

Local Saito-Kurokawa A -packets and ℓ -adic cohomology of Rapoport-Zink tower for $\mathrm{GSp}(4)$: announcement

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1 Introduction

This is an announcement of a recent joint work of Tetsushi Ito and the author on the ℓ -adic cohomology of the Rapoport-Zink tower for GSp_4 . The Rapoport-Zink tower for GSp_4 is a p -adic local counterpart of the Siegel threefold. Its ℓ -adic cohomology H_{RZ}^i is naturally equipped with actions of three groups; the Weil group of \mathbb{Q}_p , $\mathrm{GSp}_4(\mathbb{Q}_p)$ and a non-trivial inner form $J(\mathbb{Q}_p)$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$. These actions are expected to be strongly related with the local Langlands correspondence, but they are not fully understood yet. In this work, we focus on a certain class of non-tempered local A -packets of $J(\mathbb{Q}_p)$, called the local Saito-Kurokawa A -packets. We determine how these A -packets and the associated L -packets contribute to the $\mathrm{GSp}_4(\mathbb{Q}_p)$ -supercuspidal part of H_{RZ}^i . See Theorem 3.1 for the precise statement.

The outline of this article is as follows. In Section 2, we give a brief review of the local Langlands correspondence. We also recall the Lubin-Tate tower, which is essential to prove the local Langlands correspondence for GL_n . In Section 3, we introduce the Rapoport-Zink tower for GSp_4 , which is a GSp_4 -version of the Lubin-Tate tower. After that, we state our main theorem and explain the ideas of the proof.

2 Local Langlands correspondence

Throughout this article, we fix a prime number p . In this section, we briefly recall the local Langlands correspondence. Let G be a connected reductive group over \mathbb{Q}_p . We assume that G is an inner form of a split group for simplicity. We write $\Pi(G)$ for the set of the isomorphism classes of irreducible smooth representations (over \mathbb{C}) of $G(\mathbb{Q}_p)$, and $\Phi(G)$ for the set of the \widehat{G} -conjugacy classes of L -parameters $W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \widehat{G}$. Here $W_{\mathbb{Q}_p}$ denotes the Weil group of \mathbb{Q}_p , and \widehat{G} denotes the dual group of G over \mathbb{C} . The local Langlands correspondence for G is a conjectural map $\mathrm{LLC}: \Pi(G) \rightarrow \Phi(G)$ with finite fibers. The fiber Π_ϕ^G of $\phi \in \Phi(G)$ is called the L -packet of ϕ . The map LLC is expected to be surjective when G is split.

If $G = \mathrm{GL}_n$, then \widehat{G} equals $\mathrm{GL}_n(\mathbb{C})$, and an L -parameter $W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \widehat{G}$ is identified with an n -dimensional semisimple representation of $W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$. The

local Langlands correspondence for GL_n has been proved by Harris-Taylor [HT01] (see also [Hen00] and [Sch13]). In this case, every L -packet is a singleton; in other words, the map $\mathrm{LLC}: \Pi(\mathrm{GL}_n) \rightarrow \Phi(\mathrm{GL}_n)$ is bijective. Let us briefly recall the construction of $\mathrm{LLC}(\pi)$ for a supercuspidal $\pi \in \Pi(\mathrm{GL}_n)$. It is given by using the Lubin-Tate tower $\{M_K\}_{K \subset \mathrm{GL}_n(\mathbb{Z}_p)}$, which is a projective system of rigid spaces over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ indexed by compact open subgroups of $\mathrm{GL}_n(\mathbb{Z}_p)$. Here are basic geometric properties of the Lubin-Tate tower:

- $M_{\mathrm{GL}_n(\mathbb{Z}_p)} = \coprod_{\mathbb{Z}} ((n-1)\text{-dimensional open unit disk over } \widehat{\mathbb{Q}}_p^{\mathrm{ur}})$.
- $M_K/M_{\mathrm{GL}_n(\mathbb{Z}_p)}$ is a finite étale covering. In particular, each M_K is an $(n-1)$ -dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. If K is an open normal subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$, then $M_K/M_{\mathrm{GL}_n(\mathbb{Z}_p)}$ is a Galois covering with Galois group $\mathrm{GL}_n(\mathbb{Z}_p)/K$.

