On the module category of the triplet W-algebra $\mathcal{W}_{p_+,p_-}$

Hiromu Nakano
Mathematical Institute,
Tohoku University

We study the structure of the category of modules over the triplet W-algebra $\mathcal{W}_{p_+,p_-}$ defined by Feigin, Gainutdinov, Semikhatov and Tipunin [1]. Since $\mathcal{W}_{p_+,p_-}$ satisfies the C_2-cofinite condition, by Huang, Lepowsky and Zhang [2], every simple module has the projective cover and the module categories have the structure of a braided tensor category. We determine the structure of the projective covers of all simple $\mathcal{W}_{p_+,p_-}$-modules, and determine certain non-semisimple fusion rules conjectured by Rasmussen [3] and Gaberdiel, Runkel and Wood [4]. This paper is based on the thesis [5].

1 Main results on the triplet W-algebra $\mathcal{W}_{p_+,p_-}$

Fix two coprime integers p_+, p_- such that $p_- > p_+ \geq 2$ and let

$$c_{p_+,p_-} := 1 - 6\frac{(p_+ - p_-)^2}{p_+p_-}$$

be a minimal central charge of Virasoro algebra. Let us briefly rewiew the definitions of the triplet W-algebra $\mathcal{W}_{p_+,p_-}$ and the simple $\mathcal{W}_{p_+,p_-}$-modules in accordance with [6, 7, 8].

For $\alpha \in \mathbb{C}$, let F_α be the bosonic Fock module generated from the bosonic field

$$Y(|\alpha\rangle, z) = e^{\alpha\hat{a}} z^{\alpha a_0} e^{\alpha \sum_{n\geq 1} \frac{a_{-n}}{n} z^n} e^{-\alpha \sum_{n\geq 1} \frac{a_n}{n} z^{-n}},$$

where

$$[a_m, a_n] = m\delta_{m+n,0}\mathrm{id}, \qquad [\hat{a}, a_n] = \delta_{n,0}\mathrm{id}.$$

Let

$$T := \frac{1}{2}\left(a_{-1}^2 - (\alpha_+ + \alpha_-)a_{-2}\right)|0\rangle, \qquad \alpha_+ := \sqrt{\frac{2p_-}{p_+}}, \quad \alpha_- := -\sqrt{\frac{2p_+}{p_-}}$$

be a confomal vector. By T, each Fock module F_α becomes a Virasoro module whose central charge c_{p_+,p_-}.

For $r, s, n \in \mathbb{Z}$ we introduce the following symbols

$$\alpha_{r,s;n} := \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_- + \frac{\sqrt{2p_+p_-}}{2}n.$$

Let $F_{r,s;n} := F_{\alpha_{r,s;n}}$.

As detailed in [9], we can define the complex screening operators

$$Q_+^{[r]} = \oint_{z=0} \mathrm{d}z \int_{[\Delta_{r-1}]} Q_+(z)Q_+(zy_1)\cdots Q_+(zy_{r-1})\mathrm{d}y_1\cdots\mathrm{d}y_{r-1} \in \mathrm{Hom}_{\mathbb{C}}(F_{r,k;l}, F_{-r,k;l}),$$

$$Q_-^{[s]} = \oint_{z=0} \mathrm{d}z \int_{[\Delta_{s-1}]} Q_-(z)Q_-(zy_1)\cdots Q_-(zy_{s-1})\mathrm{d}y_1\cdots\mathrm{d}y_{s-1} \in \mathrm{Hom}_{\mathbb{C}}(F_{k,s;l}, F_{k,-s;l}),$$

where $Q_\pm(z) = Y(|\alpha_\pm\rangle, z)$ and $[\Delta_n]$ is a regularized cycle constructed from the simplex $\Delta_n = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid 1 > y_1 > \cdots y_n > 0\ \}$. Let $Q_+^{[r]}$ and $Q_-^{[s]}$ be the zero modes of $Q_+^{[r]}(z)$ and $Q_-^{[s]}(z)$. These zero modes commute with every Virasoro mode of $Y(T, z)$ and are called screening operators.

