COMPLETE ASYMPTOTIC EXPANSIONS FOR THE TRANSFORMED LERCH ZETA-FUNCTIONS VIA THE LAPLACE-MELLIN AND RIEMANN-LIOUVILLE OPERATORS (PRE-ANNOUNCEMENT)

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Abstract. This is a pre-announcement version of the forthcoming paper [21].

For a complex variable s, and any real parameters a and λ with $a > 0$, the Lerch zeta-function $\phi(s,a,\lambda)$ is defined by the Dirichlet series $\sum_{l=0}^{\infty} e(\lambda l)(a+l)^{-s}$ $(\operatorname{Re} s > 1)$, and its meromorphic continuation over the whole s-plane, where $e(\lambda) = e^{2\pi i\lambda}$, and the domain of the parameter a can be extended to the whole sector $|\arg z| < \pi$ through the procedure in [13]. It is the principal aim of the present article to treat asymptotic aspects of the transformed functions obtained by applying the Laplace-Mellin and Riemann-Liouville operators (in terms of the variable s), which are denoted by $\mathcal{LM}^{\alpha}_{z;\tau}$ and $\mathcal{RL}^{\alpha;\beta}_{z;\tau}$ respectively, to a slight modification, $\phi^*(s,a,\lambda)$, of $\phi(s,a,\lambda)$. For any $m \in \mathbb{Z}$, let $(\phi^*)^{(m)}(s,a,\lambda)$ denote the mth derivative with respect to s if $m \geq 0$, and the $|m|$th primitive defined with its initial point at $s + \infty$ if $m < 0$. We shall then show that complete asymptotic expansions exist, if $a > 1$, for $\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda)$ and for $\mathcal{RL}^{\alpha;\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda)$ (Theorems 1–4), as well as for their several iterated variants (Theorems 5–10), when the pivotal parameter z of the transforms tends to both 0 and ∞ through appropriate sectors. Most of our results include any vertical half-lines in their respective regions of validity; this allows us to deduce complete asymptotic expansions for the relevant transforms through arbitrary vertical half-lines, upon taking $(s,z) = (\sigma, it)$ with any $\sigma \in \mathbb{R}$, when $t \to \pm\infty$ (Corollaries 2.1, 4.1, 6.1 and 8.1).

1. Introduction

Throughout the present article, s is a complex variable, z a complex parameter, a and λ real parameters with $a > 0$, and the notation $e(z) = e^{2\pi i z}$ is frequently used. The Lerch zeta-function $\phi(s,a,\lambda)$ is defined by the Dirichlet series

$$\phi(s,a,\lambda) = \sum_{l=0}^{\infty} e(\lambda l)(a+l)^{-s} \qquad (\operatorname{Re} s > 1), \tag{1.1}$$

and its meromorphic continuation over the whole s-plane (cf. [24][25]); this reduces if $\lambda \in \mathbb{Z}$ to the Hurwitz zeta-function $\zeta(s,a)$, and so $\zeta(s,1) = \zeta(s)$ is the Riemann zeta-function.

Let $\Gamma(s)$ denote the gamma function, α and β be complex numbers with $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$, $f(z)$ a function holomorphic in the sector $|\arg z| < \pi$, and write $X_+ = \max(0, X)$ for any $X \in \mathbb{R}$. We introduce here the Laplace-Mellin and Riemann-Liouville

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(or Erdélyi-Köber) transforms of $f(z)$, given by

$$\mathcal{LM}^{\alpha}_{z;\tau} f(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(z\tau)\tau^{\alpha-1}e^{-\tau}d\tau \tag{1.2}$$

and

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau} f(\tau) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty f(z\tau)\tau^{\alpha-1}(1-\tau)_+^{\beta-1}d\tau \tag{1.3}$$

with the normalization gamma multiples, provided that the integrals converge; the factor $\tau^{\alpha-1}$ is inserted to secure the convergence of the integrals as $\tau \to 0^+$, while $e^{-\tau}$ and $(1-\tau)_+^{\beta-1}$ have effects to extract the portions of $f(\tau)$ corresponding to $\tau = O(z)$. Let $\delta(\lambda)$ denote the symbol which equals 1 or 0 according to $\lambda \in \mathbb{Z}$ or otherwise. We further introduce a slight modification, $\phi^*(s,a,\lambda)$, of $\phi(s,z,\lambda)$, defined as

$$\phi^*(s,z,\lambda) = \phi(s,z,\lambda) - \frac{\delta(\lambda)z^{1-s}}{s-1} = \begin{cases} \zeta(s,z) - \dfrac{z^{1-s}}{s-1} & \text{if } \lambda \in \mathbb{Z}, \\ \phi(s,z,\lambda) & \text{otherwise,} \end{cases} \tag{1.4}$$

by which the only possible singularity at $s=1$ can be removed. Let $f^{(m)}(s)$, for any entire function $f(s)$, denote its mth derivative if $m = 0,1,2,\ldots$, while its $|m|$th primitive if $m = -1,-2,\ldots$, defined inductively by

$$f^{(m)}(s) = \int_{s+\infty}^{s} f^{(m+1)}(w)dw = -\int_0^{0+\infty} f^{(m+1)}(s+z)dz \qquad (m=-1,-2,\ldots), \tag{1.5}$$

provided that the integral converges, where the paths of integrations are the horizontal line segments.