The group $\mathrm{GL}_n(\mathbb{Q}_p)$ acts on the projective system $\{M_K\}_{K \subset \mathrm{GL}_n(\mathbb{Z}_p)}$; it is a local analogue of the Hecke action. The group D^\times also acts on the tower, where D is the central division algebra over \mathbb{Q}_p with invariant $1/n$. Now we fix a prime number ℓ and an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. We put $H_{\mathrm{LT}}^i = \varinjlim_K H_c^i(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$. It is equipped with an action of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$. Roughly speaking, the L -parameter $\mathrm{LLC}(\pi)$ for a supercuspidal $\pi \in \Pi(\mathrm{GL}_n)$ is constructed by using the irreducible decomposition of H_{LT}^{n-1} .

Theorem 2.1 ([Car86], [HT01], [Boy09]) *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. We put $\rho = \mathrm{JL}(\pi)$, where JL denotes the Jacquet-Langlands correspondence between $\mathrm{GL}_n(\mathbb{Q}_p)$ and D^\times . Then $\mathrm{LLC}(\pi)$ is a unique irreducible n -dimensional representation of $W_{\mathbb{Q}_p}$ (which is regarded as a representation of $W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$ by the first projection) satisfying the following:*

$$\mathrm{Hom}_{D^\times}(H_{\mathrm{LT}}^{n-1}, \rho)^{\mathrm{sm}} \cong \pi \boxtimes \mathrm{LLC}(\pi) \left(\frac{n-1}{2} \right).$$

Here $(-)^{\mathrm{sm}}$ denotes the smooth part with respect to the $\mathrm{GL}_n(\mathbb{Q}_p)$ -action, and $(\frac{n-1}{2})$ denotes the Tate twist.

Remark 2.2 If $i \neq n-1$, we have $\mathrm{Hom}_{D^\times}(H_{\mathrm{LT}}^i, \rho)^{\mathrm{sm}} = 0$. See [Boy09].

The key of the proof of Theorem 2.1 is to relate $\{M_K\}_{K \subset \mathrm{GL}_n(\mathbb{Z}_p)}$ to a certain Shimura variety. Let us explain it in the case $n=2$. In the following we write \mathbb{A} for the ring of adèles of \mathbb{Q} . For a compact open subgroup $K' \subset \mathrm{GL}_2(\mathbb{A}^\infty)$, let $\mathrm{Sh}_{K'}$ denote the modular curve over \mathbb{Q} with level K' . We write $\mathrm{Sh}_{K', \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{an}}$ for the rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$ associated with $\mathrm{Sh}_{K', \widehat{\mathbb{Q}}_p^{\mathrm{ur}}} = \mathrm{Sh}_{K'} \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. We fix a sufficiently small compact open subgroup K^p of $\mathrm{GL}_2(\mathbb{A}^{\infty, p})$. We write $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \widehat{\mathbb{Z}}_p^{\mathrm{ur}}}$ for the integral modular curve over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ with level $\mathrm{GL}_2(\mathbb{Z}_p)K^p$. The supersingular locus of its mod p fiber $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \overline{\mathbb{F}}_p}$ is denoted by $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \overline{\mathbb{F}}_p}^{\mathrm{ss}}$. We have the specialization map $\mathrm{sp}: \mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{an}} \rightarrow \mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \overline{\mathbb{F}}_p}$. Let $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}}$ be the rigid analytic open

subset of $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{an}}$ obtained as the inverse image of $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \overline{\mathbb{F}}_p}^{\mathrm{ss}}$ (strictly speaking, we are in fact working in the framework of adic spaces, so we need to take the interior of the inverse image). The open subset $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}}$ is called the supersingular reduction locus, since its classical point corresponds to an elliptic curve with good supersingular reduction. Finally, for a compact open subgroup K of $\mathrm{GL}_2(\mathbb{Z}_p)$, let $\mathrm{Sh}_{KK^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}}$ be the inverse image of $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}}$ in $\mathrm{Sh}_{KK^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{an}}$. Then the following holds:

Proposition 2.3 (p -adic uniformization) *We have an isomorphism*

$$\mathrm{Sh}_{KK^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}} \cong \widetilde{D}^\times \backslash (M_K \times \mathrm{GL}_2(\mathbb{A}^{\infty, p}) / K^p),$$

where \widetilde{D} is the quaternion division algebra over \mathbb{Q} which ramifies exactly at ∞ and p .

In this work, we use the local Langlands correspondence for $G = \mathrm{GSp}_4$ and its non-trivial inner form J . Both of the dual groups \widehat{G} and \widehat{J} are equal to $\mathrm{GSp}_4(\mathbb{C})$. The local Langlands correspondence for G and J are due to Gan-Takeda [GT11] and Gan-Tantono [GT14], respectively. Unlike the GL_n -case, no geometry is needed in the proofs of them. They used the local theta lifting to reduce the local Langlands correspondence for G and J to that for GL_2 and GL_4 . However, the author is still interested in how the local Langlands correspondence for these groups interacts with geometry.

Let $\phi: W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{GSp}_4(\mathbb{C})$ be an element of $\Phi(G) = \Phi(J)$. The corresponding L -packets Π_ϕ^G and Π_ϕ^J are not necessarily singletons. We are particularly interested in the case where Π_ϕ^G contains a supercuspidal representation. Such L -parameters are classified as follows:

Proposition 2.4 *Let $r: \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$ denote the natural embedding. If Π_ϕ^G contains a supercuspidal representation, then one of the following holds:*

- (i) *There exists a 4-dimensional irreducible representation ϕ_0 of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = \phi_0 \boxtimes \mathbf{1}$, where $\mathbf{1}$ denotes the trivial representation of $\mathrm{SL}_2(\mathbb{C})$. In this case, each of Π_ϕ^G and Π_ϕ^J consists of one supercuspidal representation.*
- (ii) *There exist distinct 2-dimensional irreducible representations ϕ_0 and ϕ_1 of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\phi_1 \boxtimes \mathbf{1})$. In this case, each of Π_ϕ^G and Π_ϕ^J consists of two supercuspidal representations.*
- (iii) *There exist a 2-dimensional irreducible representation ϕ_0 of $W_{\mathbb{Q}_p}$ and a character χ of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$, where \mathbf{Std} denotes the standard representation of $\mathrm{SL}_2(\mathbb{C})$. In this case, each of Π_ϕ^G and Π_ϕ^J consists of one supercuspidal representation and one non-supercuspidal discrete series representation.*
- (iv) *There exist distinct characters χ_0, χ_1 of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = (\chi_0 \boxtimes \mathbf{Std}) \oplus (\chi_1 \boxtimes \mathbf{Std})$. In this case, Π_ϕ^G consists of one supercuspidal representation and one non-supercuspidal discrete series representation, and Π_ϕ^J consists of two non-supercuspidal discrete series representations.*

In this article we focus on the case (iii). We write π_{sc} (resp. π_{disc}) for the supercuspidal (resp. non-supercuspidal) representation belonging to Π_{ϕ}^G . Similarly, we write ρ_{sc} (resp. ρ_{disc}) for the supercuspidal (resp. non-supercuspidal) representation belonging to Π_{ϕ}^J .

