{13}-1}\zeta(2j_2)\zeta(k_{12}+k_{13}+s_{23}-2j_2)\\
&+2\sum_{j_3=0}^{[k_{13}/2]}(-1)^{k_{12}}\binom{k_{12}+k_{13}-1-2j_3}{k_{12}-1}\zeta(2j_3)\zeta(k_{12}+k_{13}+s_{23}-2j_3).
\end{aligned} \tag{6.9}$$

Example 7. Set $r=3$, $(y_1,y_2,y_3)=(0,0,0)$. For $(s_{23},s_{24},s_{34})\in\mathbb{C}^3$,

$$\begin{aligned}
&\zeta_3(k_{12},k_{13},k_{14},s_{23},s_{24},s_{34};A_3)+(-1)^{k_{12}+k_{13}}\zeta_3(s_{23},k_{12},s_{24},k_{13},s_{34},k_{14};A_3)\\
&+(-1)^{k_{12}}\zeta_3(k_{12},s_{23},s_{24},k_{13},k_{14},s_{34};A_3)+(-1)^{k_{12}+k_{13}+k_{14}}\zeta_3(s_{23},s_{24},k_{12},s_{34},k_{13},k_{14};A_3)\\
&=2\sum_{j_2=0}^{[k_{12}/2]}\sum_{\substack{l_3,l_4\ge 0\\ l_3+l_4=k_{12}-2j_2}}(-1)^{k_{12}}\binom{k_{13}+l_3-1}{l_3}\binom{k_{14}+l_4-1}{l_4}\\
&\quad\times\zeta(2j_2)\zeta_2(s_{23}+k_{13}+l_3,s_{24}+k_{14}+l_4,s_{34};A_2)\\
&+2\sum_{j_3=0}^{[k_{13}/2]}\sum_{\substack{l_2,l_4\ge 0\\ l_2+l_4=k_{13}-2j_3}}(-1)^{k_{12}+l_4}\binom{k_{12}+l_2-1}{l_2}\binom{k_{14}+l_4-1}{l_4}\\
&\quad\times\zeta(2j_3)\zeta_2(s_{23}+k_{12}+l_2,s_{24},s_{34}+k_{14}+l_4;A_2)
\end{aligned} \tag{6.10}$$

$$+2\sum_{j_4=0}^{[k_{14}/2]}\sum_{\substack{l_2,l_3\geq 0\\ l_2+l_3=k_{14}-2j_4}}(-1)^{k_{12}+k_{13}}\binom{k_{12}+l_2-1}{l_2}\binom{k_{13}+l_3-1}{l_3}$$
$$\times\zeta(2j_4)\zeta_2(s_{23},s_{24}+k_{12}+l_2,s_{34}+k_{13}+l_3;A_2).$$

6.2 Various Expressions

In particular, if $k_{12}=k_{13}=k_{14}=s_{23}=s_{24}=s_{34}=2$,

$$\begin{aligned}4\zeta_3(2,2,2,2,2,2;A_3)=&\,2\zeta(2)\{2\zeta_2(4,4,2;A_2)+\zeta_2(4,2,4;A_2)\}\\&-6\zeta_2(6,4,2;A_2)-6\zeta_2(6,2,4;A_2)-8\zeta_2(5,5,2;A_2)\\&+4\zeta_2(5,2,5;A_2)-6\zeta_2(4,6,2;A_2).\end{aligned}\tag{6.11}$$

On the other hand, we obtained already in [2, Eq. (4.28)]

$$\begin{aligned}4\zeta_3(2,2,2,2,2,2;A_3)=&\,8\zeta(2)\left\{\zeta_2(4,4,2;A_2)+\zeta_2(3,5,2;A_2)\right\}\\&-12\zeta_2(6,4,2;A_2)+12\zeta_2(5,5,2;A_2)-6\zeta_2(4,6,2;A_2).\end{aligned}\tag{6.12}$$

Remark. These two expressions are transformed into each other by use of partial fraction decompositions.

Remark. (Open Problem) However in general A_r cases, we have two different expressions of the right-hand side and we do not know whether these two expressions are transformed into each other by use of partial fraction decompositions. Thus these expressions may give new value relations.