It is the principal aim of the present paper to treat asymptotic aspects of the Laplace-Mellin and Riemann-Liouville transforms of the modified Lerch zeta-function, given by

$$\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) = \frac{1}{\Gamma(\alpha)} \int_0^\infty (\phi^*)^{(m)}(s+z\tau,a,\lambda)\tau^{\alpha-1}e^{-\tau}d\tau, \tag{1.6}$$

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty (\phi^*)^{(m)}(s+z\tau,a,\lambda)\tau^{\alpha-1}(1-\tau)_+^{\beta-1}d\tau, \tag{1.7}$$

for any $m \in \mathbb{Z}$, where the conditions $a > 1$ and $|\arg z| \le \pi/2$ are required in (1.6) for convergence of the integral. We shall show that complete asymptotic expansions exist, if $a > 1$, for (1.6) and (1.7) (Theorems 1–4), and for their iterated variants (Theorems 5–10), when the pivotal parameter z of the transforms tends to both 0 and ∞ through appropriate sectors. Most of our results include any vertical half-lines in their region of validity; this allows us to deduce the complete asymptotic expansions through any vertical half-lines, upon taking $(s,z) = (\sigma, it)$ with any $\sigma \in \mathbb{R}$, as $t \to \pm\infty$ (Corollaries 2.1, 4.1, 6.1 and 8.1).

We give here a brief overview of history of research related to asymptotic aspects of the integral transforms of zeta-functions. The study of Laplace transforms for the mean square of $\zeta(s)$ seems to be initiated by Hardy-Littlewood [6], who obtained the asymptotic relation, as $\varepsilon \to 0^+$,

$$\mathcal{L}_{1/2}(\varepsilon) = \int_0^\infty \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^2 e^{-\varepsilon t}dt \sim \frac{1}{\varepsilon}\log\frac{1}{\varepsilon},$$

say, in connection with the research of asymptotic behaviour of the upper-truncated mean square of $\zeta(s)$, in the form $\int_0^T |\zeta(1/2+it)|^2 dt$ as $T \to +\infty$. Wilton [26] then refined the result above to

$$\mathcal{L}_{1/2}(\varepsilon) = \frac{1}{\varepsilon}\log\frac{1}{\varepsilon} - \frac{\log 2\pi - \gamma_0}{\varepsilon} + O\biggl(\frac{1}{\sqrt{\varepsilon}}\log^{3/2}\frac{1}{\varepsilon}\biggr)$$

as $\varepsilon \to 0^+$, where γ_0 is the 0th Euler-Stieljes constant (cf. [4, p.34, 1.12(17)]). The last O-term was, in fact, replaced by a complete asymptotic expansion by Köber [22], who showed, for any integer $N \geq 1$,

$$\begin{aligned}\mathcal{L}_{1/2}(\varepsilon) = \frac{1}{\varepsilon}\log\frac{1}{\varepsilon} - \frac{\log 2\pi - \gamma_0}{\varepsilon} + a_0 + \sum_{n=1}^{N-1}\varepsilon^n\biggl(a_n + b_n\log\frac{1}{\varepsilon} + c_n\log^2\frac{1}{\varepsilon}\biggr)\\ + O\biggl(\varepsilon^N\log^2\frac{1}{\varepsilon}\biggr)\end{aligned}$$

as $\varepsilon \to 0^+$, with some constants a_0, a_n, b_n and c_n ($n = 1, 2, \ldots$). It was finally succeeded, through rather elementary arguments, by Atkinson [2], among other things, in dropping the terms with $\log^2(1/\varepsilon)$ in the asymptotic series above, i.e. $c_n = 0$, and in improving the error term to $O\{\varepsilon^N\log(1/\varepsilon)\}$.

It can be said that many of the recent developments into this direction have been made, to a greater or less extent, in the spirit of Atkinson's influential work [3] on the error term, $E(T)$, of the upper-truncated mean square $\int_0^T |\zeta(1/2+it)|^2 dt$ as $T \to +\infty$. A more general weighted mean square

$$\mathcal{L}_\rho(s) = \int_0^\infty |\zeta(\rho+ix)|^2 e^{-sx}dx,$$

was treated in the late 1990's by Jutila [9], who made a detailed study of $\mathcal{L}_\rho(s)$, especially on the critical line $\rho = 1/2$, while obtaining its asymptotic formula as $s \to 0$, and applied it to re-derive the classical (so-called) Atkinson's formula for $E(T)$. A further study of $\mathcal{L}_\rho(s)$ has been carried out by Kačinskaitė-Laurinčikas [10]. On the other hand, the lower-truncated Mellin transform

$$\mathcal{M}_{k,\rho}(s) = \int_1^\infty |\zeta(\rho+ix)|^{2k}x^{-s}dx \qquad (k = 1, 2, \ldots),$$

was explored by Ivić-Jutila-Motohashi [8], who applied it to investigate the higher power moments, in particular the eighth power moment, of $\zeta(s)$. A research subsequent to [8] was due to Ivić [7], while Laurinčikas [23] made a detailed study of the case $k = 1$, i.e. the mean square case, of $\mathcal{M}_{k,\rho}(s)$.