Definition 1.1.
The lattice vertex operator algebra $\mathcal{V}_{[p_+,p_-]}$ *is the tuple*

$$(\mathcal{V}_{1,1}^+, |0\rangle, T, Y),$$

where underlying vector space of $\mathcal{V}_{[p_+,p_-]}$ *is given by*

$$\mathcal{V}_{1,1}^+ = \bigoplus_{n\in\mathbb{Z}} F_{1,1;2n} = \bigoplus_{n\in\mathbb{Z}} F_{n\sqrt{2p_+p_-}},$$

and $Y(|\alpha_{1,1;2n}\rangle; z) = V_{\alpha_{1,1;2n}}(z)$ *for* $n \in \mathbb{Z}$.

It is a known fact that simple $\mathcal{V}_{[p_+,p_-]}$-modules are given by the following $2p_+p_-$ direct sum of Fock modules

$$\mathcal{V}_{r,s}^+ = \bigoplus_{n\in\mathbb{Z}} F_{r,s;2n}, \qquad \mathcal{V}_{r,s}^- = \bigoplus_{n\in\mathbb{Z}} F_{r,s;2n+1},$$

where $1 \le r \le p_+, 1 \le s \le p_-$.

Note that the two screening operators Q_+ and Q_- act on $\mathcal{V}_{1,1}^+$. We define the following vector subspace of $\mathcal{V}_{1,1}^+$:

$$\mathcal{K}_{1,1} = \mathrm{ker}Q_+ \cap \mathrm{ker}Q_- \subset \mathcal{V}_{1,1}^+.$$

Definition 1.2 ([1])**.** *The* triplet W-algebra

$$\mathcal{W}_{p_+,p_-} = (\mathcal{K}_{1,1}, |0\rangle, T, Y)$$

is a sub vertex operator algebra of $\mathcal{V}_{[p_+,p_-]}$*, where the vacuum vector, conformal vector and vertex operator map are those of* $\mathcal{V}_{[p_+,p_-]}$*.*

Let

$$r^\vee = p_+ - r, \qquad s^\vee = p_- - s.$$

For each $1 \le r \le p_+,\ 1 \le s \le p_-$, let $\mathcal{X}^\pm_{r,s}$ be the following vector subspace of $\mathcal{V}^\pm_{r,s}$:

1. For $1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1$,
$$\mathcal{X}^+_{r,s} = Q_+^{[r^\vee]}(\mathcal{V}^-_{r^\vee,s}) \cap Q_-^{[s^\vee]}(\mathcal{V}^-_{r,s^\vee}), \quad \mathcal{X}^-_{r,s} = Q_+^{[r^\vee]}(\mathcal{V}^+_{r^\vee,s}) \cap Q_-^{[s^\vee]}(\mathcal{V}^+_{r,s^\vee}).$$

2. For $1 \le r \le p_+ - 1,\ s = p_-$,
$$\mathcal{X}^+_{r,p_-} = Q_+^{[r^\vee]}(\mathcal{V}^-_{r^\vee,p_-}), \qquad \mathcal{X}^-_{r,p_-} = Q_+^{[r^\vee]}(\mathcal{V}^+_{r^\vee,p_-}).$$

3. For $r = p_+,\ 1 \le s \le p_- - 1$,
$$\mathcal{X}^+_{p_+,s} = Q_-^{[s^\vee]}(\mathcal{V}^-_{p_+,s^\vee}), \qquad \mathcal{X}^-_{p_+,s} = Q_-^{[s^\vee]}(\mathcal{V}^+_{p_+,s^\vee}).$$

4. $r = p_+,\ s = p_-$,
$$\mathcal{X}^+_{p_+,p_-} = \mathcal{V}^+_{p_+,p_-}, \qquad \mathcal{X}^-_{p_+,p_-} = \mathcal{V}^-_{p_+,p_-}.$$