6.3 B_r Case

Theorem 8 (generating function for B_r case with $I^c=\{1\}$)**.** *We use the following realization:*

$$\Delta_+=\{e_i\pm e_j\mid 1\leq i<j\leq r\}\cup\{e_j\mid 1\leq j\leq r\}.\tag{6.13}$$

Put $t_{e_1\pm e_i}=t_{\pm i}$ *for* $2\leq i\leq r$ *and* $t_{e_1}=t_1$*.*

$F(t_1,(t_{\pm i})_{2\leq i\leq r},(y_j)_{1\leq j\leq r},(m_i)_{2\leq i\leq r};\{2,\ldots,r\};B_r)$

$$\begin{aligned}=&\sum_{j=2}^{r}\prod_{2\leq i<j}\frac{t_{-i}}{t_{-i}-t_{-j}+2\pi\sqrt{-1}(m_i+\cdots+m_{j-1})}\prod_{j<i\leq r}\frac{t_{-i}}{t_{-i}-t_{-j}-2\pi\sqrt{-1}(m_j+\cdots+m_{i-1})}\\&\times\prod_{2\leq i\leq j}\frac{t_{+i}}{t_{+i}-t_{-j}-2\pi\sqrt{-1}(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r)}\\&\times\prod_{j<i\leq r}\frac{t_{+i}}{t_{+i}-t_{-j}-2\pi\sqrt{-1}(m_j+\cdots+m_{i-1}+2(m_i+\cdots+m_{r-1})+m_r)}\\&\times\frac{t_1}{t_1-2t_{-j}-2\pi\sqrt{-1}(2(m_j+\cdots+m_{r-1})+m_r)}\end{aligned}$$

$$\times \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i-y_1)+\sum_{i=j}^{r} m_i y_i\Big)\Big)\frac{t_{-j}\exp(t_{-j}\{y_1\})}{e^{t_{-j}}-1}$$

$$+\sum_{j=2}^{r}\prod_{2\le i\le j}\frac{t_{-i}}{t_{-i}-t_{+j}+2\pi\sqrt{-1}(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r)}$$

$$\times\prod_{j<i\le r}\frac{t_{-i}}{t_{-i}-t_{+j}+2\pi\sqrt{-1}(m_j+\cdots+m_{i-1}+2(m_i+\cdots+m_{r-1})+m_r)}$$

$$\times\prod_{2\le i<j}\frac{t_{+i}}{t_{+i}-t_{+j}-2\pi\sqrt{-1}(m_i+\cdots+m_{j-1})}\prod_{j<i\le r}\frac{t_{+i}}{t_{+i}-t_{+j}+2\pi\sqrt{-1}(m_j+\cdots+m_{i-1})}$$

$$\times\frac{t_1}{t_1-2t_{+j}+2\pi\sqrt{-1}(2(m_j+\cdots+m_{r-1})+m_r)}$$

$$\times \exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{j-1} m_i(y_i-y_1)+\sum_{i=j}^{r-1} m_i(y_i-2y_1)+m_r(y_r-y_1)\Big)\Big)\frac{t_{+j}\exp(t_{+j}\{y_1\})}{e^{t_{+j}}-1}$$

$$+\prod_{2\le i\le r}\frac{t_{-i}}{t_{-i}-t_1+\pi\sqrt{-1}(2(m_i+\cdots+m_{r-1})+m_r)}$$

$$\times\prod_{2\le i\le r}\frac{t_{+i}}{t_{+i}-t_1-\pi\sqrt{-1}(2(m_i+\cdots+m_{r-1})+m_r)}$$

$$\times\frac{1}{2}\Big(\exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{r-1} m_i(y_i-y_1)+m_r(y_r-\frac{1}{2}y_1)\Big)\Big)\frac{t_1\exp(t_1\{\frac{1}{2}y_1\})}{e^{t_1}-1}$$

$$+\exp\Big(2\pi\sqrt{-1}\Big(\sum_{i=2}^{r-1} m_i(y_i-(y_1+1))+m_r(y_r-\frac{1}{2}(y_1+1))\Big)\Big)\frac{t_1\exp(t_1\{\frac{1}{2}(y_1+1)\})}{e^{t_1}-1}\Big)$$

Note that by expanding this expression, we see that we obtain functional relations among $\zeta_r(\cdot;B_r)$ and $\zeta_{r-1}(\cdot;B_{r-1})$ similar to those in the case of type A_r obtained in Theorem 7.