As for the relevant asymptotic results on allied zeta-functions, the Laplace transform

$$\mathcal{L}_{1/2}(\varepsilon, \chi) = \int_0^\infty \left|L\Bigl(\frac{1}{2}+it, \chi\Bigr)\right|^2 e^{-\varepsilon t}dt,$$

where $L(s, \chi)$ denotes the L-function attached to a primitive Dirichlet character χ modulo $q \geq 2$, was treated in the same paper by Köber [22], who established, for any integer

$N \geq 1$,

$$\mathcal{L}_{1/2}(\varepsilon,\chi) = \frac{\varphi(q)}{q}\cdot\frac{1}{\varepsilon}\log\frac{1}{\varepsilon} + \frac{\varphi(q)}{q}\biggl(\gamma_0 + \log\frac{q}{2\pi} + 2\sum_{p|q}\frac{\log p}{p-1}\biggr)\frac{1}{\varepsilon}$$
$$+ a_{0,\chi} + \sum_{n=1}^{N-1}\varepsilon^n\biggl(a_{n,\chi} + b_{n,\chi}\log\frac{1}{\varepsilon} + c_{n,\chi}\log^2\frac{1}{\varepsilon}\biggr) + O\biggl(\varepsilon^N\log^2\frac{1}{\varepsilon}\biggr)$$

as $\varepsilon \to 0^+$, with some constants $a_{0,\chi}$, $a_{n,\chi}$, $b_{n,\chi}$ and $c_{n,\chi}$ $(n = 1, 2, \ldots)$. Here $\varphi(q)$ denotes Euler's totient function, p runs through all prime divisors of q, and the implied O-constant may depend on q and N. Next let a, b, μ and ν be arbitrary real parameters, and $\psi_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z)$ denote the generalized Epstein zeta-function, defined for $\operatorname{Im} z > 0$ by

$$\psi_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) = \sideset{}{'}\sum_{m,n=-\infty}^{\infty}\frac{e((a+m)\mu + (b+n)\nu)}{|a+m+(b+n)z|^{2s}} \qquad (\operatorname{Re} s > 1), \tag{1.10}$$

and its meromorphic continuation over the whole s-plane, where the possibly emerging singular term 0^{-2s} is to be excluded; the particular case $(a, b) \in \mathbb{Z}^2$ and $(\mu, \nu) = (0, 0)$ reduces to the classical Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s; z)$. The author [19] has shown that complete asymptotic expansions exist for $\zeta_{\mathbb{Z}^2}(s; z)$ when $y = \operatorname{Im} z \to +\infty$, and also for the Laplace-Mellin transform $\mathcal{LM}^{\alpha}_{Y;y}\zeta_{\mathbb{Z}^2}(s; x + iy)$ when $Y \to +\infty$. The method developed in [19] could be extended to show in his subsequent paper [20] that similar expansions still exist for $\psi_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z)$ when $y \to +\infty$, as well as for the Riemann-Liouville transform $\mathcal{RL}^{\alpha,\beta}_{Y;y}\zeta_{\mathbb{Z}^2}(s; x + iy)$ when $Y \to +\infty$.

The article is organized as follows. Various complete asymptotic expansions for the transforms (1.6) and (1.7) are presented in the next section, and those for their iterated variants in Section 3, together with their applications. The details of the proofs of Theorems 1–10 and their corollaries will be given in the forthcoming paper [21].

2. Statement of results (1)

We first introduce the Hadamard type operators with the initial point at ∞, defined for any $(r, s) \in \mathbb{C}^2$ by

$$\mathcal{I}^r_{\infty,s}f(s) = \frac{1}{\Gamma(r)\{e(r) - 1\}}\int_{\infty}^{(0+)} f(s+z)z^{r-1}dz, \tag{2.1}$$

provided that the integral converges. Here the path of integration is a contour which starts from ∞, proceeds along the real axis to a small $\varepsilon > 0$, encircles the origin counterclockwise, and returns to ∞ along the real axis; $\arg z$ varies from 0 to 2π along the contour. Then the auxiliary zeta-function $\phi^*_r(s, a, \lambda)$ is defined, for any $(r, s) \in \mathbb{C}^2$ and for any $a, \lambda \in \mathbb{R}$ with $a > 1$, by

$$\phi^*_r(s, a, \lambda) = \mathcal{I}^r_{\infty,s}\phi^*(s, a, \lambda), \tag{2.2}$$

which is crucial in describing our assertions on (1.6), (1.7) and their iterated variants. Further, let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any $n \in \mathbb{Z}$ denote the rising factorial of s, and write

$$\Gamma\binom{\alpha_1, \ldots, \alpha_m}{\beta_1, \ldots, \beta_n} = \frac{\prod_{h=1}^m \Gamma(\alpha_h)}{\prod_{k=1}^n \Gamma(\beta_k)}$$

for complex numbers α_h and β_k $(h = 1, \ldots, m;\ k = 1, \ldots, n)$.

We now state our results on the Laplace-Mellin transform (1.6) with respect to s. The following Theorems 1 and 2 assert the asymptotic expansions as $z \to 0$ and as $z \to \infty$, respectively, for $\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau, a, \lambda)$, through the sector $|\arg z| < \pi$.

Theorem 1. *Let α and s be any complex numbers with $\operatorname{Re}\alpha > 0$, a and λ any real parameters with $a > 1$, and m any integer. Then for any integer $N \geq 0$ the formula*

$$\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau, a, \lambda) = (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n (\alpha)_n}{n!} \phi^*_{-n-m}(s, a, \lambda) z^n + R^{1,+}_{\alpha,m,N}(s, a, \lambda; z) \tag{2.3}$$

holds in the sector $|\arg z| < \pi$. Here $R^{1,+}_{\alpha,m,N}(s, a, \lambda; z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^{1,+}_{\alpha,m,N}(s, a, \lambda; z) = O\{(|\operatorname{Im} s| + 1)^{\max(0, \lfloor 2-\operatorname{Re} s \rfloor)} |z|^N\} \tag{2.4}$$

as $z \to 0$ through $|\arg z| \leq \pi - \eta$ with any small $\eta > 0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re} s$, a, λ, α, m, N and η.

