Irreducible modules of the cyclotomic KLR algebras

Louise Sutton
Okinawa Institute of Science and Technology
Okinawa, Japan 904-0495

1 Introduction

The representation theory of the cyclotomic quiver Hecke algebras or cyclotomic KLR algebras $\mathscr{R}_n^\Lambda$ of type A is governed by Specht modules that are indexed by multipartitions, from which the irreducible $\mathscr{R}_n^\Lambda$-modules arise as certain quotients. One of the main open problems is to determine the dimensions of the irreducible $\mathscr{R}_n^\Lambda$-modules. A natural question is to ask when do Specht modules coincide with irreducible $\mathscr{R}_n^\Lambda$-modules. A classification of irreducible Specht modules is known for the Iwahori–Hecke algebras of type A when $q \neq -1$. In joint work with Matthew Fayers, we use this classification to study irreducible Specht modules for the Iwahori–Hecke algebras of type B. Our work also depends on joint work with Robert Muth, Thomas Nicewicz and Liron Speyer that studies core combinatorics, skew Specht modules, and analogous *skew cyclotomic quiver Hecke algebras*.

2 Background

We give an overview of KLR algebras, Specht modules and the associated combinatorics.

2.1 Lie theoretic notation

Throughout, $\mathbb{F}$ will denote an arbitrary field of characteristic $p \geqslant 0$. We denote the multiplicative order of $q \in \mathbb{F}^\times$ by $e \in \{\infty\} \cup \{2, 3, \dots\}$, called the *quantum characteristic*. We set $I := \mathbb{Z}/e\mathbb{Z}$, which we identify with the set $\{0, 1, \dots, e-1\}$ if e is finite, otherwise we identify I with $\mathbb{Z}$. We let Γ be the (type A_∞ or $A^{(1)}_{e-1}$) quiver with vertex set I and an arrow $i \to i-1$ for each $i \in I$.

We recall standard notation for the Kac–Moody algebra associated to the generalised Cartan matrix $(a_{ij})_{i,j\in I}$ following [15]. We have simple roots $\{\alpha_i \mid i \in I\}$, fundamental dominant weights $\{\Lambda_i \mid i \in I\}$, and the invariant symmetric bilinear form $(\ ,\ )$ such that $(\alpha_i, \alpha_j) = a_{i,j}$ and $(\Lambda_i, \alpha_j) = \delta_{ij}$, for all $i, j \in I$. Let Φ_+ be the set of positive roots, and let $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$ be the positive cone of the root lattice. If $\alpha = \sum_{i\in I} c_i\alpha_i \in Q_+$, then we define the *height of* α to be $\mathrm{ht}(\alpha) = \sum_{i\in I} c_i$.

An *e-multicharge* is an ordered pair $\kappa = (\kappa_1, \dots, \kappa_\ell) \in I^\ell$. We define its associated dominant weight Λ of level ℓ to be $\Lambda = \Lambda_\kappa := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell}$.

2.2 The symmetric group and Hecke algebras

Let $\mathfrak{S}_n$ be the symmetric group on n letters. Associated to each complex reflection group $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$ of type $G(\ell, 1, n)$ is an associated *Ariki–Koike algebra*, or alternatively called a *cyclotomic Hecke algebra*. For our purposes, we are interested in the Hecke algebras associated to the symmetric group $\mathfrak{S}_n$ and to the signed symmetric group $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$, namely the Iwahori–Hecke algebras of type A and B, respectively.

Definition 2.1. Let $q \in \mathbb{F}^\times$. The *Hecke algebra of the symmetric group* or *Iwahori–Hecke algebra of type A*, denoted $\mathscr{H}_{\mathbb{F},q}(\mathfrak{S}_n)$, is the unital, associative $\mathbb{F}$-algebra with generating set $\{T_1, \dots, T_{n-1}\}$ subject to the relations

$$
\begin{aligned}
(T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leqslant i < n;\\
T_i T_j &= T_j T_i && \text{for } |i-j| > 1;\\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2.
\end{aligned}
$$

Note that we recover the group algebra of the symmetric group when $q = 1$.

Definition 2.2. Let $q, Q_1, Q_2 \in \mathbb{F}^\times$. Then the *Hecke algebra of the signed symmetric group* or the *Iwahori–Hecke algebra of type B*, denoted $\mathscr{H}_{\mathbb{F},q,Q_1,Q_2}((\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n)$, is the unital associative $\mathbb{F}$-algebra with generating set $\{T_0, T_1, \dots, T_{n-1}\}$ subject to the relations

$$
\begin{aligned}
(T_0 - Q_1)(T_0 - Q_2) &= 0 &&\\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0 &&\\
(T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leqslant i < n;\\
T_i T_j &= T_j T_i && \text{for } |i-j| > 1;\\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leqslant i \leqslant n-2.
\end{aligned}
$$

2.3 Young diagrams and tableaux

A *composition* λ of n is a sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| := \sum_{i=1}^{\infty} \lambda_i = n$. We say that a *partition of* n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_i \geqslant \lambda_{i+1}$ for all $i \geqslant 1$. We write $\varnothing$ for the empty partition $(0, 0, \dots)$. We denote by $\mathscr{P}_n$ the set of all partitions of n, and let $\mathscr{P} := \bigcup \mathscr{P}_n$.

