Improvement of a regularity condition for long-range scattering for NLS with critical homogeneous nonlinearity

Hayato MIYAZAKI
Faculty of Education, Kagawa University

Abstract

In this note, we survey the results in the author with Masaki and Uriya [14, 17] for the final state problem for nonlinear Schrödinger equations with critical homogeneous nonlinearity which is not necessarily a polynomial. It is also mentioned that a regularity condition for the final data can be improved by estimates for the kind of asymptotics of the solution developed by Kawamoto and the author [12].

1 Introduction

In this note, we deal with the final state problem for the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = F(u), \tag{NLS}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $d \leqslant 3$, and $u = u(t, x)$ is a $\mathbb{C}$-valued unknown function. The nonlinearity F is homogeneous of degree $1 + 2/d$; that is, it satisfies

$$F(\alpha z) = \alpha^{1+\frac{2}{d}} F(z) \tag{H}$$

for any $\alpha > 0$ and $z \in \mathbb{C}$. The typical example is $F(u) = \mu|u|^{\frac{2}{d}}u$ ($\mu \in \mathbb{R} \setminus \{0\}$) which is called the gauge-invariant nonlinearity. Moreover, the following nonlinearity also satisfies (H):

$$F(u) = a_1|\operatorname{Re} u|^{\frac{2}{d}} \operatorname{Re} u + a_2|\operatorname{Im} u|^{\frac{2}{d}} \operatorname{Im} u, \quad a_1, a_2 \in \mathbb{C}.$$

This type of the nonlinearity is introduced by Kato [11]. When $a_1 = 1$ and $a_2 = 0$, it also appears as a main part of a generalized version of Gross-Pitaevskii equation introduced in Masaki and the author [16]. The aim of this note is to survey the result in [14, 17] and to mention that we can improve the regularity condition for the final data, necessary to discuss the asymptotic behavior of solutions to (NLS).

It is known that the exponent $2/d$ is a threshold in view of the long-time behavior of solutions. To describe why the exponent $2/d$ is the threshold, let us consider the gauge invariant nonlinearity $F(u) = \mu|u|^p u$, $p > 1$. When $p > 2/d$, it is known that (NLS) admits a nontrivial solution to possess the asmptotics of the free solution $e^{it\Delta/2}u_+$ of the form

$$(it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{2t}} \widehat{u_+}\left(\frac{x}{t}\right) \tag{1}$$

as $t \to \infty$ (e.g. [29]). However, if $p = 2/d$, then there are no nontrivial solutions that behave like the free solution in L^2-topology for large time (e.g. [1, 27]).

In the case of $p = 2/d$, Ozawa [23] and Ginibre-Ozawa [2] show that the equation has a solution that asymptotically behave like the free solution with a logarithmic phase correction

$$(it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{2t}} \widehat{u_+}\left(\frac{x}{t}\right) \exp\left(-i\mu \left|\widehat{u_+}\left(\frac{x}{t}\right)\right|^{\frac{2}{d}} \log t\right) \tag{2}$$

for large time. Here we say that the nonlinearity is *short-range* if (NLS) admits a nontrivial solution that asymptotically behaves like (1) for large time. Also the nonlinearity is said to be *long-range* if (NLS) admits a nontrivial solution that asymptotically behaves like (2) with a suitable $\mu \in \mathbb{R} \setminus \{0\}$.

Let us further focus on the case $p = 2/d$. In this case, the asymptotic behavior of the solutions depends on the shape of the nonlinearity. For instance, when $d = 2$, $F(u) = \mu|u|u + \lambda_1 u^2 + \lambda_2 \overline{u}^2$ with $\lambda_j \in \mathbb{C}$ is short-range if $\mu = 0$ and long-range if $\mu \neq 0$ (cf. [9, 10, 20, 25]). Eventually, via the Fourier series expansion, Masaki and the author [14] treat general nonlinearity satisfying (H), namely, including the non-gauge-invariant nonlinearity, and they prove that if $g_0 = 0$ and $g_1 \in \mathbb{R}$, then (NLS) admits a nontrivial solution that asymptotically behaves like (2) with $\mu = g_1$ by introducing a decomposition of the nonlinearity

$$F(u) = g_0|u|^{1+\frac{2}{d}} + g_1|u|^{\frac{2}{d}}u + \sum_{n\neq 0,1} g_n|u|^{1+\frac{2}{d}-n}u^n \tag{3}$$

with the coefficients

$$g_n := \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})e^{-in\theta}d\theta \tag{4}$$

under a summability assumption on $\{g_n\}_n$ (see also [17]). In particular, if $g_0 = g_1 = 0$ then there exists an asymptotically free solution. We note that if $g_0 \neq 0$, then there are no solutions that behave like a free solution (cf. [15, 19, 24, 26]). Also, we only consider the case where the given final data u_+ has a very small low-frequency part, because if the data has a non-negligible low-frequency part, then there appears other kind of asymptotic behavior (see [3, 4, 6–8, 21, 22]).