Theorem 2. *Let s, a, λ, α and m be as in Theorem 1. Then for any integer $N \geq 0$ the formula*

$$\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau, a, \lambda) = (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n (\alpha)_n}{n!} \phi^*_{\alpha+n-m}(s, a, \lambda) z^{-\alpha-n} + R^{1,-}_{\alpha,m,N}(s, a, \lambda; z) \tag{2.5}$$

holds in the sector $|\arg z| < \pi$. Here $R^{1,-}_{\alpha,m,N}(s, a, \lambda; z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^{1,-}_{\alpha,m,N}(s, a, \lambda; z) = O\{(|\operatorname{Im} s| + 1)^{\max(0, \lfloor 2-\operatorname{Re} s \rfloor)} |z|^{-\operatorname{Re}\alpha - N}\} \tag{2.6}$$

as $z \to \infty$ through $|\arg z| \leq \pi - \eta$ with any small $\eta > 0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re} s$, a, λ, α, m, N and η.

We write $\operatorname{sgn}(X) = X/|X|$ for any real $X \neq 0$. Then the case $(s, z) = (\sigma, it)$ with $\sigma, t \in \mathbb{R}$ of Theorem 2 yields the following asymptotic expansion as $t \to \pm\infty$ through any vertical half-lines.

Corollary 2.1. *Let s, a, λ, α and m be as in Theorem 1, and σ any real number. Then for any integer $N \geq 0$ the asymptotic expansion*

$$\begin{aligned} &\mathcal{LM}^{\alpha}_{t;\tau}(\phi^*)^{(m)}(\sigma + i\tau, a, \lambda) \\ &\quad = (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n (\alpha)_n}{n!} \phi^*_{\alpha+n-m}(\sigma, a, \lambda) (e^{\operatorname{sgn}(t)\pi i/2}|t|)^{-\alpha-n} + O(|t|^{-\operatorname{Re}\alpha - N}) \end{aligned} \tag{2.7}$$

holds as $t \to \pm\infty$, where the constant implied in the O-symbol depends at most on σ, a, λ, α, m and N.

We proceed to state our results on the Riemann-Liouville transform (1.7) with respect to the variable s. The following Theorems 3 and 4 assert complete asymptotic expansions as $z \to 0$ and as $z \to \infty$, respectively, for $\mathcal{RL}^{\alpha,\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau, a, \lambda)$, through the sector $|\arg z| < \pi$.

Theorem 3. *Let α, β and s be any complex numbers with* $\operatorname{Re}\alpha > 0$ *and* $\operatorname{Re}\beta > 0$*, a and λ any real parameters with $a > 1$, and m any integer. Then for any integer $N \geq 0$ the formula*

$$\mathcal{RL}_{z;\tau}^{\alpha,\beta}(\phi^*)^{(m)}(s+\tau,a,\lambda) = (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{(\alpha+\beta)_n n!}\phi^*_{-n-m}(s,a,\lambda)z^n + R^{2,+}_{\alpha,\beta,m,N}(s,a,\lambda;z) \tag{2.8}$$

holds in the sector $|\arg z| < \pi$*. Here $R^{2,+}_{\alpha,\beta,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate*

$$R^{2,+}_{\alpha,\beta,m,N}(s,a,\lambda;z) = O\{(|\operatorname{Im} s|+1)^{\max(0,\lfloor 2-\operatorname{Re}s\rfloor)}|z|^N\} \tag{2.9}$$

as $z \to 0$ through $|\arg z| \leq \pi - \eta$ with any small $\eta > 0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re} s$*, a, λ, α, β, m, N and η.*

We write $\epsilon(z) = \operatorname{sgn}(\arg z)$ for any complex z in the sectors $0 < |\arg z| < \pi$.

Theorem 4. *Let α, β, s, a, λ and m be as in Theorem 3. The for any integers $N_j \geq 0$ $(j = 1, 2)$ the formula*

$$\begin{aligned}
&\mathcal{RL}_{z;\tau}^{\alpha,\beta}(\phi^*)^{(m)}(s+\tau,a,\lambda)\\
&\quad= (-1)^m \Gamma\binom{\alpha+\beta}{\beta} e^{-\epsilon(z)\pi i\alpha}\left\{\sum_{n=0}^{N_1-1}\frac{(-1)^n(\alpha)_n(1-\beta)_n}{n!}\right.\\
&\qquad\left.\times\,\phi^*_{\alpha+n-m}(s,a,\lambda)(e^{-\epsilon(z)\pi i}z)^{-\alpha-n} + R^{2,-}_{1,\alpha,\beta,m,N_1}(s,a,\lambda;z)\right\}\\
&\qquad+ (-1)^m \Gamma\binom{\alpha+\beta}{\alpha} e^{\epsilon(z)\pi i\beta}\left\{\sum_{n=0}^{N_2-1}\frac{(-1)^n(\beta)_n(1-\alpha)_n}{n!}\right.\\
&\qquad\left.\times\,\phi^*_{\beta+n-m}(s+z,a,\lambda)z^{-\beta-n} + R^{2,-}_{2,\alpha,\beta,m,N_2}(s,a,\lambda;z)\right\}
\end{aligned} \tag{2.10}$$

holds in the sectors $0 < |\arg z| < \pi$*. Here $R^{2,-}_{j,\alpha,\beta,m,N_j}(s,a,\lambda;z)$ $(j = 1, 2)$ are the remainder terms expressed by certain Mellin-Barnes type integrals, respectively, and satisfy the estimates*

$$R^{2,-}_{1,\alpha,\beta,m,N_1}(s,a,\lambda;z) = O\{(|\operatorname{Im} s|+1)^{\max(0,\lfloor 2-\operatorname{Re}s\rfloor)}|z|^{-\operatorname{Re}\alpha-N_1}\} \tag{2.11}$$

and

$$R^{2,-}_{2,\alpha,\beta,m,N_2}(s,a,\lambda;z) = O\{(|\operatorname{Im}(s+z)|+1)^{\max(0,\lfloor 2-\operatorname{Re}(s+z)\rfloor)}|z|^{-\operatorname{Re}\beta-N_2}\} \tag{2.12}$$

both as $z \to \infty$ through $\eta \leq |\arg z| \leq \pi - \eta$ with any small $\eta > 0$. Here the constant implied in the O-symbol in (2.11) *depends at most on* $\operatorname{Re} s$*, a, λ, α, β, m, N_1 and η, while that in* (2.12) *at most on* $\operatorname{Re} s$*,* $\operatorname{Re} z$*, a, λ, α, β, m, N_2 and η.*