An *ℓ-multipartition* λ of n is an ℓ-tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of partitions such that $|\lambda| = \sum_{i=1}^{\ell} |\lambda^{(i)}| = n$. We refer to $\lambda^{(i)}$ as the *ith component* of λ. We abuse notation and also write $\varnothing$ for the *empty ℓ-multipartition* $(\varnothing, \dots, \varnothing)$. We denote by $\mathscr{P}_n^\ell$ the set of all ℓ-multipartitions of n, and let $\mathscr{P}^\ell := \bigcup \mathscr{P}_n^\ell$.

For $\lambda, \mu \in \mathscr{P}_n^\ell$, we say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if for all $1 \leqslant m \leqslant \ell$ and $k \geqslant 1$,

$$\sum_{i=1}^{m-1} |\lambda^{(i)}| + \sum_{j=1}^{k} \lambda_j^{(m)} \geqslant \sum_{i=1}^{m-1} |\mu^{(i)}| + \sum_{j=1}^{k} \mu_j^{(m)}.$$

The *Young diagram* of an ℓ-multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^\ell$ is defined to be

$$[\lambda] := \{(i, j, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, \ell\} \mid 1 \leqslant j \leqslant \lambda_i^{(m)}\}.$$

We refer to elements of $[\lambda]$ as *nodes* of λ. We say that a node $A \in [\lambda]$ is *removable* if $[\lambda] \setminus \{A\}$ is a Young diagram of an ℓ-multipartition, while a node $A \notin [\lambda]$ is *addable* if $[\lambda] \cup \{A\}$ is a Young diagram of an ℓ-multipartition.

If $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$ is a partition, the *conjugate partition*, denoted $\lambda' = (\lambda'_1, \lambda'_2, \dots)$, is defined by

$$\lambda'_i := |\{j \geqslant 1 \mid \lambda_j \geqslant i\}|.$$

For $\lambda \in \mathscr{P}^\ell$, we define the *conjugate ℓ-multipartition*, also denoted λ', to be $\lambda' := (\lambda^{(\ell)\prime}, \dots, \lambda^{(1)\prime})$.

Fix an e-multicharge $\kappa = (\kappa_1, \dots, \kappa_\ell) \in I^\ell$. The *$e$-residue* of a node $A = (i, j, m) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, \ell\}$ is defined to be

$$\operatorname{res} A := \kappa_m + j - i \pmod{e}.$$

We call a node of residue r an *r-node*. We define the *residue content* of $\lambda \in \mathscr{P}^\ell$ to be

$$\mathrm{cont}(\lambda) := \sum_{A \in [\lambda]} \alpha_{\mathrm{res}\, A} \in Q_+.$$

We write

$$\mathscr{P}^\ell_\alpha := \{\lambda \in \mathscr{P}^\ell \mid \mathrm{cont}(\lambda) = \alpha\}, \quad \alpha \in Q_+.$$

Let $\lambda \in \mathscr{P}^\ell_n$. Then a *$\lambda$-tableau* is a bijection $\mathtt{T} : [\lambda] \to \{1, \dots, n\}$. We say that $\mathtt{T}$ is *standard* if its entries increase down each column (i.e. column-strict) and along each row (i.e. row-strict), within each component. We denote the set of all standard λ-tableaux by $\mathrm{Std}(\lambda)$.

For $\lambda \in \mathscr{P}^\ell_n$, let $\mathtt{T}$ be a standard λ-tableau. Now define $\mathtt{T}_{\leqslant a}$ to be the μ-tableau obtained from $\mathtt{T}$ by removing all of the nodes occupied with entries greater than a, where $a \in \{1, \dots, n\}$ and $\mu \in \mathscr{P}^\ell_a$. It follows that $\mathtt{T}_{\leqslant a}$ has the *shape* of $\mu \in \mathscr{P}^\ell_a$, and we write $\mathrm{sh}(\mathtt{T}_{\leqslant a}) = \mu$.

There is a natural $\mathfrak{S}_n$-action on $\mathrm{Std}(\lambda)$. We refer the reader to [16] for further details.

2.4 KLR algebras and their cyclotomic quotients

Suppose $\alpha \in Q^+$ has height n, and set

$$I^\alpha = \{\mathbf{i} = (i_1, \dots, i_n) \in I^n \mid \alpha_{i_1} + \dots + \alpha_{i_n} = \alpha\}.$$