We define the interior Kac table $\mathcal{T}$ as the following quotient set

$$\mathcal{T} = \{(r,s)|\ 1 \le r < p_+, 1 \le s < p_-\}/\sim$$

where $(r,s) \sim (r',s')$ if and only if $r' = p_+ - r, s' = p_- - s$. Note that $\#\mathcal{T} = \frac{(p_+-1)(p_--1)}{2}$. For $(r,s) \in \mathcal{T}$, let $L(h_{r,s})$ be the Virasoro minimal simple module defined by

$$L(h_{r,s}) = \mathrm{Ker}_{F_{r,s;0}} Q_+^{[r]} / \mathrm{Im}_{F_{r^\vee,s;-1}} Q_+^{[r^\vee]}.$$

Theorem 1.3 ([6, 7, 8])**.** *The* $\frac{(p_+-1)(p_--1)}{2} + 2p_+p_-$ *vector spaces*

$$L(h_{r,s}),\ (r,s) \in \mathcal{T}, \qquad \mathcal{X}^\pm_{r,s},\ 1 \le r \le p_+,\ 1 \le s \le p_-$$

become simple $\mathcal{W}_{p_+,p_-}$*-modules and give all simple* $\mathcal{W}_{p_+,p_-}$*-modules.*

We use the following symbols for the projective covers of the simple modules.

Definition 1.4. *Let $1 \leq r < p_+$, $1 \leq s < p_-$.*

1. *Let $\mathcal{P}^+_{r,s}$ and $\mathcal{P}^-_{r,s}$ be the projective covers of the simple modules $\mathcal{X}^+_{r,s}$ and $\mathcal{X}^-_{r,s}$, respectively.*

2. *Let $\mathcal{P}(h_{r,s})$ be the projective cover of the minimal simple module $L(h_{r,s})$.*

3. *Let $\mathcal{Q}(\mathcal{X}^{\pm}_{r,p_-})_{r^\vee,p_-}$ be the projective covers of the simple modules $\mathcal{X}^+_{r,p_-}$ and $\mathcal{X}^-_{r,p_-}$, respectively.*

4. *Let $\mathcal{Q}(\mathcal{X}^{\pm}_{p_+,s})_{p_+,s^\vee}$ be the projective covers of the simple modules $\mathcal{X}^+_{p_+,s}$ and $\mathcal{X}^-_{p_+,s}$, respectively.*

Theorem 1.5 ([5])**.** *The projective modules $\mathcal{P}^{\pm}_{r,s}$, $\mathcal{Q}(\mathcal{X}^{\pm}_{r,p_-})_{r^\vee,p_-}$ and $\mathcal{Q}(\mathcal{X}^{\pm}_{p_+,s})_{p_+,s^\vee}$ have the following socle series:*

1. *For $\mathcal{P}^+_{r,s}$, we have*

$$\begin{aligned}
&S_1 = \mathcal{X}^+_{r,s},\\
&S_2/S_1 = \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee} \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s},\\
&S_3/S_2 = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r,s},\\
&S_4/S_3 = \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee},\\
&\mathcal{P}^+_{r,s}/S_4 = \mathcal{X}^+_{r,s}.
\end{aligned}$$

where $S_i = \mathrm{Soc}_i$.

2. *For $\mathcal{P}^-_{r^\vee,s}$, we have*

$$\begin{aligned}
&S_1 = \mathcal{X}^-_{r^\vee,s},\\
&S_2/S_1 = \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s},\\
&S_3/S_2 = \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r^\vee,s} \oplus \mathcal{X}^-_{r,s^\vee} \oplus L(h_{r,s}) \oplus L(h_{r,s}) \oplus \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r,s^\vee} \oplus \mathcal{X}^-_{r^\vee,s},\\
&S_4/S_3 = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\
&\mathcal{P}^-_{r^\vee,s}/S_4 = \mathcal{X}^-_{r^\vee,s}.
\end{aligned}$$

3. *Let (a,b,c,d,ϵ) be an element in $\{(r,p_-,r^\vee,+),(r^\vee,p_-,r,p_-,-),(p_+,s,p_+,s^\vee),(p_+,s^\vee,p_+,s)\}$. Then, for the socle series of $\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}$, we have*

$$\begin{aligned}
&\mathrm{Soc}_1 = \mathcal{X}^\epsilon_{a,b},\\
&\mathrm{Soc}_2/\mathrm{Soc}_1 = \mathcal{X}^{-\epsilon}_{c,d} \oplus \mathcal{X}^{-\epsilon}_{c,d},\\
&\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{c,d}/\mathrm{Soc}_2 = \mathcal{X}^\epsilon_{a,b}.
\end{aligned}$$

