An example of “resonant reflecting” for matrix Schrödinger operators in the semiclassical limit

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1 Introduction

The aim of this manuscript is to show a resonant effect for matrix Schrödinger operators which is comparable to the resonant tunneling effect for scalar Schrödinger operators. The tunneling effect is one of the most famous quantum effects (see e.g., [15]). In the single-barrier problem for a 1D scalar semiclassical Schrödinger operator

$$-h^2\frac{d^2}{dx^2}+V(x),\quad (h>0,\ V\in C_0^\infty(\mathbb{R};\mathbb{R})),\tag{1.1}$$

the transmission probability is exponentially small in the sense of semiclassical limit $h\to 0^+$ (see e.g., [11, 12, 18] and references therein). However, resonant tunneling effect says in the double-barrier problem that the transmission probability reaches almost one at resonant energies ([5, 7, 19] see also [16] for an elementary model with piecewise constant potential and the resonant tunneling for quantum walks). Conversely, in the no-barrier problem, the transmission probability is exponentially close to one (e.g. [18]). We consider no-barrier problem for one of the two coupled scalar Schrödinger operators, and show the “resonant reflecting effect” in the sense that the transmission probability reaches almost zero at resonant energies.

Let us make precise our problem. We study the scattering matrix in the semiclassical limit for a 2×2-matrix Schrödinger operator in 1D:

$$P(h)=\begin{pmatrix}P_1(h) & hW\\ hW & P_2(h)\end{pmatrix},\quad h>0,\tag{1.2}$$

where for each $j=1,2$, $P_j(h)$ stands for the scalar semiclassical Schrödinger operator and W for a multiplication operator of a function W:

$$P_j(h)=-h^2\frac{d^2}{dx^2}+V_j(x).$$

We assume the following conditions.

Condition A. *There exist real values* E_1 *and* E_2 *such that the functions* $V_1 - E_1$*,* $V_2 - E_2$*,* W *are smooth* (C^∞) *real-valued with a compact support.*

Then the scattering matrix $\mathcal{S} = \mathcal{S}(E)$ is defined for $E > \max\{E_1, E_2\}$ as a unitary 4×4-matrix. At such energy levels, behavior and estimates of the anti-diagonal 2×2-blocks $\mathcal{S}_{12}(E)$ and $\mathcal{S}_{21}(E)$ of this matrix are interesting in the context of phase space tunneling (see [1, 4, 14]):

$$\mathcal{S}(E) = \begin{pmatrix} \mathcal{S}_{11}(E) & \mathcal{S}_{12}(E) \\ \mathcal{S}_{21}(E) & \mathcal{S}_{22}(E) \end{pmatrix}.$$

Note that if the anti-diagonal part W of the operator P vanishes identically, it follows that the diagonal blocks $\mathcal{S}_{11}$ and $\mathcal{S}_{22}$ coincide with the scattering matrices of the scalar operator P_1 and P_2, and that the anti-diagonal blocks vanish $\mathcal{S}_{12} = \mathcal{S}_{21} = 0$. Our interest is the diagonal block $\mathcal{S}_{11}$ as a modification of the scattering matrix of P_1. We consider energies near E_0 between E_1 and E_2 such that P_1 and P_2 admit no-barrier and single-well at E_0, respectively.

Condition B. *Let*

$$0 = E_1 < E_0 < E_2. \tag{1.3}$$

The functions V_1*,* V_2*,* W *are even. There exists* $a^0 > 0$ *such that*

$$\frac{V_2(x) - E_0}{x - a^0} > 0 > V_1(x) - E_0 \quad \textit{for } x > 0.$$

Under Conditions A and B, the block $\mathcal{S}_{11}(E)$ is a unitary matrix for E near E_0 (see Proposition 1). We call the modulus of the diagonal and anti-diagonal entries of $\mathcal{S}_{11}(E)$ the transmission and reflection coefficients, and denote by $t(E)$ and $r(E)$. The unitarity implies the identity

$$t(E)^2 + r(E)^2 = 1. \tag{1.4}$$

Near E_0, the spectrum of P_1 is continuous, and that of P_2 consists of (real) eigenvalues. For a positive $l > 0$ independent of h, set

