Topics on the essential self-adjointness for Klein-Gordon type operators on spacetimes*

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Abstract

We discuss recent results on the essential self-adjointness of Klein-Gordon type operators on several classes of spacetimes. These results are based on joint works with Kouichi Taira (Ritsumeikan University).

1 Introduction

We consider the second order operator of the form:

$$P = \sum_{j,k=1}^{n} D_j g^{jk}(x) D_k + \frac{1}{2}\sum_{j=1}^{n}(D_j u_j(x)\xi_j + u_j(x)D_j) + u_0(x),$$

on an n-dimensional manifold X, where $D_j = -i\frac{\partial}{\partial x_j}$ (in a local coordinate system). We suppose all the coefficients are real-valued C^∞ functions. The top order coefficients $\{g^{jk}(x)\}$ is a Lorentzian cometric, and hence we suppose it is non-degenerate for all $x \in X$.

Example: (Wave operator on an asymptotically Minkowski metric) We set $X = \mathbb{R}^4$,

$$(g_0^{jk}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We suppose $g^{jk}(x) \to g_0^{jk}$ as $|x| \to \infty$. $u_0(x)$ is the mass term, and $u_1(x)$ is the electric potential, and $(u_2(x), u_3(x), u_4(x))$ is the vector potential.

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Motivations and related works:

In the construction of field theory on curved a spacetime, the Feynman propagator is essential. It is widely known the Feynman propagator is given by $(P - i0)^{-1}$, at least formally, but the self-adjointness of P had not been known before Vasy [13] and Nakamura-Taira [8], which prove the essential self-adjointness for asymptotically flat spacetimes. We note Gell-Redman-Haber-Vasy [5], Gérard-Wrochna [6, 7] and Dereziński-Siemssen [2, 3] used different method to construct Feynman propagator, but the essential self-adjointness was conjectured.

Recently, a simplified proof for the asymptotically flat spacetimes is given in Nakamura-Taira [9], and the proof for the asymptotically static spacetimes is given in Nakamura-Taira [10]. Given the essential self-adjointness, Vasy [13] and Taira [12] proved the limiting absorption principle for asymptotically flat spacetimes. There are many non essentially self-adjoint metric with pseudo Riemannian metric without boundary, and thus this problem is far from obvious, and there are many open questions related to geometry of pseudo Riemannian metric. See, e.g., Colin de Verdière and Le Bihan [1] and Taira [11].

Classical Mechanics:

In the proof of these results, the classical mechanics generated by the Hamiltonian

$$p_2(x, \xi) = \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k$$

plays essential roles. We note this is essentially the geodesic flow in the differential geometry. We denote $\exp(tH_{p_2})$ be the Hamilton flow on T^*X generated by the symbol $p_2(x, \xi)$, and we write

$$(y(t, x_0, \xi_0), \eta(t, x_0, \xi_0)) = \exp(tH_{p_2})(x_0, \xi_0)$$

for $t \in \mathbb{R}$, $(x_0, \xi_0) \in T^*X$. Along the flow, $p_2(y(t), \eta(t))$ is invariant, and since our metric is not positive definite, it can take all the real numbers. In particular, a solution (geodesic) with $p_2(y(t), \eta(t)) = 0$ is called a *null geodesic*, and especially important in the following argument. We recall microlocal singularities of a solution to $Pu = 0$ propagate along the null-geodesics (Propagation of Singularities Theorem). If $p_2(y(t), \eta(t)) > 0$ then a solution is called *time-like*, and if $p_2(y(t), \eta(t)) < 0$ then a solution is called *space-like*.

Asymptotically flat spacetimes

We consider the case $X = \mathbb{R}^n$, and suppose $g^{jk}(x) \to g_0^{ij}$ in the following sense, where g_0^{jk} is a nondegenerate symmetric matrix.

Assumption A. For all j,k, $g^{jk}(x)$, $u_j(x)$, $x\in\mathbb{R}^n$, are real-valued smooth functions. Moreover, there exists $\mu>0$ such that for any $\alpha\in\mathbb{Z}_+^n$

$$\begin{aligned}\left|\partial_x^\alpha(g^{jk}(x)-g_0^{jk})\right| &\le C_\alpha\langle x\rangle^{-\mu-|\alpha|}, \quad x\in\mathbb{R}^n,\ j,k=1,\dots,n,\\ \left|\partial_x^\alpha u_j(x)\right| &\le C_\alpha\langle x\rangle^{-\mu-|\alpha|}, \quad x\in\mathbb{R}^n,\ j=0,\dots,n,\end{aligned}$$

with some $C_\alpha>0$.