Definition 2.3. We define the *Khovanov–Lauda–Rouquier (KLR) algebra* or *quiver Hecke algebra $\mathscr{R}_\alpha$* to be the unital associative $\mathbb{F}$-algebra with generating set

$$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$$

and relations

$$\begin{aligned}
e(\mathbf{i})e(\mathbf{j}) &= \delta_{\mathbf{i},\mathbf{j}} e(\mathbf{i}); \\
\sum_{i \in I^\alpha} e(\mathbf{i}) &= 1; \\
y_r e(\mathbf{i}) &= e(\mathbf{i}) y_r; \\
\psi_r e(\mathbf{i}) &= e(s_r \mathbf{i}) \psi_r; \\
y_r y_s &= y_s y_r; \\
\psi_r y_s &= y_s \psi_r && \text{if } s \neq r, r+1; \\
\psi_r \psi_s &= \psi_s \psi_r && \text{if } |r-s| > 1; \\
y_r \psi_r e(\mathbf{i}) &= (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e(\mathbf{i}); \\
y_{r+1} \psi_r e(\mathbf{i}) &= (\psi_r y_r + \delta_{i_r, i_{r+1}}) e(\mathbf{i}); \\
\psi_r^2 e(\mathbf{i}) &= \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e(\mathbf{i}) & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r) e(\mathbf{i}) & \text{if } i_r = i_{r+1} + 1, \\ (y_r - y_{r+1}) e(\mathbf{i}) & \text{if } i_r = i_{r+1} - 1, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(\mathbf{i}) & \text{if } i_r = i_{r+1} + 1 = i_{r+1} - 1; \end{cases} \\
\psi_r \psi_{r+1} \psi_r e(\mathbf{i}) &= \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) e(\mathbf{i}) & \text{if } i_{r+2} = i_r = i_{r+1} + 1, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1) e(\mathbf{i}) & \text{if } i_{r+2} = i_r = i_{r+1} - 1, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 2y_{r+1} + y_r + y_{r+2}) e(\mathbf{i}) & \text{if } i_{r+2} = i_r = i_{r+1} + 1 = i_{r+1} - 1, \\ (\psi_{r+1} \psi_r \psi_{r+1}) e(\mathbf{i}) & \text{otherwise}; \end{cases}
\end{aligned}$$

for all admissible $r, s, \mathbf{i}, \mathbf{j}$.

The *KLR algebra* or *quiver Hecke algebra* $\mathscr{R}_n$ is defined to be the direct sum $\bigoplus_\alpha \mathscr{R}_\alpha$, where the sum is taken over all $\alpha \in Q^+$ of height n.

Definition 2.4. We define the *cyclotomic KLR algebra* or *cyclotomic quiver Hecke algebra* $\mathscr{R}_\alpha^\Lambda$ to be the quotient of $\mathscr{R}_\alpha$ by the relation:

$$y_1^{(\Lambda,\alpha_{i_1})} e(\mathbf{i}) = 0,$$

for all $\mathbf{i} \in I^\alpha$.

The *cyclotomic KLR algebra* or *cyclotomic quiver Hecke algebra* $\mathscr{R}_n^\Lambda$ is defined to be the direct sum $\bigoplus_\alpha \mathscr{R}_\alpha^\Lambda$, where the sum is taken over all $\alpha \in Q^+$ of height n.

Lemma 2.5 [3, Corollary 1]. *There is a unique $\mathbb{Z}$-grading on $\mathscr{R}_n^\Lambda$ such that, for all admissible r and $\mathbf{i}$,*

$$\deg(e(\mathbf{i})) = 0, \quad \deg(y_r) = 2, \quad \deg \psi_r(e(\mathbf{i})) = -a_{i_r, r_{r+1}}.$$

These $\mathbb{Z}$-graded algebras are connected to the Hecke algebras of complex reflection groups via Brundan and Kleshchev's *Isomorphism Theorem.*

Theorem 2.6 [2, Main Theorem]. *If $e = p$ or $p \nmid e$, then $\mathscr{R}_n^\Lambda$ of type A_∞ or $A_{e-1}^{(1)}$ is isomorphic to a cyclotomic Hecke algebra associated to a complex reflection group $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$ of type $G(\ell, 1, n)$.*

In particular, $\mathscr{R}_n^\Lambda$ is isomorphic to:

- ◇ an Iwahori–Hecke algebra $\mathscr{H}_{\mathbb{F},q}(\mathfrak{S}_n)$ of type A when $\Lambda = \Lambda_\kappa$ for some $\kappa \in I$;
- ◇ an Iwahori–Hecke algebra $\mathscr{H}_{\mathbb{F},q,Q_1,Q_2}((\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n)$ of type B when $\Lambda = \Lambda_{\kappa_1} + \Lambda_{\kappa_2}$ for some $\kappa_1, \kappa_2 \in I$ such that $Q_1 = q^{\kappa_1}$ and $Q_2 = q^{\kappa_2}$.

2.5 Affine induction and restriction

Given $\alpha, \beta \in Q_+$, we set

$$\mathscr{R}_{\alpha,\beta} := \mathscr{R}_\alpha \otimes \mathscr{R}_\beta.$$

We denote by $M \boxtimes N$ the outer tensor product of an $\mathscr{R}_\alpha$-module M and an $\mathscr{R}_\beta$-module N. There is an injective algebra homomorphism $\mathscr{R}_{\alpha,\beta} \hookrightarrow \mathscr{R}_{\alpha+\beta}$, mapping $e(\mathbf{i}) \otimes e(\mathbf{j})$ to $e(\mathbf{ij})$, where $\mathbf{ij}$ denotes the concatenation of the two sequences. The image of the identity element of $\mathscr{R}_{\alpha,\beta}$ under this map is

$$e_{\alpha,\beta} := \sum_{\mathbf{i} \in I^\alpha,\ \mathbf{j} \in I^\beta} e(\mathbf{ij}).$$

