Transcendence of zeros of certain weakly holomorphic modular forms

九州大学大学院数理学府 花元 誠一
Seiichi Hanamoto
Graduate School of Mathematics,
Kyushu University

1 Introduction

The Eisenstein series $E_k(z)$ is perhaps the easiest example of modular forms. In 1970, F.K.C. Rankin and Swinnerton-Dyer [10] showed that all zeros of the Eisenstein series $E_k(z)$ in the standard fundamental domain for $SL_2(\mathbb{Z})$ lie on the arc for all weight $k \geq 4$. Simlar results for other modular forms were proved [9, 11, 12].

In 2008, Duke and Jenkins [2] constructed a canonical basis $f_{k,m}$ for the space of weakly holomorphic modular forms for level 1. Let Δ be the Ramanujan Δ function and j be weight 0 modular function which is known simply as the j-function. The basis $f_{k,m}$ is defined by

$$f_{k,m} = \Delta^{\ell} E_{k'} F_{k,D}(j)$$

where $k = 12\ell + k'$, $k' \in \{0, 4, 6, 8, 10, 14\}$ and $F_{k,D}(x)$ is a monic polynomial in x of degree $D = \ell + m$. The basis $f_{k,m}$ have the following form

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}).$$

They considered $f_{k,m}$ as a two-parameter family of weakly holomorphic modular forms that is a canonical basis for the space and proved almost all of the basis elements have all of their zeros on a lower boundary of the standard fundamental domain for $SL_2(\mathbb{Z})$. We consider the locations of the zeros for weakly holomorphic modular forms $g_{k,m}$ of level 1 defined by

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j,$$

where $a_j \in \mathbb{R}$.

2 Definitions and statements of results

Let $k \in 2\mathbb{Z}$, N be a prime number or 1, and $\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) | c \equiv 0 \pmod{N} \right\}$. Put $\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } y > 0\}$, and $q = e^{2\pi i z}$ for $z \in \mathbb{H}$.

A holomorphic function f on $\mathbb{H}$ is a weakly holomorphic modular form of weight k with respect to $\Gamma_0(N)$ if f satisfies the following two conditions:

- $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N)$.
- $f(z) = \displaystyle\sum_{n \geq n_0} a(n) q^n$ and $\dfrac{1}{z^k} f\left(-\dfrac{1}{z}\right) = \displaystyle\sum_{n \geq n_1} b(n) q^{\frac{n}{N}}$

  with $a(n_0) \neq 0$ and $b(n_1) \neq 0$.

We define f is holomorphic if $n_0 \geq 0$ and $n_1 \geq 0$, a cusp form if $n_0 \geq 1$ and $n_1 \geq 1$. We denote the space of holomorphic modular form of weight k on $\Gamma_0(N)$ by $M_k(N)$, the space of weakly holomorphic modular forms by $M_k^!(N)$. Put $M_k = M_k(1)$ and $M_k^! = M_k^!(1)$ in this paper.

Duke and Jenkins considered an explicit basis of $M_k^!$ which is indexed by the order of the pole at ∞ in [2]. Let $k = 12\ell + k'$ where $\ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. For any integer $m \geq -\ell$, there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^!$ which has an expansion

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1}). \tag{1}$$

We note that there exists a unique $f_{k,m} \in M_k^!$ with the expansion (1). For any $f = \sum a(n) q^n \in M_k^!$, we can write

$$f = \sum_{n_0 \leq n \leq \ell} a(n) f_{k,-n}$$

when we know first few Fourier coefficients of f. Therefore we see that $\{f_{k,m}\}_{m \geq -\ell}$ form a natural basis of $M_k^!$.