Definition 1.6. *By taking quotients of $\mathcal{P}^+_{r,s}$, $\mathcal{P}^+_{r^\vee,s^\vee}$, $\mathcal{P}^-_{r^\vee,s}$ and $\mathcal{P}^-_{r,s^\vee}$, we obtain eight indecomposable modules $\mathcal{Q}(\mathcal{X}^\epsilon_{a,b})_{b,c}$ where*

$$\begin{aligned}\{(\epsilon,a,b,c,d)\} =& \{(+,r,s,r^\vee,s),(+,r,s,r,s^\vee),(+,r^\vee,s^\vee,r^\vee,s),(+,r^\vee,s^\vee,r,s^\vee),\\ &(-,r^\vee,s,r,s),(-,r^\vee,s,r^\vee,s^\vee),(-,r,s^\vee,r,s),(-,r,s^\vee,r^\vee,s^\vee)\},\end{aligned}$$

and each socle series is given by:

1. *For $\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}$,*

$$\begin{aligned}&\mathrm{Soc}_1 = \mathcal{X}^+_{a,b},\\ &\mathrm{Soc}_2/\mathrm{Soc}_1 = \mathcal{X}^-_{c,d} \oplus L(h_{a,b}) \oplus \mathcal{X}^-_{c,d},\\ &\mathcal{Q}(\mathcal{X}^+_{a,b})_{c,d}/\mathrm{Soc}_2 = \mathcal{X}^+_{a,b}.\end{aligned}$$

2. *For $\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}$,*

$$\begin{aligned}&\mathrm{Soc}_1 = \mathcal{X}^-_{a,b},\\ &\mathrm{Soc}_2/\mathrm{Soc}_1 = \mathcal{X}^+_{c,d} \oplus \mathcal{X}^+_{c,d},\\ &\mathcal{Q}(\mathcal{X}^-_{a,b})_{c,d}/\mathrm{Soc}_2 = \mathcal{X}^-_{a,b}.\end{aligned}$$

Using the structure of the center of the Zhu algebra $A(\mathcal{W}_{p_+,p_-})$ [6, 7, 8], we can determine the structure of the projective modules $\mathcal{P}(h_{r,s})$.

Theorem 1.7 ([5])**.** *Each projective module $\mathcal{P}(h_{r,s})$ has the following length five socle series:*

$$\begin{aligned}&\mathrm{Soc}_1(\mathcal{P}(h_{r,s})) = L(h_{r,s}),\\ &\mathrm{Soc}_2(\mathcal{P}(h_{r,s}))/\mathrm{Soc}_1(\mathcal{P}(h_{r,s})) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\ &\mathrm{Soc}_3(\mathcal{P}(h_{r,s}))/\mathrm{Soc}_2(\mathcal{P}(h_{r,s})) = 2\mathcal{X}^-_{r^\vee,s} \oplus L(h_{r,s}) \oplus 2\mathcal{X}^-_{r,s^\vee},\\ &\mathrm{Soc}_4(\mathcal{P}(h_{r,s}))/\mathrm{Soc}_3(\mathcal{P}(h_{r,s})) = \mathcal{X}^+_{r,s} \oplus \mathcal{X}^+_{r^\vee,s^\vee},\\ &\mathcal{P}(h_{r,s})/\mathrm{Soc}_4(\mathcal{P}(h_{r,s})) = L(h_{r,s}).\end{aligned}$$

In the following, we introduce the structure of certain fusion rules of $\mathcal{W}_{p_+,p_-}$. Let us define the following indecomposable modules.

Definition 1.8.