Let $\mathrm{Ind}_{\alpha,\beta}$ and $\mathrm{Res}_{\alpha,\beta}$ denote the induction and restriction functors, respectively, between the module categories

$$\mathrm{Ind}_{\alpha,\beta} := \mathscr{R}_{\alpha+\beta} e_{\alpha,\beta} \otimes_{\mathscr{R}_{\alpha,\beta}} ? : \quad \mathscr{R}_{\alpha,\beta}\text{-mod} \to \mathscr{R}_{\alpha+\beta}\text{-mod},$$
$$\mathrm{Res}_{\alpha,\beta} := e_{\alpha,\beta} \mathscr{R}_{\alpha+\beta} \otimes_{\mathscr{R}_{\alpha+\beta}} ? : \quad \mathscr{R}_{\alpha+\beta}\text{-mod} \to \mathscr{R}_{\alpha,\beta}\text{-mod}.$$

Both $\mathrm{Ind}_{\alpha,\beta}$ and $\mathrm{Res}_{\alpha,\beta}$ are exact functors, and send finite dimensional modules to finite dimensional modules. Moreover, $\mathrm{Ind}_{\alpha,\beta}$ is left adjoint to $\mathrm{Res}_{\alpha,\beta}$.

2.6 Specht modules and irreducible modules

The representation theory of cyclotomic KLR algebras $\mathscr{R}_n^\Lambda$ is governed by a special family of modules called *Specht modules*. There is an explicit presentation of Specht modules – we refer the reader to [16, Definition 7.11] for further details. For each $\lambda \in \mathscr{P}_n^\ell$, we denote the corresponding Specht module by $S(\lambda)$. Specht modules are $\mathbb{Z}$-graded, inherited from the non-trivial $\mathbb{Z}$-grading on the cyclotomic KLR algebras.

Theorem 2.7. [4, Corollary 4.6], [16, Proposition 7.14 and Corollary 7.20] *For $\lambda \in \mathscr{P}_n^\ell$, there is a set of vectors $\{v_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$ that forms a homogeneous $\mathbb{F}$-basis of $S(\lambda)$.*

For each $\lambda \in \mathscr{P}_n^\ell$, the head of the Specht module $S(\lambda)$ is an irreducible $\mathscr{R}_n^\Lambda$-module, denoted $D(\mu)$ for some $\mu \in \mathscr{P}_n^\ell$.

Theorem 2.8 [3, Theorem 5.10]. *The modules $\{D(\mu) \mid \mu \in \mathscr{P}_n^\ell,\ \mu$ is regular$\}$ give a complete set of graded irreducible $\mathscr{R}_n^\Lambda$-modules up to isomorphism and grading shift. Moreover, each $D(\mu)$ is self-dual as a graded module.*

The indexing set of the irreducible $\mathscr{R}_n^\Lambda$-modules is given by *regular multipartitions*. In general, there is a recursive description of regular multipartitions – see [1] for a description of the combinatorics of regular multipartitions. However, for irreducible modules $D(\mu)$ for the Iwahori–Hecke algebra of type A, which are indexed by *partitions*, there is an explicit description of the indexing set given by *e-regular partitions*.

Definition 2.9. Let $\lambda \in \mathscr{P}$.

- We say that λ is *e-singular* if $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+e} > 0$ for some $i \geqslant 1$. Otherwise we say that λ is *e-regular*.
- We say that λ is *e-restricted* if and only if its conjugate λ' is e-regular.

Definition 2.10. Let $\lambda \in \mathscr{P}_\alpha$. We say that λ is an *e-core* if and only if λ is the unique partition of residue content α. (Note that e-cores are both e-regular and e-restricted.)

Specht modules arise as the cell modules of cyclotomic KLR algebras – see [12] for details.

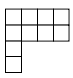
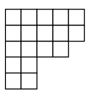
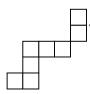
Theorem 2.11 [12, Theorem 5.8 and Corollary 5.10]. *The cyclotomic KLR algebra $\mathscr{R}_n^\Lambda$ is a graded cellular algebra, and its graded cell modules are the graded Specht modules as introduced in [4].*

3 Skew Specht modules and skew cyclotomic KLR algebras

We give a brief overview of skew Specht modules and the associated combinatorics of skew shapes, together with their connection with restrictions of Specht modules. Furthermore, we introduce joint work with Robert Muth, Thomas Nicewicz and Liron Speyer on *skew cyclotomic quiver Hecke algebras*.

3.1 Skew diagrams and tableaux

Let $\lambda, \mu \in \mathscr{P}^\ell$, with $[\mu] \subseteq [\lambda]$. Then the *skew diagram* of λ by μ is $\lambda/\mu := \lambda \backslash \mu$. Note that $\lambda/\varnothing = \lambda$ for $\lambda \in \mathscr{P}_n^\ell$, so skew diagrams generalise Young diagrams.

Example. Let $\lambda = (5^2, 4, 2^2)$, $\mu = (4^2, 1^2)$. Then $[\mu] =$  $\subset [\lambda] =$  and $[\lambda/\mu] =$ .