We define three modular forms to construct the basis $\{f_{k,m}\}_{m \geq -\ell}$. Bernoulli numbers B_k and $\sigma_{k-1}(n)$ are each defined by

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!}, \qquad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Then the Ramanujan Δ function, Eisenstein series E_k and j function are each defined by

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n,$$

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \quad (k \geq 4),\ E_0 = 1,$$

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + \sum_{n\geq 1}c(n)q^n.$$

Their weights are each 12, k, 0 and orders at ∞ are each 1, 0, -1. The function $f_{k,m}$ is constructed by

$$f_{k,m} = \Delta^{\ell}E_{k'}F_{k,D}(j)$$

where $F_{k,D}(x)$ is a monic polynomial in x of degree $D = \ell + m$.

For the group $SL_2(\mathbb{Z})$, we use a fundamental domain in the upper half-plane bounded by the lines $\Re(z) = -\frac{1}{2}$ and $\Re(z) = \frac{1}{2}$, the circles of radius 1 centered at $z = 0$. We include the boundary on the left half of this fundamental domain. The cusps of this fundamental domain can be taken to be at ∞.

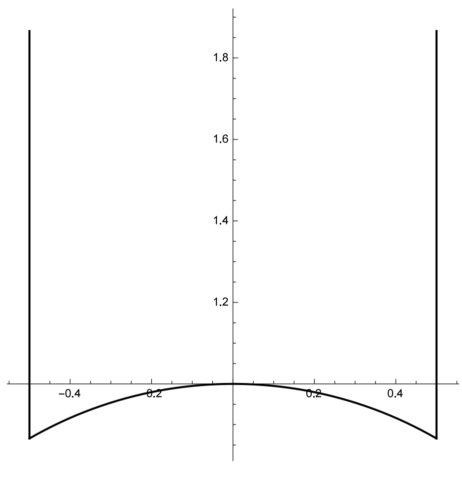

Figure 1: A fundamental domain for $SL_2(\mathbb{Z})$.

The description of the zeros of a weakly holomorphic modular form $f \in M_k^!$ on $\mathbb{H}$ is clearly equivalent to the description of the zeros of f on $\mathcal{F}$. Thus, for the remainder of this paper, when we speak of a zero z_0 of $f \in M_k^!$, we assume $z_0 \in \mathcal{F}$.

We define four constants by $\delta_1 = 0.432207, \delta_2 = 0.024975, \delta_3 = 0.004807$ and $\delta_4 = 0.257348$. Then we define $\gamma(j)$ and $A_{k'}$ by

$$\gamma(j) = \begin{cases} \delta_3^j \delta_1^{\ell - j} & \text{if } 1 \leq j \leq \ell, \\ \delta_2^j \delta_3^\ell & \text{if } \ell + 1 \leq j \leq \ell + m. \end{cases} \quad A_{k'} = \begin{cases} 2.76009 & \text{if } k' = 0, \\ 0.684214 & \text{if } k' = 4, \\ 0.950549 & \text{if } k' = 6, \\ 0.184724 & \text{if } k' = 8, \\ 0.258108 & \text{if } k' = 10, \\ 0.075404 & \text{if } k' = 14. \end{cases}$$

We note here

$$\begin{aligned} \left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x + 0.65i\right)} \right| &\leq \delta_1, \\ \left| \Delta\left(x + 0.65i\right) \right| &\leq \delta_2, \\ \left| \Delta\left(e^{i\theta}\right) \right| &\leq \delta_3, \\ e^{-2\pi m(\sin\theta - 0.65)} &\leq \delta_4 \end{aligned}$$

$$\text{and} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{1}{\Delta\left(x + 0.65i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x + 0.65i\right)}{j\left(x + 0.65i\right) - j\left(e^{i\theta}\right)} \right| dx \leq A_{k'},$$

for $\theta \in [1.9, 2\pi/3]$ and $x \in [-1/2, 1/2]$. Then we have the following theorem.

Theorem 2.1. *Let $k = 12\ell + k'$, where $\ell \in \mathbb{Z}_{\geq 0}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. Let*

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z) \Delta(z)^j,$$

where $a_j \in \mathbb{R}, m \geq 0$ and $\ell + m \geq 1$. If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j| (3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'},$$

then all of the zeros of $g_{k,m}$ in the fundamental domain for $SL_2(\mathbb{Z})$ lie on the circle $|z| = 1$.