The case $(s,z) = (\sigma, it)$ with $\sigma, t \in \mathbb{R}$ of Theorem 4 asserts the following asymptotic expansion as $t \to \pm\infty$ through any vertical half-lines.

Corollary 4.1. *Let α, β, a, λ be as in Theorem 2 and σ any real number. Then for any integers $N_j \geq 0$ $(j = 1, 2)$ the asymptotic expansion*

$$\begin{aligned}(2.13)\qquad & \mathcal{RL}^{\alpha,\beta}_{t;\tau}(\phi^*)^{(m)}(\sigma + i\tau, a, \lambda)\\ &= (-1)^m \Gamma\binom{\alpha+\beta}{\beta} e^{-\operatorname{sgn}(t)\pi i\alpha}\Biggl\{\sum_{n=0}^{N_1-1} \frac{(-1)^n(\alpha)_n(1-\beta)_n}{n!}\\ &\times \phi^*_{\alpha+n-m}(\sigma, a, \lambda)(e^{-\operatorname{sgn}(t)\pi i/2}|t|)^{-\alpha-n} + O(|t|^{-\operatorname{Re}\alpha - N_1})\Biggr\}\\ &+ (-1)^m \Gamma\binom{\alpha+\beta}{\alpha} e^{\operatorname{sgn}(t)\pi i\beta}\Biggl\{\sum_{n=0}^{N_2-1} \frac{(-1)^n((\beta)_n(1-\alpha)_n}{n!}\\ &\times \phi^*_{\beta+n-m}(\sigma + it, a, \lambda)(e^{\operatorname{sgn}(t)\pi i/2}|t|)^{-\beta-n} + O(|t|^{\max(0,\lfloor 2-\sigma\rfloor)-\operatorname{Re}\beta - N_2})\Biggr\}\end{aligned}$$

holds as $t \to \pm\infty$, where the constants implied in the O-symbols depend at most on σ, α, β, m and N_j $(j = 1, 2)$.

3. Statement of results (2)

In this section, we state our results on the iterated transforms. For this, let $K_\nu(Z)$ denote the modified Bessel function of the third kind, defined by

$$K_\nu(Z) = \frac{1}{2}\int_0^\infty \exp\Bigl\{-\frac{Z}{2}\Bigl(u + \frac{1}{u}\Bigr)\Bigr\} u^{-\nu-1}\,du$$

for any $\nu \in \mathbb{C}$ and for $|\arg Z| < \pi/2$ (cf. [5, p.82, 7.12(23)]), where the domain of Z is extended to $|\arg Z| < \pi$ by rotating appropriately the path of integration. We can then show the following expression.

Proposition 1. *For any $f(\tau)$ holomorphic in $|\arg \tau| < \pi$, we have*

$$(3.1)\qquad \mathcal{LM}^{\beta}_{z;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1} f(\tau_1) = \frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty f(z\tau)\tau^{(\alpha+\beta)/2-1}K_{\alpha-\beta}(2\sqrt{\tau})\,d\tau,$$

provided that the integral converges.

We proceed to state our results on the iteration of two Laplace-Mellin transforms. The following Theorems 5 and 6 assert complete asymptotic expansions as $z \to 0$ and $z \to \infty$, respectively, for $\mathcal{LM}^{\beta}_{z;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s + \tau_1, a, \lambda)$, through the sector $|\arg z| < \pi$.

Theorem 5. *Let α, β, λ, a and m be as in Theorem 2. Then for any integer $N \geq 0$, the formula*

$$\begin{aligned}(3.2)\qquad & \mathcal{LM}^{\beta}_{z;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s + \tau_1, a, \lambda)\\ &= (-1)^m \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n(\beta)_n}{n!}\phi^*_{-n-m}(s, a, \lambda) z^n + R^{3,+}_{\alpha,\beta,m,N}(s, a, \lambda; z)\end{aligned}$$

holds in the sector $|\arg z| < \pi$. Here $R^{3,+}_{\alpha,\beta,m,N}(s, a, \lambda; z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$(3.3)\qquad R^{3,+}_{\alpha,\beta,m,N}(s, a, \lambda; z) = O\{(|\operatorname{Im} s| + 1)^{\max(0,\lfloor 2-\operatorname{Re} s\rfloor)}|z|^N\}$$

as $z \to 0$ through $|\arg z| \le \pi - \eta$ with any small $\eta > 0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re} s$, *a, λ, α, β, m, N and η.*