For a fixed $\mu \in \mathscr{P}^\ell_n$, we denote by $\mathscr{S}^\ell_{\mu,d}$ the set of all skew diagrams λ/μ with $|\lambda/\mu| = d$. For $\lambda/\mu \in \mathscr{S}^\ell_{\mu,d}$, a λ/μ-*tableau* is a bijection $\mathtt{T} : [\lambda/\mu] \to \{1, \dots, d\}$. We denote by $\mathrm{Std}(\lambda/\mu)$ the set of all standard λ/μ-tableaux, that is, all λ/μ-tableaux that are both column- and row-strict.

3.2 Skew Specht modules

Over the affine KLR algebra $\mathscr{R}_n$, Muth [19] constructed Specht modules for skew shapes – see [19]§4 for an explicit presentation of these Specht modules. For each skew shape λ/μ, we denote the associated *skew* Specht module by $S(\lambda/\mu)$. There is an analogous standard basis theorem for skew Specht modules.

Theorem 3.1 (Proposition 4.7, [19])**.** *For* $\lambda/\mu \in \mathscr{S}^\ell_{\mu,d}$*, there is a set of vectors* $\{v_\mathtt{T} \mid \mathtt{T} \in \mathrm{Std}(\lambda/\mu)\}$ *that forms a homogeneous* $\mathbb{F}$*-basis of* $S(\lambda/\mu)$*.*

Skew Specht modules appear in the filtrations of restrictions of Specht modules.

Theorem 3.2 (Theorem 5.13, [19])**.** *Let* $\lambda \in \mathscr{P}^\ell_{\alpha+\beta}$ *and*

$$\{\mu_1, \dots, \mu_k\} = \{\mu \in \mathscr{P}^\ell_\alpha \mid [\mu] \subset [\lambda]\},$$

and assume that $\mu_i \rhd \mu_j$ *implies* $i < j$*. Write*

$$V_i := \mathbb{F}\text{-span}\{v_\mathtt{T} \in S(\lambda) \mid \mathtt{T} \in \mathrm{Std}(\lambda), \mathrm{sh}(\mathtt{T}_{\leqslant a}) = \mu_j \textit{ for some } j < i\}$$

for all i*. Then*

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = \mathrm{Res}_{\alpha,\beta} S(\lambda)$$

is a graded filtration of $\mathrm{Res}_{\alpha,\beta} S(\lambda)$ *by* $\mathscr{R}_{\alpha,\beta}$*-submodules* V_i*, with subquotients*

$$V_i/V_{i-1} \cong S(\mu_i) \boxtimes S(\lambda/\mu_i)$$

as $\mathscr{R}_{\alpha,\beta}$*-modules.*

We now consider a special case of this theorem in level $\ell = 1$. Let $\lambda \in \mathscr{P}_{\alpha+\beta}$. Assume that $\mu \in \mathscr{P}_\alpha$ is a *core* partition, in which case μ is the unique partition of content α. Hence

$$\mathrm{Res}_{\alpha,\beta} S(\lambda) \cong S(\mu) \boxtimes S(\lambda/\mu).$$

3.3 Skew cyclotomic KLR algebras

We saw before that cyclotomic KLR algebras are cellular, whose cell modules are Specht modules. It is therefore a natural question to ask if we can construct an analogous cellular algebra whose cell modules are *skew Specht modules*.

Let $\alpha, \beta \in Q_+$. We now fix $\ell = 1$, a fundamental dominant weight Λ, and a core partition $\mu \in \mathscr{P}_\alpha$. (Note that the following generalises to arbitrary levels, the details of which we have omitted here. The combinatorics of cores is more complicated in higher levels, and the results in level $\ell = 1$ are sufficient for our study of irreducible Specht modules.) The following is joint work with Robert Muth, Thomas Nicewicz and Liron Speyer.

We denote by φ the composition of algebra homomorphisms

$$\mathscr{R}_\beta \hookrightarrow \mathscr{R}_\alpha \otimes \mathscr{R}_\beta \hookrightarrow e_{\alpha,\beta}\mathscr{R}_{\alpha+\beta}e_{\alpha,\beta} \to e_{\alpha,\beta}\mathscr{R}^\Lambda_{\alpha+\beta}e_{\alpha,\beta}.$$

The kernel $\ker(\varphi)$ of this composition of maps is a two-sided ideal of $\mathscr{R}_\beta$.

Definition 3.3. The *α-skew cyclotomic KLR algebra*, or *α-skew cyclotomic quiver Hecke algebra*, is defined to be

$$\mathscr{R}_\beta^{\Lambda/\alpha} := \mathscr{R}_\beta / \ker(\varphi).$$

We obtain the analogous result to Theorem 2.11.

Theorem 3.4 (Muth, Nicewicz, Speyer, S., 2023+)**.** *The α-skew cyclotomic KLR algebra $\mathscr{R}_\beta^{\Lambda/\alpha}$ is a graded cellular algebra, and its graded cell modules are skew Specht modules.*

Moreover, there is an exact functor of $\mathbb{Z}$-graded categories $\mathscr{R}_{\alpha+\beta}^{\Lambda}$-mod $\to \mathscr{R}_\beta^{\Lambda/\alpha}$-mod that sends cell modules to cell modules.

This result connects irreducible Specht modules and irreducible skew Specht modules in the following way.