1. *For $1 \le r \le p_+ - 1$, $1 \le s \le p_- - 1$,*

$$\mathcal{K}_{r,s} := \mathcal{W}_{p_+,p_-}.\,|\alpha_{r,s}\rangle\,.$$

2. *For* $1 \le r \le p_+$, $1 \le s \le p_-$,

$$\mathcal{K}_{r,p_-} := \mathcal{X}^+_{r,p_-}, \qquad \mathcal{K}_{p_+,s} := \mathcal{X}^+_{p_+,s}.$$

Let $\mathcal{C}_{p_+,p_-}$ be the category of $\mathcal{W}_{p_+,p_-}$-modules and let $(\mathcal{C}_{p_+,p_-}, \boxtimes, \mathcal{K}_{1,1})$ be the braided tensor category on $\mathcal{C}_{p_+,p_-}$, where $\mathcal{K}_{1,1}$ is the unit object.

Similar to the arguments in [10, 11, 12], we can show the following theorem.

Theorem 1.9. *The indecomposable modules* $\mathcal{K}_{1,2}$ *and* $\mathcal{K}_{2,1}$ *are rigid and self-dual.*

Using the self-duality of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$, we obtain the following theorems.

Theorem 1.10 ([5]). *All indecomposable modules of types* $\mathcal{K}_{r,s}$, $\mathcal{Q}(\mathcal{X}^\pm_{r,s})_{\bullet,\bullet}$ *and* $\mathcal{P}^\pm_{r,s}$ *are rigid and self-dual in* $(\mathcal{C}_{p_+,p_-}, \boxtimes, \mathcal{K}_{1,1})$.

Theorem 1.11 ([5]). 1. *For* $1 \le r \le p_+$, $1 \le s \le p_-$,

$$\mathcal{K}^*_{1,1} \boxtimes \mathcal{K}_{r,s} = \mathcal{K}^*_{r,s},$$

where $\mathcal{K}^*_{r,s}$ *is the contragredient of* $\mathcal{K}_{r,s}$.

2. *For any simple modules* $\mathcal{X}^\pm_{r,s}$ *and* $\mathcal{X}^\pm_{r',s'}$, *we have*

$$\mathcal{X}^\pm_{r,s} \boxtimes \mathcal{X}^\pm_{r',s'} = (\mathcal{K}_{r,s} \boxtimes \mathcal{K}_{r',s'}) \boxtimes \mathcal{K}^*_{1,1}.$$

Let us introduce the free abelian group $P^0(\mathcal{C}_{p_+,p_-})$ of rank $8p_+p_- - 4p_+ - 4p_- + 2$

$$\begin{aligned} P^0(\mathcal{C}_{p_+,p_-}) = & \bigoplus_{r=1}^{p_+} \bigoplus_{s=1}^{p_-} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{X}^\epsilon_{r,s}]_P \oplus \bigoplus_{r=1}^{p_+-1} \bigoplus_{s=1}^{p_--1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{P}^\epsilon_{r,s}]_P \\ & \oplus \bigoplus_{r=1}^{p_+-1} \bigoplus_{s=1}^{p_--1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}^\epsilon_{r,s})_{r^\vee,s}]_P \oplus \bigoplus_{r=1}^{p_+-1} \bigoplus_{s=1}^{p_--1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}^\epsilon_{r,s})_{r,s^\vee}]_P \\ & \oplus \bigoplus_{r=1}^{p_+-1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}^\epsilon_{r,p_-})_{r^\vee,p_-}]_P \oplus \bigoplus_{s=1}^{p_--1} \bigoplus_{\epsilon=\pm} \mathbb{Z}[\mathcal{Q}(\mathcal{X}^\epsilon_{p_+,s})_{p_+,s^\vee}]_P. \end{aligned}$$

For any $M \in \mathcal{C}_{p_+,p_-}$ which have minimal simple modules in the Socle, let $\pi_0(M)$ be the quotient module of M quotiented by all the minimal simple modules in the Socle. We define a $\pi \in \mathrm{Hom}(\mathcal{C}_{p_+,p_-})$ such that for any M in $\mathcal{C}_{p_+,p_-}$

$$\pi(M) = \begin{cases} \pi_0(M) & M \text{ contains minimal simple modules in } \mathrm{Soc}(M) \\ M & \text{otherwise} \end{cases}$$