$$\mathcal{A}(E) := \int_{\mathbb{R}} \sqrt{\max\{E - V_2(x), 0\}}\, dx, \quad E \in [E_0 - lh, E_0 + lh], \tag{1.5}$$

the classical action of the well, and

$$\mathfrak{B}_h = \mathfrak{B}_h(l, E_0) := \left\{ E \in [E_0 - lh, E_0 + lh];\, \cos\left(\frac{\mathcal{A}(E)}{h}\right) = 0 \right\} \tag{1.6}$$

the set of Bohr-Sommerfeld points. Then for any $h > 0$ small enough, there exists a bijection $s_h : \mathfrak{B}_h(l, E_0) \to \sigma(P_2(h)) \cap [E_0 - lh, E_0 + lh]$ (the set $\sigma(P_2(h))$ is the spectrum of $P_2(h)$) such that

$$|s_h(E) - E| = \mathcal{O}(h^2) \tag{1.7}$$

uniformly for $E \in \mathfrak{B}_h(l, E_0)$ (see e.g., [10, 20]). Since the spectrum of $P_1(h)$ near E_0 is continuous, $\sigma(P_2(h))$ is the set of embedded eigenvalues near E_0 of $P(h)$ when W vanishes identically. For a general W, the eigenvalues of $P_2(h)$ may turn into resonances in the lower half plane (see (2.3) for the definition of resonances). The asymptotic distribution of resonances near E_0 is studied in [3, 13, 17] and [2]. When the characteristic sets $\{(x, \xi) \in \mathbb{R}^2;\ \xi^2 + V_1(x) = E_0\}$ and $\{(x, \xi) \in \mathbb{R}^2;\ \xi^2 + V_2(x) = E_0\}$ do not intersect, the imaginary part of the resonances is exponentially small with respect to h [3, 13, 17], but when they do with a finite degeneracy, it is of polynomial order [2]. To avoid a delicate estimate, we assume that they intersect.

Condition C. *The difference $V_1 - V_2$ only vanishes at $x = 0$, and*

$$V_1(0) = V_2(0) = 0,\ V_1''(0) - V_2''(0) \neq 0,\ W(0) \neq 0,\ \mathrm{supp} W \subset \{V_2 < E_0\}.$$

Theorem. *Assume Conditions A, B, and C. Let $l > 0$ be independent of h. Then there exists $h_0 > 0$ such that*

$$t(E) = \left| \frac{\tau(E) + \mathcal{O}(h)}{e^{\frac{2i\theta_1}{h}} \tau(E) - 2(E_2 - E)^{1/2} \omega^2 h^{2/3} e^{i\frac{\mathcal{A}(E)}{h}}} \right|, \tag{1.8}$$

holds with

$$\tau(E) = \left(4 + (E_2 - E)^{1/2} \omega^2 h^{2/3}\right) \cos \frac{\mathcal{A}(E)}{h},$$

$$\omega = 2W(0) E^{-1/2} \left(\frac{3!2\sqrt{E}}{V_2''(0) - V_1''(0)} \right)^{1/3} \Gamma\left(\frac{4}{3}\right) \cos \frac{\pi}{6},$$

$$\theta_1 = \int_0^{+\infty} \left(\sqrt{E - E_1} - \sqrt{E - V_1(x)} \right) dx,$$

uniformly for $0 < h < h_0$ and for $E \in [E_0 - lh, E_0 + lh]$.

It follows in particular that

$$t(E) = 1 + \mathcal{O}(h^{2/3}) \tag{1.9}$$

for E away of order h from $\mathfrak{B}_h(l, E_0)$, and that

$$t(E) = \mathcal{O}(h^{1/3}) \tag{1.10}$$

for $E \in \mathfrak{B}_h(l, E_0)$. In this sense, the "resonant energies" are close to the eigenvalues of $P_2(h)$. According to [2], the resonant energies are also close to the real part of resonances. They proved the existence of a bijection $z_h : \mathfrak{B}_h(l, E_0) \to \mathrm{Res}(P(h)) \cap ([E_0 - lh, E_0 + lh] - i[0, lh])$ such that

$$|z_h(E) - s_h(E)| < ch^{5/3} \tag{1.11}$$

uniformly for $E \in \mathfrak{B}_h(l, E_0)$, where $\mathrm{Res}(P(h))$ stands for the set of resonances of $P(h)$.