In order to present our result, let us briefly introduce the decomposition of the nonlinearity via the Fourier series expansion used by [14, 17]. The same concept can be found in [18, 28]. We identify a homogeneous nonlinearity F with 2π-periodic function g as follows: A homogeneous nonlinearity F is written as

$$F(u) = |u|^{1+\frac{2}{d}} F\left(\frac{u}{|u|}\right).$$

We then introduce a 2π-periodic function $g(\theta) = g_F(\theta)$ by $g_F(\theta) = F(e^{i\theta})$. Conversely, for a given 2π-periodic function g, one can construct a homogeneous nonlinearity $F = F_g : \mathbb{C} \to \mathbb{C}$ by $F_g(u) = |u|^{\frac{2}{d}} g(\arg u)$ if $u \neq 0$ and $F_g(u) = 0$ if $u = 0$. Since $g(\theta)$ is 2π-periodic function, at least formally, $g(\theta) = \sum_{n\in\mathbb{Z}} g_n e^{in\theta}$ holds with (4). Thus, the expansion gives us (3).

Throughout this note, we assume the following condition for $\{g_n\}$:

Assumption 1. Assume that the nonlinearity $F\colon \mathbb{C} \to \mathbb{C}$ is a homogeneous of degree $1 + 2/d$ such that g_n defined by (4) satisfies $g_0 = 0$, $g_1 \in \mathbb{R}$ and

$$\sum_{n\in\mathbb{Z}} |n|^{1+\eta}|g_n| < \infty$$

for some $\eta > 0$.

Notations

We introduce some notations used throughout this note. For any $p \geqslant 1$, $L^p = L^p(\mathbb{R}^d)$ denotes the usual Lebesgue space on $\mathbb{R}^d$. Set $\langle a \rangle = (1 + |a|^2)^{1/2}$ for $a \in \mathbb{C}$ or $a \in \mathbb{R}^d$. Let s, $m \in \mathbb{R}$. The weighted Sobolev space and the homogeneous Sobolev space on $\mathbb{R}^d$ are defined by $H^{m,s} = H^{m,s}(\mathbb{R}^d) = \{u \in \mathscr{S}' \mid \langle x \rangle^s \langle i\nabla \rangle^m u \in L^2\}$ and $\dot{H}^s = \dot{H}^s(\mathbb{R}^d) = \{u \in \mathscr{S}' \mid |\nabla|^s u \in L^2\}$, respectively. Here $\mathscr{S}'$ is the space of tempered distributions. We simply write $H^m = H^{m,0}$. $\mathcal{F}[u] = \widehat{u}$ is the usual Fourier transform of a function u on $\mathbb{R}^d$ and $\mathcal{F}^{-1}[u] = \check{u}$ is its inverse. $\|g\|_{\mathrm{Lip}}$ stands for the Lipschitz norm of g. The smallest integer n_0 such that $n_0 \geqslant \delta$ is denoted by $\lceil \delta \rceil$ for any $\delta \in \mathbb{R}$.

In [14, 17], Masaki, the author and Uriya prove the following:

Theorem 1 ([14, 17])**.** Let $d \leqslant 3$. Suppose that F satisfies Assumption 1 for some $\eta > 0$. Set $\delta > 0$ such that $d/2 < \delta < \min(d, 1 + 2/d, d/2 + 2\eta)$. Take

$$\delta' = 1 \ \ (d = 1), \quad \delta' = 2 \ \ (d = 2), \quad \delta' = 5/3 \ \ (d = 3). \tag{5}$$

Then, there exists $\varepsilon_0 = \varepsilon_0(\|g\|_{\mathrm{Lip}}) > 0$ such that for any $u_+ \in H^{0,\delta'} \cap \dot{H}^{-\delta}$ with $\|\widehat{u_+}\|_{L^\infty} < \varepsilon_0$, there exist a $T \geqslant 1$ and a solution $u \in C([T, \infty); L^2(\mathbb{R}^d))$ to (NLS) which satisfies

$$\sup_{\tau \geqslant T} \tau^b \left(\|u - u_p\|_{L^\infty_t([\tau,\infty); L^2_x)} + \|u - u_p\|_{X_d(\tau)} \right) < \infty \tag{6}$$

for any $b \in (0, \delta/2)$, where

$$u_p(t) = (it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{2t}} \widehat{u_+}\left(\frac{x}{t}\right) \exp\left(-ig_1 \left| \widehat{u_+}\left(\frac{x}{t}\right) \right|^{\frac{2}{d}} \log t \right), \tag{7}$$

and

$$\| \cdot \|_{X_d(\tau)} = \begin{cases} \| \cdot \|_{L^4_t([\tau,\infty); L^\infty_x(\mathbb{R}))} & (d = 1), \\ \| \cdot \|_{L^4_t([\tau,\infty); L^4_x(\mathbb{R}^2))} & (d = 2), \\ 0 & (d = 3). \end{cases} \tag{8}$$