6.4 X_r with $|I|=1$ Case

In the case $|I|=1$, we will see that the sum of some $\zeta_r(\cdot;X_r)$ is expressed in terms of Lerch zeta-functions. Let $\phi(u,s)$ be the Lerch zeta-function defined by

$$\phi(u,s)=\sum_{n=1}^{\infty}\frac{e^{2\pi\sqrt{-1}un}}{n^s}. \tag{6.14}$$

Theorem 9. *Let $s_\alpha=k_\alpha\in\mathbb{Z}_{\ge 2}$ for $\alpha\in\Delta_+\setminus\{\alpha_i\}$ and $s_{\alpha_i}\in\mathbb{C}$. Let $|\mathbf{k}|=\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}}k_\alpha$. Let $X_i=\{\nu=\{\langle q,\mu^{\mathbf{V}}_{\alpha_i}\rangle\}\mid \mathbf{V}\in\mathscr{V}_I, q\in Q^\vee/L(\mathbf{V}^\vee)\}\subset\mathbb{Q}$.*

$$\sum_{w\in W^I}\Big(\prod_{\alpha\in\Delta_{w^{-1}}}(-1)^{-k_\alpha}\Big)\zeta_r(w^{-1}\mathbf{s},0;\Delta)$$

$$=(-1)^{|\Delta_+|-1}\Big(\prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\Big)\sum_{\nu\in X_i}\sum_{j=0}^{|\mathbf{k}|}\frac{b_{\mathbf{k}\nu j}}{(2\pi\sqrt{-1})^j}\phi(\nu,s_{\alpha_i}+j), \tag{6.15}$$

where $b_{\mathbf{k}\nu j}\in\mathbb{Q}$ *is given by*

$$\sum_{\mathbf{k}\in\mathbb{N}_0^{|\Delta^*|}}\sum_{\nu\in X_i}\sum_{j=0}^{|\mathbf{k}|} b_{\mathbf{k}\nu j}x^j y^\nu \prod_{\alpha\in\Delta^*}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}=\sum_{\mathbf{V}\in\mathscr{V}_I}\prod_{\gamma\in\Delta^*\setminus\mathbf{V}_I}\frac{t_\gamma}{t_\gamma-\sum_{\beta\in\mathbf{V}_I}t_\beta\langle\gamma^\vee,\mu_\beta^{\mathbf{V}}\rangle-\langle\gamma^\vee,\mu_{\alpha_i}^{\mathbf{V}}\rangle/x}$$
$$\times\frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|}\sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} y^{\{\langle q,\mu_{\alpha_i}^{\mathbf{V}}\rangle\}}\prod_{\gamma\in\mathbf{V}_I}\frac{t_\gamma\exp(t_\gamma\{q\}_{\mathbf{V},\gamma})}{e^{t_\gamma}-1}. \quad (6.16)$$

7 A Remarkable Theorem

It is natural that from functional relations we obtain value relations; we have only to substitute integers into variables. However it is remarkable that the converse holds, that is, the generating function for $I=\emptyset$ knows "everything." The following theorem tells that $F(\mathbf{t}_I,\mathbf{y},\lambda;I;\Delta)$ for general I can be deduced from the case $I=\emptyset$.

Theorem 10 (Remarkable Theorem)**.** *Let* $I\subset\{1,\dots,r\}$*. For* $\lambda\in P_{I++}$*, we have*

$$F(\mathbf{t}_I,\mathbf{y},\lambda;I;\Delta)=\operatorname*{Res}_{\substack{t_\alpha=2\pi\sqrt{-1}\langle\alpha^\vee,\lambda\rangle\\ \alpha\in\Delta_{I+}}}\Bigl(\prod_{\alpha\in\Delta_{I+}}\frac{1}{t_\alpha}\Bigr)F(\mathbf{t},\mathbf{y};\Delta). \quad (7.1)$$

8 Poincaré Polynomials and Special Zeta-Values

For $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in(\mathbb{Z}_{\geq1})^{|\Delta_+|}$ satisfying $w^{-1}\mathbf{k}=\mathbf{k}$ for all $w\in W^I$, the left-hand side of (5.1) is

$$\sum_{w\in W^I}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k},\mathbf{0};\Delta)=\Bigl(\sum_{w\in W^I}\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(\mathbf{k},\mathbf{0};\Delta). \quad (8.1)$$