Assumption B (Null non-trapping condition)**.** If $(x_0,\xi_0)\in p_2^{-1}(\{0\})$ and $\xi_0\ne 0$, then $|y(t,x_0,\xi_0)|\to\infty$ as $|t|\to\infty$.

Theorem 1 (Vasy, N-Taira)**.** *Suppose Assumptions A and B. Then P is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.*

Asymptotically static spacetimes

We consider the case $X=\mathbb{R}\times M$, where M is a compact Riemannian manifold without boundary. We denote the Riemannian metric on M by $q_0=\sum_{i,j=1}^n q_{0,ij}(x)dx_idx_j$, locally in $x\in M$. We write $q_0(x,\xi)=\sum_{j,k=1}^n q_0^{ij}(x)\xi_j\xi_k$. Let $p(t,x,\tau,\xi)$ be the symbol of P on T^*X, and it has the form:

$$p(t,x,\tau,\xi)=\tau^2-q_0(x,\xi)+q(t,x,\tau,\xi),$$

Assumption C. $q(t,x,\tau,\xi)$ is smooth in (t,x), quadratic in (τ,ξ), and there is $\mu>1$ such that for any multi-index $\alpha\in\mathbb{Z}_+^n$ and $k\in\mathbb{Z}_+$,

$$\left|\partial_t^k\partial_x^\alpha q(t,x,\tau,\xi)\right|\le C_{k,\alpha}\langle t\rangle^{-\mu}(1+\tau^2+|\xi|^2).$$

We denote a geodesic by

$$(t(s),x(s),\tau(s),\xi(s))=\exp(sH_p)(t_0,x_0,\tau_0,\xi_0),\quad (t_0,x_0,\tau_0,\xi_0)\in T^*X.$$

Assumption D. If $p(t_0,x_0,\tau_0,\xi_0)=0$, then either $t(s)\to\pm\infty$ as $s\to\pm\infty$ or $t(s)\to\mp\infty$ as $s\to\pm\infty$.

Theorem 2 (N-Taira)**.** *Suppose Assumptions C and D. Then P is essentially self-adjoint on $C_0^\infty(X)$.*

Asymptotically expanding spacetimes (work in progress)

We consider the case $X=\mathbb{R}\times M$ as in the asymptotically static spacetimes. Let $0<\alpha<1$, and $p(t,x,\tau,\xi)$ be the symbol of P on T^*X, and it has the form:

$$p(t,x,\tau,\xi)=\tau^2-|t|^{-2\alpha}q_0(x,\xi)+q(t,x,\tau,\xi),\quad |t|>1.$$

Assumption E. $q(t,x,\tau,\xi)$ is smooth in (t,x), quadratic in (τ,ξ), and there is $\mu>1+\alpha$ such that for any multi-index $\alpha\in\mathbb{Z}_+^n$ and $k\in\mathbb{Z}_+$,

$$\left|\partial_t^k\partial_x^\alpha q(t,x,\tau,\xi)\right|\le C_{k,\alpha}\langle t\rangle^{-\mu}(1+\tau^2+|\xi|^2).$$

Assumption F. Any null geodesic satisfies either $t(s) \to \pm\infty$ as $s \to \pm\infty$ or $t(s) \to \mp\infty$ as $s \to \pm\infty$.

Theorem 3 (in progress)**.** *Suppose Assumptions E and F. Then P is essentially self-adjoint on $C_0^\infty(X)$.*

Remark 4. Probably the same hold if $-1 < \alpha \leq 0$, i.e., asymptotically shrinking cases, or cusp cases. The case $\alpha \geq 1$ is open for the moment. The de Sitter solution corresponds to the e^{-ct} weight case, where M is a constant curvature Riemannian manifold.

2 Asymptotically flat spacetimes

2.1 The first reduction

In order to show the essential self-adjointness of a symmetric operator P on $C_0^\infty(X)$, it is sufficient to show $\mathrm{Ker}(P^* - z_\pm) = \{0\}$ for some $z_\pm \in \mathbb{C}$, $\pm\mathrm{Im}(z_\pm) > 0$. We concentrate on the case $z = z_+$ in the following. The other case is similar. Let $\psi \in \mathrm{Ker}(P^* - z)$, then it implies

$$\psi \in L^2(\mathbb{R}^n), \quad (P - z)\psi = 0 \text{ in the distribution sense.} \tag{1}$$

Our theorem is proved if (1) implies $\psi = 0$. The first step of the proof is remark that it follows if we know ψ is a sufficiently good function. This simple remark is due to Vasy [13].