We also need to consider the A -parameter ψ obtained as the composite of

$$W_{\mathbb{Q}_p} \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \xrightarrow{\text{swap SL}_2 \text{ factors}} W_{\mathbb{Q}_p} \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \xrightarrow{\phi \boxtimes \mathbf{1}} \text{GSp}_4(\mathbb{C}).$$

Let Π_{ψ}^G (resp. Π_{ψ}^J) be the local A -packet attached to ψ . We should clarify what Π_{ψ}^G and Π_{ψ}^J mean, since local A -packets for J has not been fully constructed yet (see [GT19] for the construction of local A -packets for G). Recall that our ϕ satisfies $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$. This implies that $\det \phi_0 = \chi^2$. Therefore, the A -parameter $\psi' = \psi \otimes \chi^{-1}$ factors through $\text{Sp}_4(\mathbb{C}) \subset \text{GSp}_4(\mathbb{C})$. Since $\text{Sp}_4(\mathbb{C}) = \widehat{\text{SO}}_5$, ψ' can be regarded as an A -parameter for both $G^{\text{ad}} = \text{SO}_5(\mathbb{Q}_p)$ and J^{ad} . Local A -packets for $\text{SO}_5(\mathbb{Q}_p)$ was fully constructed by Arthur [Art13]. In particular we have the local A -packet $\Pi_{\psi'}^{\text{SO}_5}$, which can be regarded as a subset of $\Pi(G)$. We put $\Pi_{\psi}^G = \{\pi' \otimes (\chi \circ \text{sim}) \mid \pi' \in \Pi_{\psi'}^{\text{SO}_5}\}$, where $\text{sim}: G(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p^{\times}$ denotes the similitude character and χ is regarded as a character $\mathbb{Q}_p^{\times} \rightarrow \mathbb{C}^{\times}$ by the local class field theory $W_{\mathbb{Q}_p}^{\text{ab}} \cong \mathbb{Q}_p^{\times}$. As for J^{ad} , the local A -packet $\Pi_{\psi'}^{J^{\text{ad}}}$ for the particular A -parameter ψ' was constructed in [Gan08]. Therefore we get the local A -packet Π_{ψ}^J in the same way as above.

We call Π_{ψ}^G and Π_{ψ}^J the local Saito-Kurokawa A -packets. The structure of them are as follows:

- Π_{ψ}^G consists of π_{sc} and a non-tempered representation π_{nt} .
- Π_{ψ}^J consists of a supercuspidal representation ρ'_{sc} and a non-tempered representation ρ_{nt} . As a consequence of our main theorem, ρ'_{sc} turns out to be equal to ρ_{sc} (see Remark 3.2 (ii)).

3 Main Theorem

We continue to write G for GSp_4 and J for its unique non-trivial inner form over \mathbb{Q}_p . To state our main theorem, we introduce the (basic) Rapoport-Zink tower for GSp_4 , which is the GSp_4 -version of the Lubin-Tate tower. It is a projective system of rigid spaces over $\widehat{\mathbb{Q}}_p^{\text{ur}}$ indexed by compact open subgroups of $G(\mathbb{Z}_p)$. Here are basic geometric properties of the Rapoport-Zink tower for GSp_4 :

- $M_{G(\mathbb{Z}_p)}$ is a 3-dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\text{ur}}$ (unlike the Lubin-Tate case, we do not have an elementary expression of it).
- $M_K/M_{G(\mathbb{Z}_p)}$ is a finite étale covering. In particular, each M_K is a 3-dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\text{ur}}$. If K is an open normal subgroup of $G(\mathbb{Z}_p)$, then $M_K/M_{G(\mathbb{Z}_p)}$ is a Galois covering with Galois group $G(\mathbb{Z}_p)/K$.