Corollary 3.5 (Muth, Nicewicz, Speyer, S., 2023+)**.** *Suppose that $\mu \in \mathscr{P}_\alpha$ and $\lambda \in \mathscr{P}_{\alpha+\beta}$ such that μ is a core partition. If $S(\lambda)$ is an irreducible $\mathscr{R}_{\alpha+\beta}$-module, then $S(\lambda/\mu)$ is an irreducible $\mathscr{R}_\beta$-module.*

The $\mathscr{R}_\beta$-module $S(\lambda/\mu)$ will factor through *some* cyclotomic KLR algebra quotient $\mathscr{R}_\beta^\Lambda$. Therefore, it is a natural question to ask for which partition $\nu \in \mathscr{P}_\beta^\ell$ is the irreducible $\mathscr{R}_\beta^\Lambda$-module $D(\nu)$, up to inflation, isomorphic to $S(\lambda/\mu)$? We tackle this problem in level $\ell = 2$ for irreducible modules $D(\nu)$ that are precisely, up to a grading shift, irreducible Specht modules.

4 Irreducible Specht modules

In this section, we review known results on (ir)reducible Specht modules for the Iwahori–Hecke algebras of types A and B. Moreover, we introduce joint work with Matthew Fayers on work towards a classification of irreducible Specht modules for the Iwahori–Hecke algebras of type B.

4.1 Irreducible Specht modules for the Iwahori–Hecke algebras of type A

There is an almost complete classification of irreducible Specht modules for the Iwahori–Hecke algebra of type A:

- when $q \neq -1$, the classification is complete [13, 17, 18, 5, 6, 11];
- when $q = -1$ the classification is complete for the symmetric group [14], and almost complete for the Iwahori–Hecke algebra of type A [9, 7, 10].

For $h \in \mathbb{N}$, let $\nu_p(h)$ denote the usual p-adic valuation of h. That is, $\nu_p(h)$ denotes the largest power of p dividing h if p is finite, while $\nu_\infty(h) = 0$ for all h.

Definition 4.1. The *(e, p)-adic valuation* of h is defined to be

$$\nu_{e,p}(h) := \begin{cases} \nu_p(\frac{h}{e}) + 1 & \text{if } e|h; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2. Let λ be a partition. The *hook length* of a node $(a, b) \in [\lambda]$, denoted h_{ab}^λ, is defined to be

$$h_{ab}^\lambda := \lambda_a - b + \lambda'_b - a + 1.$$

The classification of irreducible Specht modules $S(\lambda)$ is almost complete thanks to a multitude of papers [13, 14, 17, 18, 5, 6, 11]. This (almost) classification depends on the (e, p)-adic valuations of hooks lengths as follows.

Theorem 4.3. *Suppose that $e \in \{3, 4, \dots\}$ and $\operatorname{char}\mathbb{F}$ is arbitrary or $e = \operatorname{char}\mathbb{F} = 2$. Then $S(\lambda)$ is irreducible if and only if there exist integers $k \geqslant 0$ and $\ell \geqslant 0$ with $(k+1, \ell+1) \notin [\lambda]$, satisfying*

i) $\nu_{e,p}(h^\lambda_{ab}) = \nu_{e,p}(h^\lambda_{ac})$ *whenever* $(a, b), (a, c) \in [\lambda]$ *and* $a > k$*;*

ii) $\nu_{e,p}(h^\lambda_{ac}) = \nu_{e,p}(h^\lambda_{bc})$ *whenever* $(a, c), (b, c) \in [\lambda]$ *and* $c > \ell$*;*

iii) $\nu_{e,p}(h^\lambda_{ac}) = 0$ *for* $1 \leqslant a \leqslant k$, $1 \leqslant b \leqslant \ell$.

Definition 4.4. We define a *JM-partition* (named after James and Mathas) to be a partition λ that indexes an irreducible Specht module $S(\lambda)$, satisfying the irreducibility condition in the above theorem.

Remarks. When $e = 2$ and $\operatorname{char}\mathbb{F} \neq 2$, (ir)reducible Specht modules for the Hecke algebra of the symmetric group have been studied in several papers [9, 7, 10].

There are *cyclotomic i-restriction functors*, denoted e_i, and *cyclotomic divided powers*, denoted $e_i^{(r)}$ that act on $\mathscr{R}^\Lambda_n$-modules, in particular on Specht modules. These functors were introduced in [3].

Definition 4.5. We say that $(\lambda, \mu) \in \mathscr{P}^2_n$ is *i-restrictable* if there exists a residue $i \in \mathbb{Z}/e\mathbb{Z}$ for which (λ, μ) has removable i-nodes but no addable i-nodes. If there exists no such i, we call (λ, μ) *unrestrictable.*

By the exactness of these functors, we have the following crucial result.

Lemma 4.6. *Suppose that $(\lambda, \mu) \in \mathscr{P}^2_n$ is i-restrictable and has r removable i-nodes. Further suppose $e_i^{(r)} S(\lambda, \mu) \cong S(\nu, \eta)$ for some $(\nu, \eta) \in \mathscr{P}^2_n$. Then $S(\lambda, \mu)$ is irreducible if and only if $S(\nu, \eta)$ is.*

To classify irreducible Specht modules for $\mathscr{R}^\Lambda_n$ it thus suffices to classify *unrestrictable* irreducible Specht modules.