The uniqueness assertion of the solution is in the following sense: If $\widetilde{u} \in C([T, \infty); L^2(\mathbb{R}^d))$ solves (NLS) and satisfies (6) for some $\widetilde{T}$ and $\widetilde{b} > d/4$, then $u = \widetilde{u}$.

Remark 1. When $F(u)$ satisfies Assumption 1 and $g_1 \neq 0$, Theorem 2 ensures that (NLS) admits a *long-range* nontrivial solution. The typical example of the nonlinearity is $F(u) = |\operatorname{Re} u|^{\frac{2}{d}} \operatorname{Re} u$. In fact, the corresponding periodic function is

$$g(\theta) = |\cos\theta|^{\frac{2}{d}} \cos\theta, \quad g_n = O(|n|^{-\frac{2}{d}-2})$$

as $n \to \infty$. Further, under Assumption 1 and $g_1 = 0$, Theorem 2 implies that (NLS) admits a *short-range* nontrivial solution. We then have the example $F(u) = |\operatorname{Re} u|^{\frac{2}{d}} \operatorname{Re} u - i|\operatorname{Im} u|^{\frac{2}{d}} \operatorname{Im} u$ with

$$g(\theta) = |\cos\theta|^{\frac{2}{d}} \cos\theta - i|\sin\theta|^{\frac{2}{d}} \sin\theta, \quad g_n = O(|n|^{-\frac{2}{d}-2})$$

as $n \to \infty$. For the computation of g_n, we refer to [17].

In the case $d = 2$, 3, applying the estimate for the asymptotic profile u_p developed by [12, Appendix], it enables us to improve the regularity condition for δ'.

Theorem 2. Let $d = 2, 3$. Assume the same condition in Theorem 1 except for that of δ'. Set $\delta' = \delta$. Then the same assertion to Theorem 1 holds.

The rest of the note is organized as follows: In section 2, we state the outline of the proof of Theorem 1 based on the contraction argument via an integral equation associated with (NLS) around a prescribed asymptotic profile (7). Section 3 is devoted to estimate the external term in the integral equation which is main issue to prove Theorem 1. We also prove Theorem 2 in which the regularity condition of the final data is improved, necessary to allow (NLS) to possess the solution of which the asymptotic behavior is determined by the shape of the nonlinearity.

2 Outline of the proof

The strategy of the proof of Theorem 1 is to solve an integral equation associated with (NLS) around a prescribed asymptotic profile. Let $v(t)$ be a given profile which is specified later. We denote a new unknown w by $w := u - v$. Then we obtain the equation

$$i\partial_t w + \frac{1}{2}\Delta w = F(u) - \left(i\partial_t v + \frac{1}{2}\Delta v\right) = F(v+w) - F(v) - \left(i\partial_t v + \frac{1}{2}\Delta v - F(v)\right).$$

Since $v(t)$ is an expected asymptotic profile of $u(t)$, we may assume that $\|w(t)\|_{L^2} \to 0$ as $t \to \infty$. Therefore it follows from the Duhamel principle that $w(t)$ solves the integral equation

$$w(t) = i\int_t^\infty U(t-s)(F(v(s)+w(s)) - F(v(s)))\,ds + \mathcal{E}(t), \tag{9}$$

where $U(t) = e^{it\Delta/2}$ and

$$\mathcal{E}(t) = -i\int_t^\infty U(t-s)\left(i\partial_t v(s) + \frac{1}{2}\Delta v(s) - F(v(s))\right)ds. \tag{10}$$

Conversely, if we find a solution to (9), then $u := v + w$ is a mild solution to (NLS), that is $u(t)$ solves

$$u(t) = v(t) + i\int_t^\infty U(t-s)(F(u(s)) - F(v(s)))\,ds + \mathcal{E}(t).$$

As for the existence of the solution to (9), we establish the following:

Proposition 3 ([13, 14, 17])**.** Suppose that g is Lipschitz continuous. Then there exists a constant $\varepsilon_0 = \varepsilon_0(d, \|g\|_{\mathrm{Lip}}) > 0$ such that if $v(t)$ satisfies

$$\sup_{t\geqslant T_0} t^{\frac{d}{2}}\|v(t)\|_{L^\infty} \leqslant \varepsilon_0 \tag{11}$$

and if an external term $\mathcal{E}$ satisfies

$$\sup_{\tau\geqslant T_0} \tau^b \left(\|\mathcal{E}\|_{L_t^\infty([\tau,\infty);L_x^2)} + \|\mathcal{E}\|_{X_d(\tau)}\right) \leqslant \varepsilon \tag{12}$$

for some $T_0 \geqslant 1$, $b > d/4$ and $\varepsilon \in (0, \varepsilon_0)$, then there exists a $T = T(\|g\|_{\mathrm{Lip}}, b) \geqslant T_0$ such that (9) admits a unique solution $w \in C([T,\infty); L^2)$ satisfying

$$\sup_{\tau\geqslant T} \tau^b \left(\|w\|_{L_t^\infty([\tau,\infty);L_x^2)} + \|w\|_{X_d(\tau)}\right) \leqslant 2\varepsilon,$$

where $\|\cdot\|_{X_d(\tau)}$ is defined by (8). Moreover, $u := v+w$ is an L^2-solution to (NLS) if $v \in C(\mathbb{R}; L^2)$.

Remark 2. Under Assumption 1, $g(\theta)$ is Lipschitz continuous. Indeed, we see that

$$|g(\theta_1) - g(\theta_2)| \leqslant \sum_{n\in\mathbb{Z}} |g_n||e^{in\theta_1} - e^{in\theta_2}| \leqslant C|\theta_1 - \theta_2| \sum_{n\in\mathbb{Z}} |n||g_n|$$

for any θ_1, $\theta_2 \in \mathbb{R}$.

The proof is based on the contraction mapping argument via the Strichartz estimate. Remark that as for $d = 3$, thanks to a use of the end-point Strichartz estimate, it enables us to remove the auxiliary space $X_d(t)$. On the other hand, if $d = 1$, 2, the end-point is not acceptable to possess the Strichartz estimate, so we need the auxiliary space to close the estimate. Theorem 1 is a consequence of Proposition 3. Namely, under the assumption of Theorem 1, once the prescribed profile v and the external term $\mathcal{E}$ satisfy the assumption (11) and (12), respectively, we establish Theorem 1.

Before dealing with the above issue, we mention the key tool of our analysis in terms of $U(t)$.

Lemma 4 (Dollard decomposition). For $\phi \in \mathscr{S}(\mathbb{R}^d)$, define

$$\left(\mathcal{M}(t)\phi\right)(x) = e^{\frac{i|x|^2}{2t}}\phi(x), \quad \left(\mathcal{D}(t)\phi\right)(x) = \frac{1}{(it)^{d/2}}\phi(x/t).$$

Then $U(t) = \mathcal{M}(t)\mathcal{D}(t)\mathscr{F}\mathcal{M}(t)$.

Using the above notations, we can simply write the asymptotics u_p as

$$u_p(t) = \mathcal{M}(t)\mathcal{D}(t)\widehat{w}(t), \quad \widehat{w}(t) = \widehat{u_+}\exp\left(-ig_1|\widehat{u_+}|^{\frac{2}{d}}\log t\right). \tag{13}$$

Taking $v = u_p$, let us verify that v and $\mathcal{E}$ fulfill (11) and (12). It is immediate to see that

$$\sup_{t\geqslant T} t^{\frac{d}{2}}\|u_p(t)\|_{L^\infty} \leqslant C\|\widehat{u_+}\|_{L^\infty}$$

for any $T \geqslant 0$. This yields (11) under the assumption in Theorem 1. As for (12), we deal with the latter section.

2.1 Proof of Theorem 1

In the end of this section, we prove Theorem 1 when $\mathcal{E}$ fulfills (12). Take $b_1 \in (d/4, \delta/2)$. Then there exists $\varepsilon_0 > 0$ such that for any $u_+ \in H^{0,\delta'}$ with $\|\widehat{u_+}\|_{L^\infty} < \varepsilon_0$, there exist a $T_1 = T(b_1) \geqslant 1$ and an unique solution $u_1 \in C([T_1,\infty); L^2)$ to (NLS) with (6). Let us show that u_1 satisfies (6) for any $b \in (0, \delta/2)$. It is clear that the solution u_1 satisfies (6) for any $b \leqslant b_1$. Fix $b_2 \in (b_1, \delta/2)$. then there exist a $T_2 = T(b_2) \geqslant 1$ and an unique solution $u_2 \in C([T_2,\infty); L^2)$. Since $b_1 \leqslant b_2$, by the uniqueness property of the solution space, we have $u_1 = u_2$ on $[T_2,\infty)$. Hence u_1 satisfies (6) with b_2. This completes the proof.