As in the Lubin-Tate case, the tower $\{M_K\}_{K \subset G(\mathbb{Z}_p)}$ is equipped with an action of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$. We put $H_{\text{RZ}}^i = \varinjlim_K H_c^i(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\text{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_{\ell})$, which is a representation of

$G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$. For an irreducible smooth representation ρ of $J(\mathbb{Q}_p)$, we put $H_{\text{RZ}}^{i,j}[\rho] := (\text{Ext}_{J(\mathbb{Q}_p)}^j(H_{\text{RZ}}^i, \rho)^{\mathcal{D}_{c\text{-sm}}})_{\text{sc}}$, where $(-)\text{sc}$ denotes the $G(\mathbb{Q}_p)$ -supercuspidal part. For the definition of $(-)^{\mathcal{D}_{c\text{-sm}}}$, see [Mie14, Notation]. Note that $H_{\text{RZ}}^{i,j}[\rho]$ is a representation of $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$. Since the split semisimple rank of J is 1, we have $H_{\text{RZ}}^{i,j}[\rho] = 0$ for $j \geq 2$.

Let $\phi \in \Phi(G)$ be an L -parameter satisfying Proposition 2.4 (iii); namely, there exist a 2-dimensional irreducible representation ϕ_0 of $W_{\mathbb{Q}_p}$ and a character χ of $W_{\mathbb{Q}_p}$ such that $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$. We use the same notation as in the previous section. We are interested in how $\Pi_\phi^G, \Pi_\phi^J, \Pi_\psi^G$ and Π_ψ^J contribute to H_{RZ}^i . Now we can state our main theorem:

Theorem 3.1 (joint work with Tetsushi Ito) *We have the following:*

$$\begin{aligned}
 \text{(i)} \quad & H_{\text{RZ}}^{i,0}[\rho_{\text{sc}}] \cong \begin{cases} \pi_{\text{sc}} \boxtimes \phi_0(\frac{3}{2}) & i = 3, \\ 0 & i \neq 3, \end{cases} \quad H_{\text{RZ}}^{i,1}[\rho_{\text{sc}}] = 0, \\
 & H_{\text{RZ}}^{i,0}[\rho'_{\text{sc}}] \cong \begin{cases} \pi_{\text{sc}} \boxtimes \phi_0(\frac{3}{2}) & i = 3, \\ 0 & i \neq 3, \end{cases} \quad H_{\text{RZ}}^{i,1}[\rho'_{\text{sc}}] = 0. \\
 \text{(ii)} \quad & H_{\text{RZ}}^{i,0}[\rho_{\text{disc}}] \cong \begin{cases} \pi_{\text{sc}} \boxtimes \chi(1) & i = 3, \\ 0 & i \neq 3, \end{cases} \quad H_{\text{RZ}}^{i,1}[\rho_{\text{disc}}] \cong \begin{cases} \pi_{\text{sc}} \boxtimes \chi(2) & i = 4, \\ 0 & i \neq 4. \end{cases} \\
 \text{(iii)} \quad & H_{\text{RZ}}^{i,0}[\rho_{\text{nt}}] \cong \begin{cases} \pi_{\text{sc}} \boxtimes \chi(2) & i = 4, \\ 0 & i \neq 4, \end{cases} \quad H_{\text{RZ}}^{i,1}[\rho_{\text{nt}}] \cong \begin{cases} \pi_{\text{sc}} \boxtimes \chi(1) & i = 3, \\ 0 & i \neq 3. \end{cases}
 \end{aligned}$$

Here are very rough summary of the main theorem:

- A piece of the local Langlands correspondence for G and J appears in H_{RZ}^3 . This is similar to the Kottwitz conjecture (see [Rap95]).
- The non-tempered local A -packet Π_ψ^J contributes to H_{RZ}^4 .
- There exists a supercuspidal representation of $G(\mathbb{Q}_p)$ appearing outside the middle degree. In fact, it happens only when its L -parameter has non-trivial $\text{SL}_2(\mathbb{C})$ -part (see Remark 3.2 (iv)).