Lemma 5. *If ψ satisfies* (1) *and $\psi \in H^{1/2,-1/2}(\mathbb{R}^n)$ then $\psi = 0$.*

Here $H^{s,t}(\mathbb{R}^n)$ is the weighted Sobolev space $H^{s,t}(\mathbb{R}^n) = \langle x\rangle^{-t}\langle D_x\rangle^{-s}\big[L^2(\mathbb{R}^n)\big]$.

2.2 Semiclassical quantization of the symbol:

We quantize the symbols (functions on $\mathbb{R}^{2n}$) using the semiclassical quantization:

$$\mathrm{Op}_h(a)u(x) = a^W(x, hD_x)u(x) = (2\pi h)^{-n} \iint a(\tfrac{x+y}{2}, \xi)e^{i(x-y)\xi/h}u(x)dyd\xi$$

for $a \in C^\infty(\mathbb{R}^{2n})$ and $u \in \mathcal{S}(\mathbb{R}^n)$, $h > 0$.

2.3 Asymptotically free classical mechanics:

We recall the symbol of the operator P is $p(x, \xi)$, and it is asymptotically free in the sense $p(x, \xi) \to p_0(\xi)$ as $|x| \to \infty$, where $p_0(\xi) = \sum_{j,k=1}^n g_0^{jk}\xi_j\xi_k$. This implies, with the non-trapping condition, the Hamilton flow asymptotically converges to a free motion, i.e.,

$$y(t) \sim tv(\xi), \quad \text{as } |x| \to \infty, \quad \text{where } v(\xi) = \partial_\xi p_0(\xi) = \left(2\sum_{k=1}^n g_0^{jk}\xi_k\right)_j.$$

2.4 In-coming escaping function:

With this observation, we can construct a function $b_-(x,\xi)$ with the following properties: We write $\hat{x} = x/|x|$, and we denote

$$\Gamma_-(\delta,\sigma,R) = \big\{(x,\xi)\;\big|\; 1-\delta \le |\xi|^2 \le 1+\delta, \hat{x}\cdot\hat{v}(\xi) \le \sigma, |x| \ge R\big\}$$

where $0 < \delta \ll 1$, $\sigma \in [-1,1]$ and $R \gg 0$.

Lemma 6. *For $0 < \gamma \ll 0$, $0 < \delta_0 \ll 1$ and $0 < \sigma' < \sigma < 1$, there are R_0 and $C_0 > 0$ such that for $R \ge R_0$ there exists $b_- \in S^\infty(\mathbb{R}^{2n})$ such that* $\mathrm{supp}[b_-] \subset \Gamma_-(4\delta_0,\sigma,C_0R)$*, $b_-(x,\xi) \le C\langle x\rangle^\gamma$, $(x,\xi)\in\mathbb{R}^{2n}$, and*

$$C^{-1}\langle x\rangle^\gamma \le b_-(x,\xi), \quad (x,\xi)\in\Gamma_-(\delta_0,\sigma',R),$$

$$\{p_2, b_-\} \le -c_0\langle x\rangle^{-1}b_-, \quad \textit{with } c_0 > 0.$$

2.5 In-coming smoothness:

We set $B_- = \mathrm{Op}_h(b_-)$. Quantization of the inequality in Lemma 6 implies an inequality of the form (with the sharp Gårding inequality) :

$$-i[B_-^*B_-, P] \le -\frac{c}{h}B_-^*\langle x\rangle^{-1}B_- + E^*E,$$

where E is a lower order error term (order 0 in h, with suitable microlocal support conditions). We note the principal term of P is given by $h^{-2}p_x(x,\xi)$ with respect to the semiclassical quantization. Then, by standard algebraic computations, we learn

$$\frac{c}{2h}\big\|\langle x\rangle^{-1/2}B_-\varphi\big\|^2 + 2(\mathrm{Im}z)\|B_-\varphi\|^2 \le \frac{h}{2c}\big\|\langle x\rangle^{1/2}B_-(P-z)\varphi\big\|^2 + \|E\varphi\|^2$$