Besides, we consider transcendence of zeros of $g_{k,m}$. We have the following theorem.

Theorem 2.2. *Let $k = 12\ell + k'$, where $\ell \in \mathbb{Z}_{\geq 0}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. Let*

$$g_{k,m}(z) = f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j,$$

where $a_j \in \mathbb{Q}, m \geq 0$ and $\ell + m \geq 1$. If $\{a_j\}_{j=1}^{\ell+m}$ satisfy

$$\sum_{j=1}^{\ell+m} |a_j|(3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'},$$

then all of zeros of $g_{k,m}$ in the fundamental domain for $SL_2(\mathbb{Z})$ are transcendental or equal to i or $\rho = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$.

3 Sketch of proof of Theorem 2.1

Applying the valence formula for $k = 12\ell + k'$, there are at most $\ell + m$ zeros on $\mathcal{F} - \{\rho, i\}$. Thus if $g_{k,m} \in M_k^!$ satisfies the hypotheses of Theorem 2.1, then to prove Theorem 2.1 it suffices to demonstrate that $g_{k,m}$ has $\ell + m$ simple zeros in $\{e^{i\theta} : \frac{\pi}{2} < \theta < \frac{2\pi}{3}\}$.

An easy argument [4, Proposition 2.1] shows that for any weakly holomorphic modular form f of weight k with real coefficients, the quantity $e^{ik\theta/2} f(e^{i\theta})$ is real for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$. Thus, we approximate $e^{ik\theta/2} g_{k,m}(e^{i\theta})$ by an elementary function having the required number of zeros on the arc.

Suppose $\ell \geq 1$ and $m \geq 1$. Then we set

$$H(\theta) = e^{ik\theta/2} e^{-2\pi m \sin\theta} g_{k,m}\left(e^{i\theta}\right) = H_{0,m}(\theta) + \sum_{j=1}^{\ell+m} a_j e^{12ji\theta/2} \Delta\left(e^{i\theta}\right)^j H_{j,m}(\theta),$$

where $H_{j,m}(\theta) = e^{(k-12j)i\theta/2} e^{-2\pi m \sin\theta} f_{k-12j,m}\left(e^{i\theta}\right)$. We define the function $R_{j,m}(\theta)$ for $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$ by

$$H_{j,m}(\theta) = 2\cos\left(\frac{(k-12j)\theta}{2} - 2\pi m \cos\theta\right) + R_{j,m}(\theta).$$

We seek a bound for the function $R_{j,m}(\theta)$. Details for the computation of the numerical bounds is given as with [3, 5]. By the argument in [2],

$$|R_{j,m}(\theta)| = \left| e^{-2\pi m \sin\theta} \int_{-\frac{1}{2}+\alpha'}^{\frac{1}{2}+\alpha'} \frac{\Delta^{\ell-j}(z)}{\Delta^{1+\ell-j}(\tau)} \frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)} e^{-2\pi i m\tau} d\tau \right|.$$

When $1.9 \leq \theta \leq 2\pi/3$, we have

$$|R_{j,m}(\theta)| \leq \frac{e^{-\pi m(2\sin\theta - \tan(\theta/2))}}{(2\cos(\theta/2))^k} + e^{-2\pi m(\sin\theta - 0.65)} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| G_j(x+0.65i, e^{i\theta}) \right| dx,$$

where

$$G_j(\tau, z) = \frac{\Delta^{\ell-j}(z)}{\Delta^{1+\ell-j}(\tau)} \frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)}.$$

Looking at the first term, for $\theta \in [1.9, 2\pi/3]$ and $m \geq 0$, we have

$$\left| \frac{e^{-\pi m(2\sin\theta - \tan(\theta/2))}}{(2\cos(\theta/2))^k} \right| \leq 1.$$

Considering the exponential term $e^{-2\pi m(\sin\theta - 0.65)}$, it is bounded above by 0.257348 for $\theta \in [1.9, 2\pi/3]$. We set $\delta_4 = 0.257348$.