These facts are similar to the double-barrier problem for scalar Schrödinger operators. The resonant energies in the double-barrier problem are close to the eigenvalues produced by the well between two barriers, that is, eigenvalues for the modified operator to a single-well conserving the potential well, potential V on $\{V < E_0\}$. There are also resonances of the original operator called "shape resonances" exponentially close to each eigenvalues of the modified one.

In this manuscript, we prove Theorem by a simple method of continuation of the Jost solutions. In the preparing paper [9], we will employ microlocal analysis for more general situations.

2 Jost solutions and scattering matrix

We here define the scattering matrix by using the Jost solutions. It coincides with the analytic continuation through the upper half plane of the scattering matrix defined for $E > E_2$. For $S = L, R$, we denote by $J^{\sharp}_{1,S}$, $J^{\flat}_{1,S}$, $J^{-}_{2,S}$, $J^{+}_{2,S}$ the solutions to the equation $(P(h) - E)w = 0$ characterized by

$$\begin{aligned} J^{\sharp}_{1,L} &= e^{-i\sqrt{E-E_1}\,x/h} v_1, \quad & J^{\flat}_{1,L} &= e^{i\sqrt{E-E_1}\,x/h} v_1, \\ J^{-}_{2,L} &= e^{\sqrt{E_2-E}\,x/h} v_2, \quad & J^{+}_{2,L} &= e^{-\sqrt{E_2-E}\,x/h} v_2, \end{aligned} \tag{2.1}$$

for $x \ll -1$ (left) and

$$\begin{aligned} J^{\sharp}_{1,R} &= e^{i\sqrt{E-E_1}\,x/h} v_1, \quad & J^{\flat}_{1,R} &= e^{-i\sqrt{E-E_1}\,x/h} v_1, \\ J^{-}_{2,R} &= e^{-\sqrt{E_2-E}\,x/h} v_2, \quad & J^{+}_{2,R} &= e^{\sqrt{E_2-E}\,x/h} v_2, \end{aligned} \tag{2.2}$$

for $x \gg 1$ (right) with $v_1 = {}^t(1, 0)$ and $v_2 = {}^t(0, 1)$. Such solutions exist under Conditions A and B. We call $J^{\sharp}_{1,S}$ and $J^{\flat}_{1,S}$ ($S = L, R$) outgoing and incoming Jost solutions, respectively. The two tuples $\boldsymbol{J}^{\flat} := (J^{\flat}_{1,L}, J^{\flat}_{1,R}, J^{+}_{2,L}, J^{+}_{2,R})$ and $\boldsymbol{J}^{\sharp} := (J^{\sharp}_{1,R}, J^{\sharp}_{1,L}, J^{-}_{2,R}, J^{-}_{2,L})$ are bases of the solution space if $E \in]E_1, E_2[$ is not a resonance, that is, the Wronskian

$$\mathcal{W}_h(\boldsymbol{J}^{\sharp}) := \det \begin{pmatrix} \boldsymbol{J}^{\sharp} \\ h\partial_x(\boldsymbol{J}^{\sharp}) \end{pmatrix} \tag{2.3}$$

does not vanish. Under Conditions A, B, and C, there is no real resonance near E_0 [2]. Note that the Wronskian of four solutions is independent of x. Define a 4×4-matrix $\mathcal{S}(E)$ by

$$\boldsymbol{J}^\flat = \boldsymbol{J}^\sharp \mathcal{S}(E). \tag{2.4}$$

We are interested in the 2×2-block

$$\mathcal{S}_{11} = \begin{pmatrix} I_2 & O_{2\times 2} \end{pmatrix} \mathcal{S} \begin{pmatrix} I_2 \\ O_{2\times 2} \end{pmatrix}. \tag{2.5}$$

Proposition 1. *Let $l > 0$ be independent of h. Under Conditions A, B, and C, for each $E \in [E_0 - lh, E_0 + lh]$, the 2×2-block $\mathcal{S}_{11}(E)$ is unitary.*