From this expression, we notice that the coefficient of $\zeta_r(\mathbf{k},\mathbf{0};\Delta)$ coincides with the special value $W^I(((-1)^{k_\alpha})_{\alpha\in\Delta_+})$ of the Poincaré polynomial for W^I, where the Poincaré polynomials due to Macdonald are defined as follows [15]: For indeterminates $\mathbf{u}=(u_\alpha)_{\alpha\in\Delta_+}$ and for $X\subset W$

$$X(\mathbf{u})=\sum_{w\in X}\prod_{\alpha\in\Delta_+\cap w\Delta_-}u_\alpha. \quad (8.2)$$

Since generally it is very difficult to calculate special values of these Poincaré polynomials, we need their simple descriptions.

8.1 Poincaré polynomials

It is known [15] that if $u_\alpha=u$ for all $\alpha\in\Delta_+$,

$$W^I(\mathbf{u})=\frac{W(\mathbf{u})}{W_I(\mathbf{u})}, \quad (8.3)$$

with

$$W(\mathbf{u})=\prod_{i=1}^{r}\frac{u^{d_i}-1}{u-1},\qquad W_I(\mathbf{u})=\prod_{i\in I}\frac{u^{d'_i}-1}{u-1},\tag{8.4}$$

where d_i and d'_i are the degrees of the Weyl groups W and W_I, and these degrees are given as in the following table.

Type	$\{d_1,\ldots,d_r\}$
A_r	$2,3,4,\ldots,r+1$
B_r,C_r	$2,4,\ldots,2r$
D_r	$2,4,\ldots,2r-2,r$
E_6	$2,5,6,8,9,12$

Type	$\{d_1,\ldots,d_r\}$
E_7	$2,6,8,10,12,14,18$
E_8	$2,8,12,14,18,20,24,30$
F_4	$2,6,8,12$
G_2	$2,6$

From these facts, we see that if $u_\alpha=u$ for all $\alpha\in\Delta_+$,

$$W^I(\mathbf{u})=\sum_{w\in W^I}\prod_{\alpha\in\Delta_+\cap w\Delta_-}u_\alpha=\frac{\prod_{i=1}^{r}(u^{d_i}-1)/(u-1)}{\prod_{i\in I}(u^{d'_i}-1)/(u-1)}.\tag{8.5}$$

8.2 Case 1 (all even)

Consider the case $u_\alpha=(-1)^{k_\alpha}=1$ for all $\alpha\in\Delta_+$. Then by l'Hôpital's rule, we obtain

$$W^I(1)=|W^I|=\frac{\prod_{i=1}^{r}d_i}{\prod_{i\in I}d'_i}\in\mathbb{Z}_{\geq 1}.\tag{8.6}$$

Example 8 (A_2 with $I=\{2\}$)**.** In this case, Δ is of type A_2 and hence $d_1=2,d_2=3$ and Δ_I is of type A_1 and hence $d'_1=2$. Put $s_{ij}=k_{ij}=2m$ (even). Then the left-hand side of (5.1) is directly calculated as

$$\begin{aligned}&1\cdot\zeta_2(k_{12},s_{23},k_{13};A_2)+(-1)^{k_{12}}\zeta_2(k_{12},k_{13},s_{23};A_2)+(-1)^{k_{12}+k_{13}}\zeta_2(s_{23},k_{13},k_{12};A_2)\\&=(1+(-1)^{k_{12}}+(-1)^{k_{12}+k_{13}})\zeta_2(2m,2m,2m;A_2)\\&=3\cdot\zeta_2(2m,2m,2m;A_2).\end{aligned}\tag{8.7}$$

On the other hand this coefficient is calculated via Poincaré polynomials as

$$W^I(1)=\frac{d_1d_2}{d'_1}=3.\tag{8.8}$$

8.3 Case 2 (all odd)

Consider the case $u_\alpha=(-1)^{k_\alpha}=-1$ for all $\alpha\in\Delta_+$. Let $K=\{i\mid 1\leq i\leq r,d_i\in 2\mathbb{Z}\}$, $K_I=\{i\mid i\in I,d'_i\in 2\mathbb{Z}\}$. Then

$$W^I(-1)=\begin{cases}\dfrac{\prod_{i\in K}d_i}{\prod_{i\in K_I}d'_i}\in\mathbb{Z}_{\geq 1} & (|K|=|K_I|)\\ 0 & (|K|\neq|K_I|).\end{cases}\tag{8.9}$$

The following is a table of several examples where $W^I(-1)$ survives.