We next seek a bound for $\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| G_j(x+0.65i, e^{i\theta}) \right| dx$. This integral is equal to

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)} \right|^{\ell-j} \left| \frac{1}{\Delta\left(x+0.65i\right)} \right| \left| \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right) - j\left(e^{i\theta}\right)} \right| dx.$$

First, we consider

$$\left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)} \right|^{\ell-j}.$$

We have

$$0.002691 \leq \left| \Delta\left(e^{i\theta}\right) \right| \leq 0.004807.$$

We set $\delta_3 = 0.004807$. We compute that

$$0.011122 \leq \left| \Delta\left(x+0.65i\right) \right| \leq 0.024975.$$

We set $\delta_2 = 0.024975$. Putting this together, we have, for $\ell \geq j$,

$$\left| \frac{\Delta\left(e^{i\theta}\right)}{\Delta\left(x+0.65i\right)} \right|^{\ell-j} \leq \left|0.432207\right|^{\ell-j}.$$

We set $\delta_1 = 0.432207$.

Next, we consider

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{1}{\Delta\left(x+0.65i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right) - j\left(e^{i\theta}\right)} \right| dx.$$

We will break our path of integration into small pieces, and consider $j(\tau)$ in relation to $j(z)$ on each. We can bound the quotient by

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{1}{\Delta\left(x+0.65i\right)} \frac{E_{k'}\left(e^{i\theta}\right) E_{14-k'}\left(x+0.65i\right)}{j\left(x+0.65i\right) - j\left(e^{i\theta}\right)} \right| dx \leq A_{k'},$$

where

$$A_{k'} = \begin{cases} 2.76009 & \text{if } k' = 0, \\ 0.684214 & \text{if } k' = 4, \\ 0.950549 & \text{if } k' = 6, \\ 0.184724 & \text{if } k' = 8, \\ 0.258108 & \text{if } k' = 10, \\ 0.075404 & \text{if } k' = 14. \end{cases}$$

Putting all of these pieces together, we see that

$$|R_{j,m}(\theta)| \leq 1 + \delta_4^m \delta_1^{\ell-j} A_{k'}$$

for $1 \leq j \leq \ell$ and

$$|R_{j+\ell,m}(\theta)| \leq 1 + \delta_4^m \left| \frac{\delta_2}{\Delta\left(e^{i\theta}\right)} \right|^j A_{k'}$$

for $1 \leq j \leq m$.

Similarly, for $\theta \in [\pi/2, 1.9)$, we can bound $|R_{j,m}(\theta)|$. We note that the bound of $|R_{j+\ell,m}(\theta)|$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$ is larger than for $\frac{\pi}{2} \leq \theta < 1.9$ since $|R_{j+\ell,m}(\theta)| > 1$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$. Therefore we also use the bound of $|R_{j+\ell,m}(\theta)|$ for $1.9 \leq \theta \leq \frac{2\pi}{3}$ when $\frac{\pi}{2} \leq \theta < 1.9$.

$\left|H(\theta) - 2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)\right|$ is bounded above by

$$
\begin{aligned}
& |R_{0,m}(\theta)| + \sum_{j=1}^{\ell} |a_j| \left(2 + |R_{j,m}(\theta)|\right) \left|\Delta\left(e^{i\theta}\right)\right|^j \\
& \qquad\qquad + \sum_{j=1}^{m} |a_{j+l}| \left(2 + |R_{j+\ell,m}(\theta)|\right) \left|\Delta\left(e^{i\theta}\right)\right|^{j+\ell} \\
\leq\ & 1 + \delta_4^m \delta_1^\ell A_{k'} + \sum_{j=1}^{\ell+m} |a_j| \left(3\delta_3^j + \delta_4^m \gamma^j(j) A_{k'}\right).
\end{aligned}
$$