3 Estimates for the external term

In this section, we shall show that the external term $\mathcal{E}$, defined by (10), satisfies (12) under the assumption in Theorem 1. To this end, we use the following identity:

Lemma 5 ([10, 13, 14, 17])**.** Denote $\mathcal{E}$ by (10). Then it holds that

$$\mathcal{E}(t) = \mathcal{E}_{\mathrm{r}}(t) + \mathcal{E}_{\mathrm{nr}}(t),$$

where $\widehat{w}$ is defined by (13),

$$\mathcal{E}_{\mathrm{r}}(t) = R(t)\widehat{w}(t) - i\int_t^\infty U(t-s)R(s)\mathcal{G}(\widehat{w}(s))\frac{ds}{s},$$

$$\mathcal{E}_{\mathrm{nr}}(t) = i\int_t^\infty U(t-s)\mathcal{N}(u_p(s))\,ds,$$

and

$$\mathcal{G}(u(t)) = g_1|u(t)|^{\frac{2}{d}}u(t), \quad \mathcal{N}(u(t)) = \sum_{n\neq 0,1} g_n|u(t)|^{1+\frac{2}{d}-n}u(t)^n,$$

$$R(t) = \mathcal{M}(t)\mathcal{D}(t)\left(\mathscr{F}\mathcal{M}(t)\mathscr{F}^{-1} - 1\right).$$

In what follows, we set $\phi_n = |\widehat{w}|^{1+\frac{2}{d}-n}\widehat{w}^n$. Define $\delta > 0$ and $\delta' > 0$ as in the assumption of Theorem 1. Also, fix $n \in \mathbb{Z}$ and we handle estimates for each n.

By the choice of the constant g_1 in the phase correction in $\widehat{w}$, the resonant part $\mathcal{E}_r$ has the factor $R(t)$ which gives us the necessary decay in time. Hence we see that $\mathcal{E}_r$ satisfies (12) as follows:

Lemma 6 ([10])**.** Let $u_+ \in H^{0,\delta'}$. It holds that

$$\begin{aligned}&\|R(t)\widehat{w}(t)\|_{L_t^\infty(\tau,\infty;L^2)} + \|R(t)\widehat{w}(t)\|_{X_d(\tau)}\\ &\leqslant C\tau^{-\frac{\delta}{2}}\langle g_1\log\tau\rangle^{\lceil\delta\rceil}\|u_+\|_{H^{0,\delta}}\left(1+\|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}}\right)^{\lceil\delta\rceil}\end{aligned}$$

and

$$\begin{aligned}&\left\|\int_t^\infty U(t-s)R(s)\mathcal{G}(\widehat{w}(s))\frac{ds}{s}\right\|_{L_t^\infty(\tau,\infty;L^2)} + \left\|\int_t^\infty U(t-s)R(s)\mathcal{G}(\widehat{w}(s))\frac{ds}{s}\right\|_{X_d(\tau)}\\ &\leqslant C|g_1|\int_\tau^\infty s^{-1-\frac{\delta}{2}}\|\phi_1(s)\|_{H^\delta}\,ds\end{aligned}$$

for any $\tau \geqslant 1$.

Let us deal with the treatment of $\mathcal{E}_{\mathrm{nr}}$. To obtain necessary time-decay, we pay attention to the difference between the phase oscillation of $\mathcal{N}(u_p)$ and that of asymptotics to the free solution $U(t)u_+$. Using the integration by part in time, one can extract the necessary decay from the difference. The following holds:

Proposition 7 ([14, 17])**.** Let $u_+ \in H^{0,\delta'} \cap \dot{H}^{-\delta}$. Then there exists $C > 0$ such that

$$\begin{aligned}&\|\mathcal{E}_{\mathrm{nr}}\|_{L_t^\infty(\tau,\infty;L^2)} + \|\mathcal{E}_{\mathrm{nr}}\|_{X_d(\tau)}\\ &\leqslant C\sum_{n\neq 0,1}|n|^{-\delta+1+\eta-\varepsilon}|g_n|\int_\tau^\infty s^{-1-\frac{\delta}{2}}\left(\|\phi_n(s)\|_{H^\delta} + \left\||\xi|^{-\delta}\phi_n(s)\right\|_{L^2}\right)ds\\ &\quad + C\sum_{n\neq 0,1}|n|^{-\delta+\eta-\varepsilon}|g_n|\int_\tau^\infty s^{-1-\frac{\delta}{2}}\left(\|\partial_s\phi_n(s)\|_{H^\delta} + \left\||\xi|^{-\delta}\partial_s\phi_n(s)\right\|_{L^2}\right)ds\\ &\quad + C\sum_{n\neq 0,1}|n|^{\frac{1}{2}-\delta+\eta-\varepsilon}|g_n|\left(\int_\tau^\infty s^{-1-2\delta}\left(\|\phi_n(s)\|_{H^\delta} + \left\||\xi|^{-\delta}\phi_n(s)\right\|_{L^2}\right)^4 ds\right)^{1/4}\end{aligned}$$

for all $\tau \geqslant 1$, where $\eta > (\delta - d/2)/2$ and $\varepsilon > 0$ is a sufficiently small constant.