for $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $z\in\mathbb{C}$, $\mathrm{Im}z > 0$. If $\psi \in L^2$ and $(P-z)\psi = 0$, then, at least formally, we have

$$\frac{c}{2}\big\|\langle x\rangle^{-1/2}B_-\psi\big\|^2 + 2h(\mathrm{Im}z)\|B_-\psi\|^2 \le h\|E\psi\|^2$$

and this implies $\big\|\langle x\rangle^{-1/2}B_-\psi\big\| = O(h^{1/2})$ and $\|B_-\psi\| = O(1)$. In fact, the error operator E has the same form as B_-, and we can iterate this procedure to show

$$\big\|\langle x\rangle^{-1/2}B_-\psi\big\| = O(h^N), \quad \|B_-\psi\| = O(h^N) \quad \text{with any } N.$$

These estimates imply ψ is smooth in the in-coming region, and $\langle x\rangle^\gamma\psi \in H^\infty$, in a microlocal sense.

2.6 Propagation of singularities and the local smoothness:

By the nontrapping condition, each null geodesic eventually arrive at the in-coming region as $t \to -\infty$. Thus, by the propagation of singularities theorem (of Hörmander) and the above smoothness in the in-coming region, we learn ψ is smooth locally overall.

2.7 Out-going escaping function and out-going smoothness

It remains to show the smoothness and the Sobolev estimate for the out-going region. We modify the argument for the incoming region. We set

$$\Gamma_+(\delta, \sigma, R) = \big\{(x, \xi) \;\big|\; 1-\delta \le |\xi|^2 \le 1+\delta, \hat{x} \cdot \hat{v}(\xi) \ge \sigma, |x| \ge R\big\}$$

where $0 < \delta \ll 1$, $\sigma \in [-1, 1]$ and $R \gg 0$.

Lemma 7. *For $0 < \gamma \ll 0$, $0 < \delta_0 \ll 1$ and $-1 < \sigma < \sigma' < 0$, there is R_0 and $C_0 > 0$ such that for $R \ge R_0$ there exists $b_+ \in S^\infty(\mathbb{R}^{2n})$ s.t. $\operatorname{supp}[b_+] \subset \Gamma_+(4\delta_0, \sigma, C_0 R)$, $b_+(x, \xi) \le C\langle x\rangle^{-\gamma}$, and*

$$C^{-1}\langle x\rangle^{-\gamma} \le b_+(x, \xi), \quad \textit{for } (x, \xi) \in \Gamma_+(\delta_0, \sigma', R),$$

$$\{p_2, b_+\} \le -c_0\langle x\rangle^{-1} b_+ + f, \quad \textit{with } c_0 > 0,$$

where $f \in C^\infty(\mathbb{R}^{2n})$ such that f is bounded and

$$\operatorname{supp}[f] \subset \big\{(x, \xi) \;\big|\; 1-4\delta \le |\xi|^2 \le 1+4\delta, |x| \le C_0 R \textit{ or } \sigma \le \beta(x, \xi) \le \sigma'\big\}.$$

By a quantization, we have

$$-i[B_+^* B_+, P] \le -\frac{c}{h} B_+^* \langle x\rangle^{-1} B_+ + F + E^* E,$$

where $F = \mathrm{Op}_h(\mathrm{Re}(f b_+))$, and E is a lower order error term (as in the in-coming case). This implies

$$\frac{c}{2}\big\|\langle x\rangle^{-1/2} B_+\psi\big\|^2 + 2h(\mathrm{Im} z)\|B_+\psi\|^2 \le h\langle \psi, F\psi\rangle + h\|E\psi\|^2$$

where $\psi \in L^2$, $(P - z)\psi = 0$. By the support property of f, we know $\langle \psi, F\psi\rangle = O(h^\infty)$, and the same argument as in the in-coming case applies. Thus, we have $\|B_+\psi\| = H(h^\infty)$. This implies $\langle x\rangle^{-\gamma}\psi \in H^\infty$ in the out-going region.