Now suppose

$$\sum_{j=1}^{\ell+m} |a_j|(3\delta_3^j + \delta_4^m \gamma(j)^j A_{k'}) < 1 - \delta_4^m \delta_1^\ell A_{k'}.$$

Then we have

$$\left|H(\theta) - 2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)\right| < 2.$$

This inequality is enough to prove the theorem. To see this, note that as θ increases from $\pi/2$ to $2\pi/3$, the quantity

$$\frac{k\theta}{2} - 2\pi m \cos\theta$$

increases from $\pi\left(3\ell + k'/4\right)$ to $\pi\left(3\ell + k'/3 + D\right)$, where $D = \ell + m$, hitting $D + 1$ distinct consecutive integer multiples of π (this is independent of the choice of k'). A short computation shows that if $D \geq |\ell|$, then the quantity $\frac{k\theta}{2} - 2\pi m \cos\theta$ is strictly increasing on this interval. Thus, there are exactly $D + 1$ values of θ in the interval $[\pi/2, 2\pi/3]$ where the function

$$2\cos\left(\frac{k\theta}{2} - 2\pi m \cos\theta\right)$$

has absolute value 2, alternating between $+2$ and -2 as θ increases. Then real-valued function $H(\theta)$ must have at least D distinct zeros as θ moves through the interval $(\pi/2, 2\pi/3)$. This accounts for all D nontrivial zeros of $g_{k,m}$.

4 Proof of Theorem 2.2

For the proof of Theorem 2.2, we use the following lemma of Schneider.

Lemma 4.1. *[8, Corollary 3.4] If $z \in \mathbb{H}$ and $j(z)$ is algebraic, then either z is transcendental or z is imaginary quadratic, i.e. $\mathbb{Q}(z)$ is a degree 2 extension of $\mathbb{Q}$, with $z \notin \mathbb{R}$.*

We can prove Theorem 2.2 as with [6]. We have the following lemma.

Lemma 4.2. *[6, Lemma 2.2] Let $a, b, c \in \mathbb{Z}$ such that $a > 0$, $\gcd(a, b, c) = 1$, and $D = b^2 - 4ac < 0$. If $z \in \mathbb{H}$ is a root of the polynomial $ax^2 + bx + c$, then the lattice $[1, z]$ is a proper fractional ideal of the order $\mathfrak{O} = [1, az]$ of $K = \mathbb{Q}(\sqrt{D})$. Moreover,*

$$\mathfrak{O} = \begin{cases} \frac{i\sqrt{D}}{2} & \text{if } D \equiv 0 \pmod 4, \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

We find that the order $\mathfrak{O}$ does not depend on z, but instead on the discriminant D of the reduced integer polynomial that has z as a root. Recall, if Λ is a lattice of C we define $j(\Lambda) = j(z)$, where $z \in \mathbb{H}$ and $\Lambda = [1, z]$. The choice of $z \in \mathbb{H}$ is well defined. By Lemma 4.2, we see that we can map a point $z \in \mathbb{H}$ to the proper fractional ideal $\Lambda = [1, z]$ of $\mathfrak{O}$, where $j([1, z]) = j(z)$.

The following lemma follows from [1, Theorem 11.1 and Proposition 13.2], and is the last result we need before the proof of Theorem 2.2.