Type of Δ	Type of Δ_I	$W^I(-1)$
A_{2m}	A_{2m-1}	$2 \cdot 4 \cdots 2m/2 \cdot 4 \cdots 2m = 1$
A_3	A_1^2	$2 \cdot 4/2 \cdot 2 = 2$
D_{2m+1}	D_{2m}	$2 \cdot 4 \cdots 4m/2 \cdot 4 \cdots (4m-2) \cdot 2m = 2$
E_6	D_4	$2 \cdot 6 \cdot 8 \cdot 12/2 \cdot 4 \cdot 6 \cdot 4 = 6$

Example 9 (A_2 with $I = \{2\}$)**.** In this case, Δ is of type A_2 and Δ_I is of type A_1 as in the previous example. Put $s_{ij} = k_{ij} = 2n+1$ (odd). Then the left-hand side of (5.1) is directly calculated as

$$\begin{aligned} &1 \cdot \zeta_2(k_{12}, s_{23}, k_{13}; A_2) + (-1)^{k_{12}} \zeta_2(k_{12}, k_{13}, s_{23}; A_2) + (-1)^{k_{12}+k_{13}} \zeta_2(s_{23}, k_{13}, k_{12}; A_2) \\ &= (1 + (-1)^{k_{12}} + (-1)^{k_{12}+k_{13}}) \zeta_2(2m, 2m, 2m; A_2) \\ &= 1 \cdot \zeta_2(2m, 2m, 2m; A_2). \end{aligned} \tag{8.10}$$

On the other hand this coefficient is obtained from the above table as

$$W^I(-1) = 1. \tag{8.11}$$

8.4 Case 3 (Mixture)

Let Δ_1 be the set of all long roots and Δ_2, that of all short roots. Assume k_α are odd for $\alpha \in \Delta_1$ and k_β are even for $\beta \in \Delta_2$, and hence $u_\alpha = -1$ for $\alpha \in \Delta_1$ and $u_\beta = 1$ for $\beta \in \Delta_2$.

Lemma 11. *Let* $\mathbf{u} = (u, 1)$. *Then we have*

$$W^I(\mathbf{u}) = \frac{W(u,1)}{W_I(u,1)} = \frac{|W_J| W(\Delta_1)(u)}{|W_{I \cap J}| W(\Delta_1 \cap \Delta_I)(u)}. \tag{8.12}$$

The following is a table of some examples, where $W^I(-1, 1)$ survives.

Type of Δ	Type of Δ_I	$W^I(-1,1)$
B_{2k+1}	B_{2k}	$2 \cdot 2 \cdot 4 \cdots 4k/2 \cdot 2 \cdot 4 \cdots (4k-2) \cdot 2k = 2$
C_{2k+1}	C_{2k}	$2 \cdot 2 \cdot 4 \cdots 4k/2 \cdot 2 \cdot 4 \cdots (4k-2) \cdot 2k = 2$
G_2	A_1	$2 \cdot 2/2 = 2$

Example 10. Let Δ be of type G_2, and Δ_I be of type A_1. Let $p = u = v$ be even and $s = q = r$, odd. Then the left-hand side of (5.1) is directly calculated as

$$\begin{aligned} &\zeta_2(p, s, q, r, u, v; G_2) + (-1)^p \zeta_2(p, q, s, r, v, u; G_2) + (-1)^{p+q} \zeta_2(v, q, r, s, p, u; G_2) \\ &\quad + (-1)^{p+q+v} \zeta_2(v, r, q, s, u, p; G_2) + (-1)^{p+q+r+v} \zeta_2(u, r, s, q, v, p; G_2) \\ &\quad + (-1)^{p+q+r+u+v} \zeta_2(u, s, r, q, p, v; G_2) \\ &= 2\zeta_2(p, q, q, q, p, p; G_2). \end{aligned} \tag{8.13}$$

On the other hand this coefficient is obtained from the above table as

$$W^I(-1,1) = 2. \tag{8.14}$$

This recovers the result in [13].

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