Remark 3.2 (i) By working in a suitable derived category, we may also consider the derived version $H_{\text{RZ}}^*[\rho] := (\text{Ext}_{J(\mathbb{Q}_p)}^*(R\Gamma_{\text{RZ}}, \rho)^{\mathcal{D}_{c\text{-sm}}})_{\text{sc}}$ of $H_{\text{RZ}}^{i,j}[\rho]$. We can recover ϕ and ψ from the $W_{\mathbb{Q}_p}$ -action and the Lefschetz operator on $H_{\text{RZ}}^*[\rho_{\text{disc}}]$ and $H_{\text{RZ}}^*[\rho_{\text{nt}}]$, respectively (*cf.* [Dat12] in the GL_n case).

- (ii) By using Theorem 3.1, we can prove that the semisimple L -parameters attached to $\pi_{\text{sc}}, \rho_{\text{sc}}$ and ρ'_{sc} by Fargues-Scholze [FS] are equal to $\phi|_{W_{\mathbb{Q}_p}}$. This implies that $\rho_{\text{sc}} \cong \rho'_{\text{sc}}$.
- (iii) By using recent results of Fargues-Scholze [FS], we can improve the theorem above. We will explain it elsewhere.
- (iv) For the L -packets of type (i) and (ii) in Proposition 2.4, we can obtain similar results as Theorem 3.1 (i). On the other hand, up to now we cannot treat the

L -packets of type (iv) in Proposition 2.4. The reason is that the theory of local A -packets for J (or J^{ad}) is not available in this case.

The proof of Theorem 3.1 is given by combination of local and global methods. First we recall some results obtained from local geometry.

Theorem 3.3 ([IM]) *Unless $2 \leq i \leq 4$, $H_{\text{RZ,sc}}^i = 0$.*

Here 2 (resp. 4) appears in the statement since it is equal to $\dim M_{G(\mathbb{Z}_p)} - \dim \mathcal{M}_{\text{red}}$ (resp. $\dim M_{G(\mathbb{Z}_p)} + \dim \mathcal{M}_{\text{red}}$), where \mathcal{M} is the natural formal model of $M_{G(\mathbb{Z}_p)}$. The equality $\dim \mathcal{M}_{\text{red}} = 1$ is related to the fact that the supersingular locus of the Siegel threefold is 1-dimensional. The method of the proof of Theorem 3.3 is similar to the author's proof of $H_{\text{LT,sc}}^i = 0$ for $i \neq n - 1$ (see [Mie10]), but it is much more complicated, mainly because connected components of \mathcal{M} are not affine (even not quasi-compact).

Theorem 3.4 *The representation $H_{\text{RZ,sc}}^2$ of $J(\mathbb{Q}_p)$ does not contain non-supercuspidal subquotient.*

This is a consequence of Theorem 3.3 and the fact that $H_{\text{RZ},G(\mathbb{Q}_p)\text{-sc},J(\mathbb{Q}_p)\text{-non-sc}}^2$ and $H_{\text{RZ},G(\mathbb{Q}_p)\text{-sc},J(\mathbb{Q}_p)\text{-non-sc}}^5$ are related by the Zelevinsky involution (see [Mie]).

Theorem 3.5 ([Mie20]) *Assume that the central character of π_{sc} is trivial on $p^{\mathbb{Z}} \subset \text{GSp}_4(\mathbb{Q}_p)$ (we can always twist π_{sc} by a character so that it satisfies this condition). Then, the representation $(\varinjlim_K H_c^i((M_K/p^{\mathbb{Z}}) \otimes_{\widehat{\mathbb{Q}}_p} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell))[\pi_{\text{sc}}^\vee]$ of $J(\mathbb{Q}_p)$ has finite length.*

This was proved by using the duality isomorphism between the Rapoport-Zink tower for G and that for J due to [KW] and [CFS].

Next we discuss the global aspect. As in the Lubin-Tate case, we use the relation between the Rapoport-Zink tower $\{M_K\}_{K \subset G(\mathbb{Z}_p)}$ and the Siegel threefold. For a compact open subgroup $K' \subset G(\mathbb{A}^\infty)$, let $\text{Sh}_{K'}$ denote the Siegel threefold over \mathbb{Q} with level K' . We put $H_c^i(\text{Sh}) = \varinjlim_{K'} H_c^i(\text{Sh}_{K'} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$, which is a representation of $G(\mathbb{A}^\infty) \times \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This representation is rather understood by using the global Langlands correspondence for GSp_4 (see [Tay93] and [Wei09]).