Theorem 6. *Let α, β, λ, a and m be as in Theorem 2, and $\alpha - \beta \notin \mathbb{Z}$. Then for any integer $N \ge 0$, upon setting $N' = N - \lfloor \operatorname{Re}(\beta - \alpha) \rfloor$, the formula*

$$\begin{aligned}(3.4)\qquad & \mathcal{LM}^{\beta}_{z;\tau_2} \mathcal{LM}^{\alpha}_{\tau_2;\tau_1} (\phi^*)^{(m)}(s + \tau_1, a, \lambda) \\ &= (-1)^m \Gamma\binom{\beta - \alpha}{\beta} \sum_{n=0}^{N-1} \frac{(\alpha)_n}{(1 + \alpha - \beta)_n n!} \phi^*_{\alpha+n-m}(s, a, \lambda) z^{-\alpha-n} \\ &\quad + (-1)^m \Gamma\binom{\alpha - \beta}{\alpha} \sum_{n=0}^{N'-1} \frac{(\beta)_n}{(1 - \alpha + \beta)_n n!} \phi^*_{\beta+n-m}(s, a, \lambda) z^{-\beta-n} \\ &\quad + R^{3,-}_{\alpha,\beta,m,N}(s, a, \lambda; z)\end{aligned}$$

holds in the sector $|\arg z| < \pi$. Here $R^{3,-}_{\alpha,\beta,m,N}(s, a, \lambda; z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$(3.5)\qquad R^{3,-}_{\alpha,\beta,m,N}(s, a, \lambda; z) = O\{(|\operatorname{Im} s| + 1)^{\max(0, \lfloor 2 - \operatorname{Re} s \rfloor)} |z|^{-\operatorname{Re}\alpha - N}\}$$

as $z \to \infty$ through $|\arg z| \le \pi - \eta$ with any small $\eta > 0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re} s$, *a, λ, α, β, m, N and η.*

The case $(s, z) = (\sigma, it)$ with $\sigma, t \in \mathbb{R}$ of Theorem 6 yields the following asymptotic expansion as $t \to \pm\infty$ through any vertical half-lines.

Corollary 6.1. *Let σ be arbitrary, α, β, m, a, λ as in Theorem 6, and suppose that $\alpha - \beta \notin \mathbb{Z}$. Then for any integer $N \ge 0$, upon setting $N' = N - \lfloor \operatorname{Re}(\beta - \alpha) \rfloor$, the asymptotic expansion*

$$\begin{aligned}(3.6)\qquad & \mathcal{LM}^{\beta}_{t;\tau_2} \mathcal{LM}^{\alpha}_{\tau_2;\tau_1} (\phi^*)^{(m)}(\sigma + i\tau_1, a, \lambda) \\ &= (-1)^m \Gamma\binom{\beta - \alpha}{\beta} \sum_{n=0}^{N-1} \frac{(\alpha)_n}{(1 + \alpha - \beta)_n n!} \phi^*_{\alpha+n-m}(s, a, \lambda) (e^{\operatorname{sgn}(t)\pi i/2} |t|)^{-\alpha-n} \\ &\quad + (-1)^m \Gamma\binom{\alpha - \beta}{\alpha} \sum_{n=0}^{N'-1} \frac{(\beta)_n}{(1 - \alpha + \beta)_n n!} \phi^*_{\beta+n-m}(s, a, \lambda) (e^{\operatorname{sgn}(t)\pi i/2} |t|)^{-\beta-n} \\ &\quad + O(|t|^{-\operatorname{Re}\alpha - N})\end{aligned}$$

holds as $t \to \pm\infty$, where the constant implied in the O-symbol depends at most on σ, a, λ, α, β, m and N.

We next proceed to state our results on the iteration of the Laplace-Mellin and Riemann-Liouville transforms. For this, let $U(\lambda; \nu; Z)$ be Kummer's confluent hypergeometric function of the second kind, defined by

$$U(\lambda; \nu; Z) = \frac{1}{\Gamma(\lambda)\{e(\lambda) - 1\}} \int_{\infty}^{(0+)} e^{-Zw} w^{\lambda-1} (1 + w)^{\nu-\lambda-1} dw$$

for any $(\lambda, \nu) \in \mathbb{C}^2$ and for $|\arg Z| < \pi/2$ (cf. [4, p.237, 6.11.2(9)]), where the domain of Z is extended to $|\arg Z| < \pi$ by rotating appropriately the path of integration. We can then show the following expression.

Proposition 2. *For any $f(\tau)$ holomorphic in $|\arg\tau|<\pi$, we have*

$$\mathcal{RL}^{\beta,\gamma}_{z;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1}f(\tau_1)=\Gamma\binom{\beta+\gamma}{\alpha,\beta}\int_0^\infty f(z\tau)\tau^{\gamma-1}e^{-\tau}U(\gamma;\alpha-\beta+1;\tau)d\tau, \tag{3.7}$$

provided that the integral converges.

The following Theorems 7 and 8 assert complete asymptotic expansions as $z\to 0$ and $z\to\infty$, respectively, for $\mathcal{RL}^{\beta,\gamma}_{z;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s+\tau_1,a,\lambda)$, through the sector $|\arg z|<\pi$.

Theorem 7. *Let s, a, λ, α, β, and m be as in Theorem 2, and $\gamma\in\mathbb{C}$ arbitrary with* $\operatorname{Re}\gamma>0$. *Then for any integer $N\geq 0$, upon seting $N'=N-\lfloor\operatorname{Re}(\beta-\alpha)\rfloor$, the formula*

$$\begin{aligned}&\mathcal{RL}^{\beta,\gamma}_{z;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s+\tau_1,a,\lambda)\\&\quad=(-1)^m\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n(\beta)_n}{(\beta+\gamma)_n}\phi^*_{-n-m}(s,a,\lambda)z^n+R^{4,+}_{\alpha,\beta,\gamma,m,N}(s,a,\lambda;z)\end{aligned} \tag{3.8}$$

holds in the sector $|\arg z|<\pi$. Here $R^{4,+}_{\alpha,\beta,\gamma,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^{4,+}_{\alpha,\beta,\gamma,m,N}(s,a,\lambda;z)=O\{(|\operatorname{Im}s|+1)^{\max(0,\lfloor 2-\operatorname{Re}s\rfloor)}|z|^N\} \tag{3.9}$$

as $z\to 0$ through $|\arg z|\leq\pi-\eta$ with any small $\eta>0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re}s$, *a, λ, α, β, γ, m, N and η.*