4.2 Irreducible Specht modules for the Iwahori–Hecke algebras of type B

Fayers began the study of irreducible Specht modules for the Iwahori–Hecke algebras of type B in [8]. Throughout, we will identify the Iwahori–Hecke algebras of type B with the corresponding cyclotomic KLR algebra $\mathscr{R}^\Lambda$. When $e = \infty$, Fayers gave a complete classification of irreducible Specht modules in this case [8][Corollary 3.6], using *one-runner abacus displays* for partitions. We refer the interested reader to [8] for details.

Fayers also began the study of irreducible Specht modules for $\mathscr{R}^\Lambda_n$ in finite quantum characteristic $e \in \{2, 3, \dots\}$.

Proposition 4.7 (Proposition 2.4, [8])**.** *Let $\lambda \in \mathscr{P}_n$. $S(\lambda, \varnothing)$ or $S(\varnothing, \lambda)$ is an irreducible $\mathscr{R}^\Lambda_n$-module if and only if $S(\lambda)$ is an irreducible $\mathscr{H}_{\mathbb{F},q}(\mathfrak{S}_n)$-module.*

When $e = 2$, Fayers reduced the classification of irreducible Specht modules for $\mathscr{R}^\Lambda_n$ to those of the Hecke algebra of the symmetric group.

Theorem 4.8 (Theorem 4.1, [8])**.** *Let $(\lambda, \mu) \in \mathscr{P}^2_n$, $e = 2$ and suppose that $S(\lambda, \mu)$ is unrestrictable. Then $S(\lambda, \mu)$ is an irreducible $\mathscr{R}^\Lambda_n$-module if and only if $(\lambda, \mu) = (\nu, \varnothing)$ or $(\varnothing, \nu)$ and $S(\nu)$ is an irreducible $\mathscr{H}_{\mathbb{F},-1}(\mathfrak{S}_n)$-module for some partition $\nu \in \mathscr{P}_n$.*

Thus, when $e = 2$, there is no unrestrictable Specht module $S(\lambda, \mu)$ with $\lambda \neq \varnothing \neq \mu$ that is irreducible. We thus suppose from now on that $e \in \{3, 4, \dots\}$.

The author previously studied the family of Specht modules indexed by *hook bipartitions* for the Iwahori–Hecke algebra with $e \in \{3, 4, \dots\}$. A *hook bipartition* of n is a bipartition of the form $((n-m), (1^m))$ for some $m \in \{0, \dots, n\}$. In [22, 23], the complete graded module structure of Specht modules indexed by hook bipartitions was determined. Of course, the one-dimensional Specht modules $S((n), \varnothing)$ and $S(\varnothing, (1^n))$ are irreducible. Besides these one-dimensional Specht modules indexed by hook bipartitions, one can obtain from their module structure a classification of irreducible Specht modules for hook bipartitions.

Theorem 4.9 (Theorems 5.1–5.4, [23]). *Let $\mathbb{F}$ be arbitrary, $e \in \{3, 4, \dots\}$ and $\kappa = (\kappa_1, \kappa_2)$. Suppose that $n - m \geqslant 1$ and $m \geqslant 1$. Then the Specht module $S((n-m), (1^m))$ is an irreducible $\mathscr{R}_n^\Lambda$-module if and only if $n \not\equiv \kappa_2 - \kappa_1 + 1 \pmod e$ and $\kappa_2 \not\equiv \kappa_1 - 1 \pmod e$.*

For arbitrary e, Speyer and the author [21] studied the decomposability of Specht modules indexed by *bihooks*, pairs of hook partitions, for the Iwahori–Hecke algebra of type B. Moreover, Muth, Speyer and the author [20] classified semisimple Specht modules indexed by bihooks, giving an explicit decomposition of these modules into irreducible summands. One can observe from the decompositions of these semisimple Specht modules, that the presentations of the irreducible summands coincide with the presentations of certain skew Specht modules. In fact, the irreducible summands are isomorphic, up to inflation, to the corresponding skew Specht modules. This observation inspired the study of irreducible Specht modules in higher levels.

In current joint work with Fayers, the author is working towards a classification of irreducible Specht modules for the Iwahori–Hecke algebra of type B with $e \in \{3, 4, \dots\}$ and over a field of arbitrary characteristic. Observe that a hook bipartition, many of which index irreducible Specht modules, is a pair of partitions whose first component is an e-regular partition and whose second component is an e-restricted partition. We generalise pairs of partitions with this property that index irreducible Specht modules.

Proposition 4.10 (Fayers–S. 2023+). *Let λ and μ be JM-partitions. Suppose $S(\lambda, \mu)$ is irreducible. Then*

- *at least one of λ and μ is e-regular,*
- *and at least one of λ and μ is e-restricted.*

Recall that JM-partitions index irreducible Specht modules for the Iwahori–Hecke algebra of type A with $e \in \{3, 4, \dots\}$. JM-partitions satisfy an "e-restricted condition" given in Theorem 4.3(i) and an "e-regular condition" given in Theorem 4.3(ii). Notice that the previous result mimics these conditions, allowing us to understand more precisely which skew diagrams correspond to bipartitions (λ, μ) that index irreducible Specht modules.