Let us fix a sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$. As in Section 2, for a compact open subgroup $K \subset G(\mathbb{Q}_p)$ we can define a rigid analytic open subset $\text{Sh}_{KK^p, \widehat{\mathbb{Q}}_p}^{\text{ss-red}}$ of $\text{Sh}_{KK^p, \widehat{\mathbb{Q}}_p}^{\text{an}}$, which is called the supersingular reduction locus. The following is an analogue of Proposition 2.3:

Proposition 3.6 (p -adic uniformization, [RZ96]) *We have an isomorphism*

$$\text{Sh}_{KK^p, \widehat{\mathbb{Q}}_p}^{\text{ss-red}} \cong \widetilde{J}(\mathbb{Q}) \backslash (M_K \times G(\mathbb{A}^{\infty,p}) / K^p),$$

where \widetilde{J} is a suitable inner form of GSp_4 over \mathbb{Q} such that $\widetilde{J} \otimes_{\mathbb{Q}} \mathbb{R}$ is anisotropic modulo center, $\widetilde{J} \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \cong G \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ and $\widetilde{J} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong J$.

We put $H^i(\mathrm{Sh}_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}}) = \varinjlim_{K, K^p} H^i(\mathrm{Sh}_{KK^p, \widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}} \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$, which is a representation of $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$. By Proposition 3.6, we have the Hochschild-Serre spectral sequence

$$E_2^{r,s} = \mathrm{Ext}_{J(\mathbb{Q}_p)}^r(H_{\mathrm{RZ}}^{6-s}(3), \mathcal{A}(\widetilde{J})_1)_{\mathrm{sc}} \Rightarrow H^{r+s}(\mathrm{Sh}_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}})_{\mathrm{sc}},$$

which is due to [Far04]. Here $\mathcal{A}(\widetilde{J})_1$ is the space of automorphic forms on $\widetilde{J}(\mathbb{A})$ which are trivial on $\widetilde{J}(\mathbb{R})$. By Boyer's trick and a result in [IM20] or [LS18], we have $H^{r+s}(\mathrm{Sh}_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}^{\mathrm{ss-red}})_{\mathrm{sc}} \cong H_c^{r+s}(\mathrm{Sh})_{\mathrm{sc}}$. Therefore we obtain:

Proposition 3.7 *We have a spectral sequence*

$$E_2^{r,s} = \mathrm{Ext}_{J(\mathbb{Q}_p)}^r(H_{\mathrm{RZ}}^{6-s}(3), \mathcal{A}(\widetilde{J})_1)_{\mathrm{sc}} \Rightarrow H_c^{r+s}(\mathrm{Sh})_{\mathrm{sc}}.$$

Now we are ready to sketch the proof of Theorem 3.1. The point is that we begin with $H_{\mathrm{RZ}}^{i,j}[\rho_{\mathrm{nt}}]$. By using Gan's result [Gan08], we can choose

- a cuspidal automorphic representation Π of $G(\mathbb{A})$
- and a cuspidal automorphic representation Σ of $\widetilde{J}(\mathbb{A})$

such that

- $\Pi_p \cong \pi_{\mathrm{sc}}$ and Π^∞ contributes to $H_c^2(\mathrm{Sh})$ and $H_c^4(\mathrm{Sh})$.
- if Π' is an automorphic representation of $G(\mathbb{A})$ such that $\Pi'_v \cong \Pi_v$ for all places $v \neq p, \infty$ and Π'_p is supercuspidal, then $\Pi = \Pi'$. It is a kind of the strong multiplicity one theorem.
- $\Sigma_p \cong \rho_{\mathrm{nt}}$ and $\Sigma_\infty \cong \mathbf{1}$.
- if Σ' is an automorphic representation of $\widetilde{J}(\mathbb{A})$ such that $\Sigma'_v \cong \Sigma_v$ for all places $v \neq p$, then $\Sigma = \Sigma'$. It is a kind of the strong multiplicity one theorem.
- $\Pi^{\infty,p} = \Sigma^{\infty,p}$; recall that we have $G(\mathbb{A}^{\infty,p}) = \widetilde{J}(\mathbb{A}^{\infty,p})$.