Sketch of the proof of Proposition 1. Let $T(E)$ be the 4×4 transfer matrix between the two bases $\boldsymbol{J}_L := (J^\sharp_{1,L}, J^\flat_{1,L}, J^-_{2,L}, J^+_{2,L})$ and $\boldsymbol{J}_R := (J^\sharp_{1,R}, J^\flat_{1,R}, J^-_{2,R}, J^+_{2,R})$ of the solution space: $\boldsymbol{J}_L T(E) = \boldsymbol{J}_R$. We represent the scattering matrix $\mathcal{S}(E)$ in terms of the entries of $T(E)$. By putting

$$A^\flat = \begin{pmatrix} 0 & t_{12} & 0 & t_{14} \\ 1 & t_{22} & 0 & t_{24} \\ 0 & t_{32} & 0 & t_{34} \\ 0 & t_{42} & 1 & t_{44} \end{pmatrix}, \quad A^\sharp = \begin{pmatrix} t_{11} & 1 & t_{13} & 0 \\ t_{21} & 0 & t_{23} & 0 \\ t_{31} & 0 & t_{33} & 1 \\ t_{41} & 0 & t_{43} & 0 \end{pmatrix}, \tag{2.6}$$

we have $\boldsymbol{J}^\sharp = \boldsymbol{J}_L A^\sharp$, $\boldsymbol{J}^\flat = \boldsymbol{J}_L A^\flat$, and $\mathcal{S} = (A^\sharp)^{-1} A^\flat$. The matrix $A^\sharp$ is invertible if and only if $\boldsymbol{J}^\sharp$ is linearly independent, that is, E is not a resonance. A straightforward computation implies in particular that

$$\begin{aligned} &\mathcal{S}_{11} = \zeta^{-1} \begin{pmatrix} t_{43} & \eta \\ -\overline{\eta} & t^{43} \end{pmatrix}, \quad \zeta = \det \begin{pmatrix} t_{21} & t_{23} \\ t_{41} & t_{43} \end{pmatrix}, \\ &\eta = \det \begin{pmatrix} t_{22} & t_{24} \\ t_{42} & t_{43} \end{pmatrix} = \overline{\det \begin{pmatrix} t_{11} & t_{14} \\ t_{41} & t_{43} \end{pmatrix}}, \end{aligned} \tag{2.7}$$

where we denote by $t_{jk} = t_{jk}(E)$ and by $t^{jk} = t^{jk}(E)$ the (j,k)-entry of $T(E)$ and of $T(E)^{-1}$, respectively. In the computation, we used the equalities $\det T(E) = 1$ and $\overline{t_{jk}} = t_{\sigma_{(1,2)}(j)\sigma_{(1,2)}(k)}$ where $\sigma_{1,2}$ stands for the transposition $(1,2)$: $\sigma_{(1,2)}(1,2,3,4) = (2,1,3,4)$. The identity

$$\mathcal{W}_h(\boldsymbol{J}_L) = \mathcal{W}_h(\boldsymbol{J}_R) = 4i\sqrt{E - E_1}\sqrt{E_2 - E}, \tag{2.8}$$

implies $\det T(E) = 1$. The complex conjugacy of $(J^\sharp_{1,L}, J^\flat_{1,L})$ and $(J^\sharp_{1,R}, J^\flat_{1,R})$, and the real-valuedness of $J^\bullet_{2,L}$, $J^\bullet_{2,R}$ imply $\overline{t_{jk}} = t_{\sigma_{(1,2)}(j)\sigma_{(1,2)}(k)}$. We also have

$$t^{43} = -\det \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{41} & t_{42} & t_{43} \end{pmatrix}.$$

By the symmetry of V_1, V_2, and W, it follows that $t^{43} = t_{43}$, and $\mathcal{S}_{11}$ is unitary. In fact, we can check the following equalities: $t_{43}t^{43} + |\eta|^2 = |\zeta|^2$ and

$$\mathcal{S}_{11}\mathcal{S}_{11}^* = |\zeta|^{-2}\begin{pmatrix}(t_{43})^2 + |\eta|^2 & (t^{43} - t_{43})\eta \\ (t^{43} - t_{43})\overline{\eta} & (t^{43})^2 + |\eta|^2\end{pmatrix}.$$

$\square$

3 Outline of the proof

We compute the scattering matrix by substituting into (2.7) the value of the entries of the transfer matrix $T(E)$. To obtain these values, we continue the Jost solutions $J^\sharp_{1,R}$ and $J^-_{2,R}$.