Lemma 4.3. *[6, Lemma 2.3] If $\mathfrak{A}$ is a proper fractional ideal of an order $\mathfrak{O}$ of an imaginary quadratic field K, then $j(\mathfrak{A})$ is an algebraic over $\mathbb{Q}$. If $\mathfrak{B}$ is any other proper fractional ideal of $\mathfrak{O}$, then $K(j(\mathfrak{A})) = K(j(\mathfrak{B}))$ and $j(\mathfrak{A})$ and $j(\mathfrak{B})$ are conjugate over K. Furthermore, the degree of $j(\mathfrak{A})$ is the class number of $\mathfrak{O}$.*

Let $g_{k,m}(z)$ satisfy the assumption of Theorem 2.2. Then we can write

$$\begin{aligned} g_{k,m}(z) &= f_{k,m}(z) + \sum_{j=1}^{\ell+m} a_j f_{k-12j,m}(z)\Delta(z)^j \\ &= \Delta(z)^\ell E_{k'}(z) F_{k,L}(j(z)), \end{aligned}$$

where $F_{k,L}(j(z))$ is a monic polynomial in $j(z)$ of degree $L = \ell + m$ with rational number coefficients. By Kohnen [7], the only possible zeros of $E_{k'}(z)$

are i and ρ. Also, we see from the valence formula that $\Delta(z)$ is never zero on $\mathbb{H}$. Thus, the only zeros of $g_{k,m}(z)$ in $\mathcal{F}$ other than i, ρ are the zeros of $F_{k,L}(j(z))$.

Suppose $z_0 \in \mathcal{F}$ such that $F_{k,L}(j(z_0)) = 0$. Since $F_{k,L}(x)$ is a polynomial with rational number coefficients, $j(z_0)$ is algebraic. Thus from Lemma 4.1, z_0 is either transcendental or imaginary quadratic.

If z_0 is imaginary quadratic, then z_0 is a root of a polynomial $P(x) = ax^2 + bx + c$, where $\gcd(a, b, c) = 1, a > 0$, and the discriminant $D_0 = b^2 - 4ac < 0$. Let $K = \mathbb{Q}(\sqrt{D_0})$.

We consider the order $\mathfrak{O} = [1, az_0]$ of K. From Lemma 4.2, the lattice $[1, z_0]$ is a proper fractional ideal of $\mathfrak{O}$, and the order $\mathfrak{O}$ has the form

$$\mathfrak{O} = \begin{cases} \left[1, \frac{i\sqrt{D_0}}{2}\right] & \text{if } D_0 \equiv 0 \pmod 4, \\ \left[1, \frac{1+\sqrt{D_0}}{2}\right] & \text{if } D_0 \equiv 1 \pmod 4. \end{cases}$$

Thus by Lemma 4.3, if $\mathfrak{A}$ is any other proper fractional ideal of $\mathfrak{O}$, $j(z_0) = j([1, z_0])$ and $j(\mathfrak{A})$ are conjugate.

We consider the point $z_1 \in \mathbb{C}$ defined by

$$z_1 = \begin{cases} \frac{i\sqrt{|D_0|}}{2} & \text{if } D_0 \equiv 0 \pmod 4, \\ \frac{1+i\sqrt{|D_0|}}{2} & \text{if } D_0 \equiv 1 \pmod 4. \end{cases}$$

Then $z_1 \in \mathcal{F}$ and we have $[1, z_1] = \mathfrak{O}$. Thus by definition $[1, z_1]$ is a proper fractional ideal of $\mathfrak{O}$, and so $j(z_0)$ and $j(z_1)$ are conjugate.

We take an automorphism σ of $K(j(\mathfrak{O}))$ such that $\sigma(j(z_0)) = j(z_1)$. Since σ acts as the identity on $\mathbb{Q}$ and $F_{k,L}$ is a polynomial with rational number coefficients, we have that

$$\begin{aligned} 0 &= \sigma(0) \\ &= \sigma(F_{k,L}(j(z_0))) \\ &= F_{k,L}(\sigma(j(z_0))) \\ &= F_{k,L}(j(z_1)). \end{aligned}$$