2.8 Proof of Theorem 1

Combining the estimates in in-coming region, out-going region and compact region, respectively, we conclude $\psi \in H^{N,-\gamma}$ for any N. Thus, by the first reduction step, we conclude $\psi = 0$ and hence P is essentially self-adjoint. □

3 Asymptotically static spacetimes

3.1 Classical mechanics

We note, since $t = x_0 \in \mathbb{R}$ is a coordinate, we use $s \in \mathbb{R}$ as the parameter of the geodesics/classical trajectories. The classical Hamiltonian asymptotically converges to

$$p_0(t, x, \tau, \xi) = \tau^2 - q_0(x, \xi) \quad \text{on } T^*(\mathbb{R} \times M) = \{(t, x, \tau, \xi)\},$$

as $t \to \pm\infty$, and hence p_0 determines the asymptotic motion. It is easy to see

$$\exp sH_{p_0}(t_0, x_0, \tau_0, \xi_0) = (t_0 + 2s, \tau_0, \exp sH_{q_0}(x_0, \xi_0)), \quad s \in \mathbb{R},$$

(with an obvious change of order of variables). Thus, the asymptotic motion is *not* free, but the free motion in t times (tensor product) the Hamilton flow on M generated by q_0.

We need to construct an escaping function which is decreasing along the asymptotic motion, but the asymptotic motion is not free, so we need some care. On the other hand, q_0 is an invariant for the flow, and we can handle it as if it is a constant.

3.2 In-coming escape function and its quantization

We can construct an escaping function of the form: $b_- = b_{-,-} + b_{-,+}$ such that

$$\text{supp}[b_{-,\pm}] \subset \{\mp t \geq R\} \cap \{|\tau - (\pm 1)| \leq 2\delta\} \cap \{|q_0(x, \xi) - 1| \leq 2\delta\},$$

$$b_{-,\pm} = |t|^\gamma \quad \text{on} \quad \{\mp t \geq 2R\} \cap \{|\tau - (\pm 1)| \leq \delta\} \cap \{|q_0(x, \xi) - 1| \leq \delta\},$$

and

$$\{p_2, b_{-,\pm}\} \leq -c_0 \langle t \rangle^{-1} b_{-,\pm}, \quad c_0 > 0.$$

Moreover, they have the form

$$b_{-,\pm}(t, x, \tau, \xi) = b^1_{-,\pm}(t, \tau) b^2_{-,\pm}(t, q_0(x, \xi)).$$

We note the support is inside the in-coming regions. Moreover, since we are only interested in null geodesics, we may suppose $\tau^2 - q_0(x, \xi) \sim 0$, and the regions cover the relevant regions after semiclassical scaling. Here, the only large parameter of the coordinate is t, and we consider the decay in t, and not in x. We quantize $b_{-,\pm}$ by

$$B_{-,\pm} = \text{Op}_h(b^1_{-,\pm}) b^2_{-,\pm}(t, h^2 Q_0),$$

where Q_0 is the Laplace-Beltrami operator on M corresponding to q_0, and $\text{Op}_h(\cdot)$ is the semiclassical quantization with respect to the t-variable. $b^2_{-,\pm}(t, h^2 Q_0)$ is defined by the functional calculus.

We note $b^2_{-,\pm}(t, h^2 Q_0)$ is a pseudodifferential operator, and its principal symbol is given by $b^2_{-,\pm}(t, q_0(x,\xi))$. However, it is not the same as the quantization of the principal symbol, and the lower order error does *not* decay in t. We need to use this quantization, since the asymptotic motion is still affected by q_0, and not free. In practice, it is important that Q_0 commutes with $B_{-,\pm}$.

3.3 Proof of Theorem 2

Using B_- and the argument analogous to the asymptotically flat case, we can show, if

$$\psi \in L^2, \quad (P - z)\psi = 0, \quad \operatorname{Im} z > 0,$$

then ψ is smooth on the incoming region:

$$\Omega_- = \big\{(t,\tau) \,\big|\, \mp t \geq 2R, \pm\tau > 0\big\},$$

and $\langle t\rangle^{\gamma}\psi \in H^\infty(\Omega_-)$. Then by the propagation of singularities theorem and the nontrapping condition again, we learn the local smoothness, i.e., $\psi \in C^\infty(X)$. We then use a similar construction of the out-going escaping function, and an argument analogous to the asymptotic free case to show ψ is smooth on the out-going region:

$$\Omega_+ = \big\{(t,\tau) \,\big|\, \pm t \geq 2R, \pm\tau > 0\big\},$$

and $\langle t\rangle^{-\gamma}\psi \in H^\infty(\Omega_+)$. Now we have $\psi \in H^{-\gamma,N}(X)$ with any N, and by the reduction argument, we conclude $\psi = 0$. Thus, P is essentially self-adjoint. □

4 Asymptotically expanding spacetimes

This part is still work in progress, and we omit the incomplete explanation in this proceeding.

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