For a basis $(u_1, \check{u}_1)$ of the solution space to the scalar equation $(P_1(h) - E)u = 0$, put

$$k_1(x,y) := \frac{W(y)}{h\mathcal{W}(u_1,\check{u}_1)}\left(u_1(x)\check{u}_1(y) - \check{u}_1(x)u_1(y)\right), \tag{3.1}$$

where we denote the Wronskian by

$$\mathcal{W}(u_1,\check{u}_1) = \det\begin{pmatrix}u_1 & \check{u}_1 \\ u_1' & \check{u}_1'\end{pmatrix}.$$

We can easily check that $k_1(x,y)$ is independent of the choice of the basis. We also define the function $k_2(x,y)$ in the same manner. The Jost solutions are represented as follows;

$$J^\sharp_{1,R} = \begin{pmatrix}\mathcal{K}_1 u^\sharp_{1,R} \\ K_2\mathcal{K}_1 u^\sharp_{1,R}\end{pmatrix}, \quad J^-_{2,R} = \begin{pmatrix}K_1\mathcal{K}_2 u^-_{2,R} \\ \mathcal{K}_2 u^-_2\end{pmatrix}, \tag{3.2}$$

where we introduce the integral operators

$$K_j f(x) = -\int_x^{+\infty} k_j(x,y)f(y)dy \quad (f \in C(\mathbb{R}),\ j \in \{1,2\}), \tag{3.3}$$

the infinite sums

$$\mathcal{K}_1 = \sum_{k\geq 0}(K_1K_2)^k, \quad \mathcal{K}_2 = \sum_{k\geq 0}(K_2K_1)^k,$$

and the solutions $u^\sharp_{1,R}$ to $(P_1(h) - E)u = 0$ and $u^-_{2,R}$ to $(P_2(h) - E)u = 0$ characterized by

$$u^\sharp_{1,R} = e^{i\sqrt{E-E_1}x/h}, \quad u^-_{2,R} = e^{-\sqrt{E_2-E}\,x/h} \quad x \gg 1.$$

These infinite sums converge to bounded linear operators on $C(\mathbb{R})$ equipped with the supremum norm. On one hand, by definition, we have

$$J_{1,R}^{\sharp} = \begin{pmatrix} t_{11}e^{-i\sqrt{E-E_1}x/h} + t_{21}e^{i\sqrt{E-E_1}x/h} \\ t_{31}e^{\sqrt{E_2-E}\,x/h} + t_{41}e^{-\sqrt{E_2-E}\,x/h} \end{pmatrix}, \tag{3.4}$$

for $x \ll -1$. On the other hand, we have

$$\mathcal{K}_1 u_{1,R}^{\sharp} = \beta^{\sharp} u_{1,L}^{\sharp} + \beta^{\flat} u_{1,L}^{\flat}, \quad K_2\mathcal{K}_1 u_{1,R}^{\sharp} = \alpha^- u_{2,L}^- + \alpha^+ u_{2,L}^+, \tag{3.5}$$

for $x \ll -1$ with

$$\begin{aligned}
\beta^{\sharp} &= \tau^{\sharp} + \frac{-1}{h\mathcal{W}(u_{1,L}^{\sharp}, u_{1,L}^{\flat})} \int_{\mathbb{R}} W(x) u_{1,L}^{\flat}(x) \mathcal{K}_1 K_2 u_{1,R}^{\sharp}(x) dx, \\
\beta^{\flat} &= \tau^{\flat} + \frac{1}{h\mathcal{W}(u_{1,L}^{\sharp}, u_{1,L}^{\flat})} \int_{\mathbb{R}} W(x) u_{1,L}^{\sharp}(x) \mathcal{K}_1 K_2 u_{1,R}^{\sharp}(x) dx, \\
\alpha^- &= \frac{-1}{h\mathcal{W}(u_{2,L}^-, u_{2,L}^+)} \int_{\mathbb{R}} W(x) u_{2,L}^+(x) \mathcal{K}_1 u_{1,R}^{\sharp}(x) dx, \\
\alpha^+ &= \frac{1}{h\mathcal{W}(u_{2,L}^-, u_{2,L}^+)} \int_{\mathbb{R}} W(x) u_{2,L}^-(x) \mathcal{K}_1 u_{1,R}^{\sharp}(x) dx.
\end{aligned}$$