3.1 Improvement of the regularity condition

To complete the proof of Theorem 1, it remains to estimate the terms for ϕ_n. In [14, 17], in order to weaken the condition on the nonlinearity for the summability of g_n, they use an interpolation technique to see

$$\left\| |\widehat{w}|^{1+\frac{2}{d}-n}\widehat{w}^n \right\|_{H^\delta} \leqslant \left\| |\widehat{w}|^{1+\frac{2}{d}-n}\widehat{w}^n \right\|_{L^2}^{1-\frac{\delta}{2}} \left\| |\widehat{w}|^{1+\frac{2}{d}-n}\widehat{w}^2 \right\|_{H^{\delta'}}^{\frac{\delta}{d}},$$

where δ' is in (5). This technique enables us to improve the upper bound of $\left\| |\widehat{w}|^{1+\frac{2}{d}-n}\widehat{w}^n \right\|_{H^\delta}$ from $O(n^{\lceil\delta\rceil})$ into $O(n^\delta)$. However, we then need stronger regularity conditions for the final data. Thus Theorem 1 has been proven under (5). Later on, Kawamoto and the author [12] improve the technique by combining the fractional Leibniz rule and an intermediate use of the interpolation. Owing to the improvement, we can weaken the regularity condition into $\delta' = \delta$. The key of the improvement is $\delta > 1$, so it is only for $d = 2$, 3.

In what follows, fix $\delta' = \delta$ and we only consider the case $d = 2$, 3.

Proposition 8 ([12])**.** Suppose $u_+ \in H^{0,\delta'}$. Take $\gamma = \delta$ if $d = 2$ and $\gamma \in (\delta, 1 + 2/d)$ if $d = 3$. Then there exists $C > 0$ such that

$$\|\phi_n(t)\|_{H^\delta} \leqslant C \langle n\rangle^\gamma \langle g_1 \log t\rangle^{\lceil\delta\rceil} \|u_+\|_{H^{0,\delta'}} \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \left(1 + \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}}\right)^{\lceil\delta\rceil}$$

for any $t \geqslant 1$.

Proposition 9 ([12])**.** Suppose $u_+ \in H^{0,\delta'}$. Then there exists $C > 0$ such that

$$\|\partial_t\phi_n(t)\|_{H^\delta} \leqslant C|g_1| \langle n\rangle^{1+\delta} t^{-1} \langle g_1 \log t\rangle^{\lceil\delta\rceil} \|u_+\|_{H^{0,\delta'}} \|\widehat{u_+}\|_{L^\infty}^{\frac{4}{d}} \left(1 + \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}}\right)^{\lceil\delta\rceil}$$

for any $t \geqslant 1$.

To prove the above propositions, we need the following:

Proposition 10 ([5, 12])**.** Suppose $u_+ \in H^{0,\delta}$. Then there exists $C > 0$ such that

$$\|\widehat{w}(t)\|_{H^\delta} \leqslant C \langle g_1 \log t\rangle^{\lceil\delta\rceil} \|u_+\|_{H^{0,\delta}} \left(1 + \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}}\right)^{\lceil\delta\rceil}$$

for any $t \geqslant 1$.

In order to mention about the idea of the improvement of the regularity condition, we give a sketch of the proof of Proposition 8.

Sketch of the proof of Proposition 8. For simplicity, we only treat the case $d = 2$. A direct calculation shows that

$$\nabla\phi_n(s) = \frac{1}{2}\left(1 + \frac{2}{d} + n\right) |w|^{\frac{2}{d}-n+1} w^{n-1}\nabla w + \frac{1}{2}\left(1 + \frac{2}{d} - n\right) |w|^{\frac{2}{d}-n-1} w^{n+1}\nabla \overline{w}.$$

The fractional Leibniz rule gives us

$$\left\| |w|^{\frac{2}{d}-n+1} w^{n-1}\nabla w \right\|_{\dot{H}^{\delta-1}} \leqslant C\left\| |\nabla|^{\delta-1}\left(|w|^{\frac{2}{d}-n+1} w^{n-1}\right) \right\|_{L^{\frac{2\delta}{\delta-1}}} \|\nabla w\|_{L^{2\delta}}$$

$$+\left\||w|^{\frac{2}{d}-n+1}w^{n-1}\right\|_{L^\infty}\|w\|_{\dot H^\delta}.$$