Definition 4.11. We say that a bipartition (λ, μ) is *skewable* if there are partitions η and ν such that η is a JM-partition, ν is a core, $\eta_{\nu_1'+1} \leqslant \nu_1$, and either

1. $\lambda = C^{\nu_1}\eta$, $\mu = R^{\nu_1'}\eta$ and $\nu_1 + \nu_1' + e\mathbb{Z} = \kappa_1 - \kappa_2$, or
2. $\lambda = R^{\nu_1'}\eta$, $\mu = C^{\nu_1}\eta$ and $\nu_1 + \nu_1' + e\mathbb{Z} = \kappa_2 - \kappa_1$,

recalling R and C are row- and column-removal operators on partitions.

Fayers and the author conjecture that the only irreducible Specht modules for the Iwahori–Hecke algebra of type B are, up to taking cyclotomic divided powers, precisely those that are isomorphic, up to inflation, to irreducible skew Specht modules as determined in Corollary 3.5.

Conjecture 4.12 (Fayers–S.). *Suppose that $(\lambda, \mu) \in \mathscr{P}_n^2$ is unrestrictable. Then $S(\lambda, \mu)$ is irreducible if and only if (λ, μ) is skewable.*

References

[1] J. Brundan and A. Kleshchev. Representation theory of symmetric groups and their double covers. In *Groups, Combinatorics and Geometry*, pages 31–53, Durham, 2001, 2003. World Sci. Publishing, River Edge, NJ. [Page 5.]

[2] J. Brundan and A. Kleshchev. Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras. *Invent. Math.*, 178(3):451–484, 2009. [Page 4.]

[3] J. Brundan and A. Kleshchev. Graded decomposition numbers for cyclotomic Hecke algebras. *Adv. Math.*, 222(6):1883–1942, 2009. [Pages 4, 5, 8.]

[4] J. Brundan, A. Kleshchev, and W. Wang. Graded Specht modules. *J. Reine Angew. Math.*, 655:61–87, 2011. [Page 5.]

[5] M. Fayers. Reducible Specht modules for Hecke algebras of type A. *J. Algebra*, 280(2):500–504, 2004. [Page 7.]

[6] M. Fayers. Irreducible Specht modules for Hecke algebras of type A. *Adv. Math.*, 193(2):438–452, 2005. [Page 7.]

[7] M. Fayers. On the irreducible Specht modules for Iwahori–Hecke algebra of type A with $q = -1$. *J. Algebra*, 323(6):1839–1844, 2010. [Pages 7, 8.]

[8] M. Fayers. Irreducible Specht modules for Iwahori–Hecke algebras of type B. *Represent. Theory*, 16:108–126, 2012. [Page 8.]

[9] M. Fayers and S. Lyle. Some reducible Specht modules for Iwahori–Hecke algebras of type A with $q = -1$. *J. Algebra*, 321(3):912–933, 2009. [Pages 7, 8.]

[10] M. Fayers and S. Lyle. The reducible Specht modules for the Hecke algebra $\mathcal{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$. *J. Algebraic Combin.*, 37:201–241, 2013. [Pages 7, 8.]

[11] S. Lyle G. D. James and A. Mathas. Rouquier blocks. *Math. Z.*, 252(3):511–531, 2006. [Page 7.]

[12] J. Hu and A. Mathas. Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A. *Adv. Math.*, 225(2):598–642, 2010. [Page 5.]

[13] G. D. James and A. Mathas. A q-analogue of the Jantzen–Schaper theorem. *Proc. London Math. Soc.*, 74(2):241–274, 1997. [Page 7.]

[14] G. D. James and A. Mathas. The irreducible Specht modules in characteristic 2. *Bull. London Math. Soc.*, 31(4):457–462, 1999. [Page 7.]

[15] V. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990. [Page 1.]

[16] A. Kleshchev, A. Mathas, and A. Ram. Universal graded Specht modules for cyclotomic Hecke algebras. *Proc. London Math. Soc.*, 105:1245–1289, 2012. [Pages 3, 5.]

[17] S. Lyle. Some reducible Specht modules. *J. Algebra*, 269(2):536–543, 2003. [Page 7.]

[18] S. Lyle. Some q-analogues of the Carter–Payne theorem. *J. Reine Angew. Math.*, 608:93–121, 2007. [Page 7.]

[19] R. Muth. Graded skew Specht modules and cuspidal modules for Khovanov–Lauda–Rouquier algebras of affine type A. *Algebr. Represent. Theory*, 22(4):977–1015, 2019. [Page 6.]

[20] R. Muth, L. Speyer, and L. Sutton. Decomposable Specht modules indexed by bihooks II. *Algebr. Represent. Theory*, 26(1):241–280, 2023. [Page 9.]

[21] L. Speyer and L. Sutton. Decomposable Specht modules indexed by bihooks. *Pacific J. Math.*, 304(2):655–711, 2020. [Page 9.]

[22] L. Sutton. Specht modules labelled by hook bipartitions I. *J. Algebra*, 515:456–510, 2018. [Page 9.]

[23] L. Sutton. Specht modules labelled by hook bipartitions II. *J. Pure Appl. Algebra*, 224(4):106204, 38, 2020. [Page 9.]