Here, we denote by $u_{1,L}^{\sharp}$, $u_{1,L}^{\flat}$, $u_{2,L}^-$, and $u_{2,L}^+$ the Jost solutions to the scalar equations $(P_j(h) - E)u = 0$ $(j = 1, 2)$ characterized by

$$\begin{aligned}
u_{1,L}^{\sharp} &= e^{-i\sqrt{E-E_1}x/h}, \quad u_{1,L}^{\flat} = e^{i\sqrt{E-E_1}x/h}, \\
u_{2,L}^- &= e^{\sqrt{E_2-E}\,x/h}, \quad u_{2,L}^+ = e^{-\sqrt{E_2-E}\,x/h},
\end{aligned}$$

for $x \ll -1$. The constants $\tau^{\sharp}$ and $\tau^{\flat}$ are determined by

$$u_{1,R}^{\sharp} = \tau^{\sharp} u_{1,L}^{\sharp} + \tau^{\flat} u_{1,L}^{\flat}. \tag{3.6}$$

By combining (3.4), (3.5), and similar ones for $J_{2,R}^-$, we obtain the integral representation of the entries t_{j1}, t_{j3} $(j = 1, 2, 3, 4)$ of transfer matrix $T(E)$. The difference between each infinite sum and its first term is suitably estimated. For example, we have

$$t_{41} = \alpha^+ = -\frac{1}{2} \int_{\mathbb{R}} W u_{2,L}^- u_{1,R}^{\sharp} dx + \mathcal{O}(h).$$

The rest of this manuscript is devoted to show how we obtain the asymptotics of the integral by using the degenerate stationary phase method. On the support of W, $u_{2,R}^-$ is written as a linear combination of another basis of solutions. There exist constants τ^+ and τ^- such that

$$\begin{aligned}
e^{\theta_2/h} u_{2,R}^- &= (E - V_2(x))^{-1/4} \sum_{\pm} \tau^{\pm} e^{\pm i \int_0^x \sqrt{E - V_2(t)} dt/h} + \mathcal{O}(h), \\
\theta_2 &= \int_{a(E)}^{+\infty} \left(\sqrt{E_2 - E} - \sqrt{V_2(x) - E} \right) dx + \sqrt{E_2 - E}\, a(E),
\end{aligned}$$

uniformly on each compact set contained in $]-a^0, a^0[$. Here, $a(E)$ is the unique solution near a^0 of $V_2(a(E)) = E$ ($a^0 = a(E_0)$). Then we have

$$e^{\theta_2/h}\int_{\mathbb{R}} W u_{2,L}^- u_{1,R}^\sharp dx = \sum_{\pm} \tau^\pm \int_{\mathbb{R}} e^{i\phi^\pm(x)/h}\sigma(x,h)dx + \mathcal{O}(h)$$

with

$$\begin{aligned}
\phi^\pm(x) &= \int_0^x \left(\sqrt{E - V_1(t)} \pm \sqrt{E - V_2(t)}\right) dt,\\
\sigma(x,h) &= e^{i\theta_1/h}\left(\frac{E - E_1}{(E - V_1(x))(E - V_2(x))}\right)^{1/4} W(x),\\
\theta_1 &= \int_0^{+\infty}\left(\sqrt{E - E_1} - \sqrt{E - V_1(x)}\right) dx.
\end{aligned}$$

Then $x = 0$ is the critical point of ϕ^- whereas $(\phi^+)'$ does not vanish. The degenerate stationary phase at $x = 0$ implies

$$\int_{\mathbb{R}} e^{i\phi^-(x)/h}\sigma(x,h)dx = e^{i\theta_1/h}(E - E_1)^{1/4}\omega h^{1/3} + \mathcal{O}(h^{2/3}), \tag{3.7}$$

with

$$\omega = 2W(0)E^{-1/2}\left(\frac{3!2\sqrt{E}}{V_1''(0) - V_2''(0)}\right)^{1/3}\Gamma\left(\frac{4}{3}\right)\cos\frac{\pi}{6}. \tag{3.8}$$

Therefore, we obtain

$$t_{41} = -\frac{e^{(i\theta_1 - \theta_2)/h}}{2}\left((E - E_1)^{1/4}\omega h^{1/3} + \mathcal{O}(h^{2/3})\right). \tag{3.9}$$

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