We then see from the interpolation that

$$\begin{aligned}\left\||\nabla|^{\delta-1}\left(|w|^{\frac{2}{d}-n+1}w^{n-1}\right)\right\|_{L^{\frac{2\delta}{\delta-1}}} &\leqslant C\left\||w|^{\frac{2}{d}-n+1}w^{n-1}\right\|_{L^\infty}^{2-\delta}\left\|\nabla\left(|w|^{\frac{2}{d}-n+1}w^{n-1}\right)\right\|_{L^{2\delta}}^{\delta-1}\\ &\leqslant C\langle n\rangle^{\delta-1}\|w\|_{L^\infty}^{(\frac{2}{d}+1)(2-\delta)}\left\||w|^{\frac{2}{d}+1-n}\nabla w\right\|_{L^{2\delta}}^{\delta-1}\\ &\leqslant C\langle n\rangle^{\delta-1}\|w\|_{L^\infty}^{\frac{2}{d}+1-\delta}\|\nabla w\|_{L^{2\delta}}^{\delta-1}.\end{aligned}$$

Combining the above with Proposition 10, together with the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{L^{2\delta}}\leqslant C\|w\|_{L^\infty}^{1-\frac{1}{\delta}}\|w\|_{\dot H^\delta}^{\frac{1}{\delta}},$$

the desired assertion holds since the last term is similar. □

By Lemma 6 and Proposition 8, there exist a constant $C=C\left(g_1,d,\|u_+\|_{H^{0,\delta'}}\right)>0$ and $b_0=b_0(\delta)>0$ such that

$$\tau^b\left(\|\mathcal{E}_{\mathrm{r}}\|_{L_t^\infty([\tau,\infty);L_x^2)}+\|\mathcal{E}_{\mathrm{r}}\|_{X_d(\tau)}\right)\leqslant C\tau^{-b_0} \tag{14}$$

for any $\tau\geqslant 1$. Further, by Lemma 5, collecting (14), Proposition 7, Proposition 8 and Proposition 9, there exist $C=C\left(\{g_n\},\eta,d,\|u_+\|_{H^{0,\delta'}\cap\dot H^{-\delta}}\right)>0$ and $b_0=b_0(\delta)>0$ such that

$$\tau^b\left(\|\mathcal{E}\|_{L_t^\infty([\tau,\infty);L_x^2)}+\|\mathcal{E}\|_{X_d(\tau)}\right)\leqslant Ct^{-b_0}$$

for any $\tau\geqslant 1$. Hence we conclude that there exists $T_0\geqslant 1$ such that $\mathcal{E}$ fulfills (12) under the assumption of Theorem 2.

Acknowledgements

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. Also, the author was supported by JSPS KAKENHI Grant Numbers 19K14580 and 22K13941.

References

[1] Jacqueline E. Barab, *Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation*, J. Math. Phys. **25** (1984), no. 11, 3270–3273. MR761850

[2] J. Ginibre and T. Ozawa, *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n\geq 2$*, Comm. Math. Phys. **151** (1993), no. 3, 619–645. MR1207269

[3] Nakao Hayashi and Pavel I. Naumkin, *Large time behavior for the cubic nonlinear Schrödinger equation*, Canad. J. Math. **54** (2002), no. 5, 1065–1085. MR1924713

[4] ———, *On the asymptotics for cubic nonlinear Schrödinger equations*, Complex Var. Theory Appl. **49** (2004), no. 5, 339–373. MR2073463

[5] ———, *Domain and range of the modified wave operator for Schrödinger equations with a critical nonlinearity*, Comm. Math. Phys. **267** (2006), no. 2, 477–492. MR2249776

[6] ———, *Nongauge invariant cubic nonlinear Schrödinger equations*, Pac. J. Appl. Math. **1** (2008), no. 1, 1–16. MR2467127

[7] ———, *Global existence for the cubic nonlinear Schrödinger equation in lower order Sobolev spaces*, Differential Integral Equations **24** (2011), no. 9-10, 801–828. MR2850366

[8] ———, *Logarithmic time decay for the cubic nonlinear Schrödinger equations*, Int. Math. Res. Not. IMRN **14** (2015), 5604–5643. MR3384451

[9] Nakao Hayashi, Pavel I. Naumkin, Akihiro Shimomura, and Satoshi Tonegawa, *Modified wave operators for nonlinear Schrödinger equations in one and two dimensions*, Electron. J. Differential Equations (2004), No. 62, 16 pp. (electronic). MR2047418

[10] Nakao Hayashi, Huimei Wang, and Pavel I. Naumkin, *Modified wave operators for nonlinear Schrödinger equations in lower order Sobolev spaces*, J. Hyperbolic Differ. Equ. **8** (2011), no. 4, 759–775. MR2864547

[11] Tosio Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. **46** (1987), no. 1, 113–129. MR877998

[12] Masaki Kawamoto and Hayato Miyazaki, *Long-range scattering for a critical homogeneous type nonlinear Schrödinger equation with time-decaying harmonic potentials*, preprint, available at `arXiv:1801.07895`.