Theorem 8. *Let s, a, λ, α, β, γ and m be as in Theorem 7, and suppose that $\alpha-\beta\notin\mathbb{Z}$. Then for any integer $N\geq 0$, upon setting $N'=N-\lfloor\operatorname{Re}(\beta-\alpha)\rfloor$, the formula*

$$\begin{aligned}&\mathcal{RL}^{\beta,\gamma}_{z;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s+\tau_1,a,\lambda)\\&\quad=(-1)^m\Gamma\binom{\beta+\gamma,\beta-\alpha}{\beta,\beta+\gamma-\alpha}\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n(1-\beta-\gamma+\alpha)_n}{(1-\beta+\alpha)_n n!}\\&\qquad\times\phi^*_{\alpha+n-m}(s,a,\lambda)z^{-\alpha-n}\\&\qquad+(-1)^m\Gamma\binom{\beta+\gamma,\alpha-\beta}{\alpha,\gamma}\sum_{n=0}^{N'-1}\frac{(-1)^n(\beta)_n(1-\gamma)_n}{(1-\alpha+\beta)_n n!}\\&\qquad\times\phi^*_{\beta+n-m}(s,a,\lambda)z^{-\beta-n}+R^{4,-}_{\alpha,\beta,\gamma,m,N}(s,a,\lambda;z)\end{aligned} \tag{3.10}$$

holds in the sector $|\arg z|<\pi$. Here $R^{4,-}_{\alpha,\beta,\gamma,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^{4,-}_{\alpha,\beta,\gamma,m,N}(s,a,\lambda;z)=O\{(|\operatorname{Im}s|+1)^{\max(0,\lfloor 2-\operatorname{Re}s\rfloor)}|z|^{-\operatorname{Re}\alpha-N}\} \tag{3.11}$$

as $z\to\infty$ through $|\arg z|\leq\pi-\eta$ with any small $\eta>0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re}s$, *a, λ, α, β, γ, m, N and η.*

The case $(s,z)=(\sigma,it)$ with $\sigma,t\in\mathbb{R}$ of Theorem 8 yields the following asymptotic expansion as $t\to\pm\infty$ through any vertical half-lines.

Corollary 8.1. *Let $\sigma\in\mathbb{R}$ be arbitrary, a, λ, α, β, γ and m as in Theorem 8. Then for any integer $N\geq 0$, upon setting $N'=N-\lfloor\mathrm{Re}(\beta-\alpha)\rfloor$, the asymptotic expansion*

$$
\begin{aligned}
(3.12)\qquad &\mathcal{RL}^{\beta,\gamma}_{t;\tau_2}\mathcal{LM}^{\alpha}_{\tau_2;\tau_1}(\phi^*)^{(m)}(\sigma+i\tau_1,a,\lambda)\\
&=(-1)^m\Gamma\binom{\beta+\gamma,\beta-\alpha}{\beta,\beta+\gamma-\alpha}\\
&\quad\times\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n(1-\beta-\gamma+\alpha)_n}{(1-\beta+\alpha)_n n!}\phi^*_{\alpha+n-m}(s,a,\lambda)(e^{\mathrm{sgn}(t)\pi i/2}|t|)^{-\alpha-n}\\
&\quad+(-1)^m\Gamma\binom{\beta+\gamma,\alpha-\beta}{\alpha,\gamma}\\
&\quad\times\sum_{n=0}^{N'-1}\frac{(-1)^n(\beta)_n(1-\gamma)_n}{(1-\alpha+\beta)_n n!}\phi^*_{\beta+n-m}(s,a,\lambda)(e^{\mathrm{sgn}(t)\pi i/2}|t|)^{-\beta-n}\\
&\quad+O(|t|^{-\mathrm{Re}\,\alpha-N})
\end{aligned}
$$

holds as $t\to\pm\infty$, where the constant implied in the O-symbol depends at most on σ, a, λ, α, β, γ, m and N.

We next proceed to state our results on the iteration of two Riemann-Liouville transforms. For this, let ${}_2F_1\left({\lambda,\mu\atop\nu};Z\right)$ be Gauß' hypergeometric function defined by

$$
{}_2F_1\left({\lambda,\mu\atop\nu};Z\right)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n n!}Z^n
$$

for any $(\lambda,\mu,\nu)\in\mathbb{C}^2\times(\mathbb{C}\setminus\{0,-1,-2,\ldots\})$ and for $|Z|<1$ (cf. [4, p.56, 2.1.1(2)]), where the domain of Z is extended to $|\arg(1-Z)|<\pi$ by Euler's integral formula for ${}_2F_1\left({\lambda,\mu\atop\nu};Z\right)$ (cf. [4, p.59, 2.1.3(10)]). We can then show the following expression.

Proposition 3. *For any $f(\tau)$ holomorphic in $|\arg\tau|<\pi/2$, we have*

$$
\begin{aligned}
(3.13)\qquad \mathcal{RL}^{\gamma,\delta}_{z;\tau_2}\mathcal{RL}^{\alpha,\beta}_{\tau_2;\tau_1}f(\tau_1)&=\Gamma\binom{\alpha+\beta,\gamma+\delta}{\alpha,\gamma,\beta+\delta}\int_0^1 f(z\tau)\tau^{\gamma-1}(1-\tau)^{\beta+\delta-1}\\
&\quad\times{}_2F_1\left({\gamma+\delta-\alpha,\beta\atop\beta+\delta};1-\tau\right)d\tau,
\end{aligned}
$$

provided that the integral converges.