Thus z_1 is also a zero of $F_{k,L}$ and hence a zero of $g_{k,m}$. Since $z_1 \in \mathcal{F}$, by Theorem 2.1 we have that z_1 must lie on the arc of the unit circle given by

$$\left\{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3} \right\}.$$

Suppose $D_0 \equiv 0 \pmod 4$, so that $D_0 = -4n$ for some positive integer n. Then $z_1 = i\sqrt{n}$, but since z_1 must lie on the unit circle we must have $n = 1$. Thus, $D_0 = -4$. Since $z_0 \in \mathbb{H}$, we have by the quadratic formula that

$$z_0 = \frac{-b + 2i}{2a}.$$

But $z_0 \in \mathcal{F}$, and so $\Im(z_0) \geq \frac{\sqrt{3}}{2}$. Thus $a = 1$, and so

$$z_0 = -\frac{b}{2} + i.$$

But again by Theorem 2.1 we have that z_0 must lie on the unit circle, so $b = 0$ and $z_0 = i$.

.If $D_0 \equiv 1 \pmod 4$, then $D_0 = -4n + 1$ for some positive integer n. Hence,

$$z_1 = \frac{-1 + i\sqrt{4n - 1}}{2},$$

and thus $|z_1|^2 = n$. Again, since z_1 must lie on the unit circle we must have $n = 1$. Therefore $D_0 = -3$. Since $z_0 \in \mathbb{H}$, we have that

$$z_0 = \frac{-b + i\sqrt{3}}{2a}$$

by the quadratic formula. And again since $z_0 \in \mathcal{F}$, we have $a = 1$ so that

$$z_0 = -\frac{b}{2} + i\frac{\sqrt{3}}{2}.$$

But again by Theorem 2.1 we have that z_0 must lie on the unit circle, so $b = 1$ and $z_0 = \rho$. Thus, we completed Theorem 2.2.

Acknowledgement The author is grateful to the conference organizers for giving him an opportunity of talk. He also would like to thank Professor Yasuro Gon for initial advice and many useful comments over the course of this work.

References

[1] D. Cox, Primes of the form $x^2 + ny^2$, *Fermat, class field theory and complex multiplication. A Wiley-Interscience Publication* (1989).

[2] W. Duke and P. Jenkins, On the zeros and coefficients of certain weakly holomorphic modular forms, *Pure Appl. Math. Q.* **4**(4) (2008) 1327–1340.

[3] S. Garthwaite and P. Jenkins, Zeros of weakly holomorphic modular forms of levels 2 and 3, *Math. Res. Lett.* **20**(4) (2013) 657–674

[4] J. Getz, A generalization of a theorem of Rankin and Swinnerton-Dyer on zeros of modular forms, *Proc. Amer. Math. Soc.* **132**(8) (2004) 2221–2231

[5] A. Haddock and P. Jenkins, Zeros of weakly holomorphic modular forms of levels 4, *Int. J. Number Theory* **10**(2) (2014) 455–470

[6] C. Jennings-Shaffer and H. Swisher, A note on the transcendence of zeros of a certain family of weakly holomorphic forms, *Int. J. Number Theory* **10**(2) (2014) 309–317

[7] W. Kohnen, Transcendence of zeros of Eisenstein series and other modular functions, *Comment. Math. Univ. St. Pauli* **52**(1) (2003) 55–57

[8] S. Lang, Introduction to Transcendental Numbers , *Addison-Wesley, Reading, MA,* (1966)

[9] T. Miezaki, H. Nozaki, and J. Shigezumi, On the zeros of Eisenstein series for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$, *J. Math. Soc. Japan* **59**(3) (2007) 693–706

[10] F. K. C. Rankin and H. P. F. Swinnerton-Dyer, On the zeros of Eisenstein series, *Bull. London Math. Soc.* **2** (1970) 169–170

[11] J. Shigezumi, On the zeros of Eisenstein series for $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$, *Kyushu J. Math.* **61**(2) (2007) 527–549

[12] J. Shigezumi, On the zeros of Poincaré series for $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$, *Osaka J. Math.* **47**(2) (2010) 487–505