[13] Satoshi Masaki, *A survey on long range scattering for Schrödinger equation and Klein-Gordon equation with critical nonlinearity of non-polynomial type*, Regularity and asymptotic analysis for critical cases of partial differential equations, 2020, pp. 103–135. MR4279656

[14] Satoshi Masaki and Hayato Miyazaki, *Long range scattering for nonlinear Schrödinger equations with critical homogeneous nonlinearity*, SIAM J. Math. Anal. **50** (2018), no. 3, 3251–3270. MR3815545

[15] ———, *Nonexistence of scattering and modified scattering states for some nonlinear Schrödinger equation with critical homogeneous nonlinearity*, Differential Integral Equations **32** (2019), no. 3-4, 121–138. MR3909981

[16] ———, *Global Behavior of Solutions to Generalized Gross-pitaevskii equation*, Differ. Equ. Dyn. Syst. (2022).

[17] Satoshi Masaki, Hayato Miyazaki, and Kota Uriya, *Long-range scattering for nonlinear Schrödinger equations with critical homogeneous nonlinearity in three space dimensions*, Trans. Amer. Math. Soc. **371** (2019), no. 11, 7925–7947. MR3955539

[18] Satoshi Masaki and Jun-ichi Segata, *Existence of a minimal non-scattering solution to the mass-subcritical generalized Korteweg–de Vries equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **35** (2018), no. 2, 283–326. MR3765544

[19] Hayato Miyazaki and Motohiro Sobajima, *Lifespan of solutions to nonlinear Schrödinger equations with general homogeneous nonlinearity of the critical order*, Advances in harmonic analysis and partial differential equations, 2020, pp. 197–207. MR4174747

[20] Kazunori Moriyama, Satoshi Tonegawa, and Yoshio Tsutsumi, *Wave operators for the nonlinear Schrödinger equation with a nonlinearity of low degree in one or two space dimensions*, Commun. Contemp. Math. **5** (2003), no. 6, 983–996. MR2030566

[21] P. I. Naumkin, *The dissipative property of a cubic non-linear Schrödinger equation*, Izv. Ross. Akad. Nauk Ser. Mat. **79** (2015), no. 2, 137–166. MR3352593

[22] Pavel I. Naumkin and Isahi Sánchez-Suárez, *On the critical nongauge invariant nonlinear Schrödinger equation*, Discrete Contin. Dyn. Syst. **30** (2011), no. 3, 807–834. MR2784622

[23] Tohru Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys. **139** (1991), no. 3, 479–493. MR1121130

[24] Akihiro Shimomura, *Nonexistence of asymptotically free solutions for quadratic nonlinear Schrödinger equations in two space dimensions*, Differential Integral Equations **18** (2005), no. 3, 325–335. MR2122723

[25] Akihiro Shimomura and Satoshi Tonegawa, *Long-range scattering for nonlinear Schrödinger equations in one and two space dimensions*, Differential Integral Equations **17** (2004), no. 1-2, 127–150. MR2035499

[26] Akihiro Shimomura and Yoshio Tsutsumi, *Nonexistence of scattering states for some quadratic nonlinear Schrödinger equations in two space dimensions*, Differential Integral Equations **19** (2006), no. 9, 1047–1060. MR2262096

[27] Walter Strauss, *Nonlinear scattering theory*, Scattering theory in mathematical physics, 1974, pp. 53–78.

[28] Hideaki Sunagawa, *Large time behavior of solutions to the Klein-Gordon equation with nonlinear dissipative terms*, J. Math. Soc. Japan **58** (2006), no. 2, 379–400. MR2228565

[29] Yoshio Tsutsumi and Kenji Yajima, *The asymptotic behavior of nonlinear Schrödinger equations*, Bull. Amer. Math. Soc. (N.S.) **11** (1984), no. 1, 186–188. MR741737

Faculty of Education
Kagawa University
1-1 Saiwai-cho, Takamatsu, Kagawa
JAPAN
E-mail address: `miyazaki.hayato@kagawa-u.ac.jp`