The following Theorems 9 and 10 assert complete asymptotic expansions as $z\to 0$ and $z\to\infty$, respectively, for $\mathcal{RL}^{\gamma,\delta}_{z;\tau_2}\mathcal{RL}^{\alpha,\beta}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s+\tau_1,a,\lambda)$, through the sector $|\arg z|<\pi/2$.

Theorem 9. *Let s, a, λ, α, β, γ and m be as in Theorem 8, and $\delta\in\mathbb{C}$ arbitrary with $\mathrm{Re}\,\delta>0$. Then for any integer $N\geq 0$ the formula*

$$
\begin{aligned}
(3.14)\qquad &\mathcal{RL}^{\gamma,\delta}_{z;\tau_2}\mathcal{RL}^{\alpha,\beta}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s+\tau_1,a,\lambda)\\
&=(-1)^m\sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n(\gamma)_n}{(\alpha+\beta)_n(\gamma+\delta)_n n!}\phi^*_{-n-m}(s,a,\lambda)z^n+R^{5,+}_{\alpha,\beta,\gamma,\delta,m,N}(s,a,\lambda;z)
\end{aligned}
$$

holds in the sector $|\arg z|<\pi/2$. Here $R^{5,+}_{\alpha,\beta,\gamma,\delta,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$
(3.15)\qquad R^{5,+}_{\alpha,\beta,\gamma,\delta,m,N}(s,a,\lambda;z)=O\{(|\mathrm{Im}\,s|+1)^{\max(0,\lfloor 2-\mathrm{Re}\,s\rfloor)}|z|^N\}
$$

as $z \to 0$ through $|\arg z| \le \pi/2-\eta$ with any small $\eta>0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re} s$, *a, λ, α, β, γ, δ, m, N and η.*

Theorem 10. *Let s, a, λ, α, β, γ, δ and m be as in Theorem 8, and suppose that $\alpha-\gamma\notin\mathbb{Z}$. Then for any integer $N\ge 0$, upon setting $N'=N-\lfloor \operatorname{Re}(\beta-\alpha)\rfloor$, the formula*

$$\begin{aligned}(3.16)\qquad &\mathcal{RL}^{\gamma,\delta}_{z;\tau_2}\mathcal{RL}^{\alpha,\beta}_{\tau_2;\tau_1}(\phi^*)^{(m)}(s+\tau_1,a,\lambda)\\ &=(-1)^m\Gamma\binom{\alpha+\beta,\gamma-\alpha,\gamma+\delta}{\beta,\gamma,\gamma+\delta-\alpha}\\ &\quad\times\sum_{n=0}^{N-1}\frac{(\alpha)_n(1-\beta)_n(1+\alpha-\gamma-\delta)_n}{(1+\alpha-\gamma)_n n!}\phi^*_{\alpha+n-m}(s,a,\lambda)z^{-\alpha-n}\\ &\quad+(-1)^m\Gamma\binom{\alpha+\beta,\alpha-\gamma,\gamma+\delta}{\alpha,\alpha+\beta-\gamma,\gamma+\delta-\beta}\\ &\quad\times\sum_{n=0}^{N'-1}\frac{(\beta)_n(1-\alpha-\beta+\gamma)_n(1+\beta-\gamma-\delta)_n}{(1-\alpha+\gamma)_n n!}\phi^*_{\beta+n-m}(s,a,\lambda)z^{-\beta-n}\\ &\quad+R^{5,-}_{\alpha,\beta,\gamma,\delta,m,N}(s,a,\lambda;z)\end{aligned}$$

holds in the sector $|\arg z|<\pi/2$. Here $R^{5,-}_{\alpha,\beta,\gamma,\delta,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$(3.17)\qquad R^{5,-}_{\alpha,\beta,\gamma,\delta,m,N}(s,a,\lambda;z)=O\{(|\operatorname{Im} s|+1)^{\max(0,\lfloor 2-\operatorname{Re} s\rfloor)}|z|^{-\operatorname{Re}\alpha-N}\}$$

as $z\to\infty$ through $|\arg z|\le\pi/2-\eta$ with any small $\eta>0$, where the constant implied in the O-symbol depends at most on $\operatorname{Re} s$, *a, λ, α, β, γ, δ, m, N and η.*

Remark. Theorem 10, in fact, fails to imply the complete asymptotic expansion for $\mathcal{RL}^{\gamma,\delta}_{t;\tau_2}\mathcal{RL}^{\alpha,\beta}_{\tau_2;\tau_1}(\phi^*)^{(m)}(\sigma+i\tau_1,a,\lambda)$ as $t\to\pm\infty$, since any vertical half-lines are not included in the region, $|\arg z|<\pi/2$, of its validity. It is still an open problem to deduce complete asymptotic expansions for $\mathcal{RL}^{\gamma,\delta}_{t;\tau_2}\mathcal{RL}^{\alpha,\beta}_{\tau_2;\tau_1}(\phi^*)^{(m)}(\sigma+i\tau_1,a,\lambda)$ through any vertical half-lines $s=\sigma+it$ as $t\to\pm\infty$. One plausible clue to the solution seems to introduce a further class of auxiliary zeta-functions, such as $\mathcal{RL}^{\beta,\gamma}_{z;\tau}\phi^*_r(s+i\tau,a,\lambda)$, and to investigate their analytic properties.

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