By taking the $\Pi^{\infty,p}$ -isotypic part of the spectral sequence in Proposition 3.7, we get a short exact sequence

$$0 \rightarrow H_{\mathrm{RZ}}^{i+1,1}[\rho_{\mathrm{nt}}] \rightarrow \pi_{\mathrm{sc}} \boxtimes H_c^{6-i}(\mathrm{Sh})[\Pi^\infty](3) \rightarrow H_{\mathrm{RZ}}^{i,0}[\rho_{\mathrm{nt}}] \rightarrow 0.$$

By assumption, $H_c^{6-i}(\mathrm{Sh})[\Pi^\infty](3) \neq 0$ only if $i = 2, 4$. On the other hand, by Theorems 3.3 and 3.4, we have $H_{\mathrm{RZ}}^{5,1}[\rho_{\mathrm{nt}}] = H_{\mathrm{RZ}}^{2,0}[\rho_{\mathrm{nt}}] = 0$. Hence we conclude

$$H_{\mathrm{RZ}}^{4,0}[\rho_{\mathrm{nt}}] \cong \pi_{\mathrm{sc}} \boxtimes H_c^2(\mathrm{Sh})[\Pi^\infty](3), \quad H_{\mathrm{RZ}}^{3,1}[\rho_{\mathrm{nt}}] \cong \pi_{\mathrm{sc}} \boxtimes H_c^4(\mathrm{Sh})[\Pi^\infty](3).$$

Next we investigate $H_{\mathrm{RZ}}^{i,j}[\rho_{\mathrm{disc}}]$. We choose Π and Σ similarly as above, but so that Π^∞ contributes to $H_c^3(\mathrm{Sh})$. Then we get a short exact sequence

$$0 \rightarrow H_{\mathrm{RZ}}^{4,1}[\rho_{\mathrm{disc}}] \rightarrow \pi_{\mathrm{sc}} \boxtimes H_c^3(\mathrm{Sh})[\Pi^\infty](3) \rightarrow H_{\mathrm{RZ}}^{3,0}[\rho_{\mathrm{disc}}] \rightarrow 0.$$

Since $H_c^3(\mathrm{Sh})[\Pi^\infty](3)$ is 2-dimensional indecomposable as a $W_{\mathbb{Q}_p}$ -representation, it suffices to determine $\dim H_{\mathrm{RZ}}^{i,j}[\rho_{\mathrm{disc}}][\pi_{\mathrm{sc}}]$. This is done by using the following facts:

- $[\rho_{\text{nt}}] + [\rho_{\text{disc}}] = [\text{induced}]$ in the Grothendieck group of finite length representations of $J(\mathbb{Q}_p)$.
- $\sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}_{J(\mathbb{Q}_p)/p^{\mathbb{Z}}}^i(V, \text{induced}) = 0$ for every $J(\mathbb{Q}_p)/p^{\mathbb{Z}}$ -representation V of finite length ([SS97]).

To apply the second fact, we need the finiteness result in Theorem 3.5.

We can treat $H_{\text{RZ}}^{i,j}[\rho_{\text{sc}}]$ and $H_{\text{RZ}}^{i,j}[\rho'_{\text{sc}}]$ in the same way. These cases are the simplest because $H_{\text{RZ}}^{i,1}[\rho_{\text{sc}}] = H_{\text{RZ}}^{i,1}[\rho'_{\text{sc}}] = 0$.

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