Theorem 1.12 ([5]). *$P^0(\mathcal{C}_{p_+,p_-})$ has the structure of a commutative ring where the product as a ring is given by*

$$[\bullet]_P \cdot [\bullet]_P = [\pi(\bullet \boxtimes \bullet)]_P.$$

The three operators

$$X = \pi(\mathcal{X}^+_{1,2} \boxtimes -), \qquad Y = \pi(\mathcal{X}^+_{2,1} \boxtimes -), \qquad Z = \pi(\mathcal{X}^-_{1,1} \boxtimes -)$$

define $\mathbb{Z}$-linear endomorphism of $P^0(\mathcal{C}_{p_+,p_-})$. Thus $P^0(\mathcal{C}_{p_+,p_-})$ is a module over $\mathbb{Z}[X,Y,Z]$. We define the following $\mathbb{Z}[X,Y,Z]$-module map

$$\begin{aligned} \psi : \mathbb{Z}[X,Y,Z] &\to P^0(\mathcal{C}_{p_+,p_-}), \\ f(X,Y,Z) &\mapsto f(X,Y,Z) \cdot [\mathcal{X}^+_{1,1}]_P. \end{aligned}$$

Theorem 1.13 ([5]). *The $\mathbb{Z}[X,Y,Z]$-module map ψ is surjective, and, through ψ, we have the isomorphism of rings*

$$P^0(\mathcal{C}_{p_+,p_-}) \simeq \frac{\mathbb{Z}[X,Y] \oplus \mathbb{Z}[X,Y]Z}{\langle Z^2 - 1, U_{2p_- -1}(X) - 2ZU_{p_- -1}(X), U_{2p_+ -1}(Y) - 2ZU_{p_+ -1}(Y)\rangle},$$

where $U_n(A)$ is the Chebyshev polynomials defined recursively

$$\begin{aligned} &U_0(A) = 1, \qquad U_1(A) = A, \\ &U_{n+1}(A) = AU_n(A) - U_{n-1}(A). \end{aligned}$$

Remark 1.14. *By using this theorem, we can obtain the non-semisimple fusion rules conjectured by [3] and [4].*

References

[1] B. L. Feigin, A.M. Gainutdinov, A.M. Semikhatov, and I. Yu Tipunin, "Logarithmic extensions of minimal models: characters and modular transformation", *Nuclear Phys. B* 757(2006),303-343.

[2] Y. Z. Huang, J. Lepowsky and L. Zhang, "Logarithmic tensor product theory for generalized modules for a conformal vertex algebra", arXiv:0710.2687v3 [math.QA].

[3] J. Rasmussen, "W-extended logarithmic minimal models", *Nucl. Phys. B* **807** (2009) 495 [0805.2991 [hep-th]].

[4] M. Gaberdiel, I. Runkel and S. Wood, "Fusion rules and boundary conditions in the $c = 0$ triplet model", *J.Phys.* **A42** (2009) 325403, arXiv:0905.0916 [hep-th].

[5] H. Nakano, "The category of modules of the triplet W-algebras associated to the Virasoro minimal models", *the Doctor Thesis, Mathematical Institute, Tohoku University.*

[6] D. Adamović and A. Milas, "On $\mathcal{W}$-algebras associated to $(2, p)$ minimal models for certain vertex algebras", *International Mathematics Research Notices* 2010 (2010) 20 : 3896-3934, arXiv:0908.4053.

[7] D. Adamović and A. Milas, "On W-algebra extensions of (2, p) minimal models: p >3", *Journal of Algebra* **344** (2011) 313-332. arXiv:1101.0803.

[8] A. Tsuchiya and S. Wood, "On the extended W-algebra of type sl_2 at positive rational level", *International Mathematics Research Notices*, Volume 2015, Issue 14, 1 January 2015, Pages 5357-5435.

[9] A. Tsuchiya and Y. Kanie. "Fock space representations of the Virasoro algebra - Intertwining operators", *Publ. RIMS, Kyoto Univ.* 22(1986) 259-327.

[10] A. Tsuchiya, S. Wood. "The tensor structure on the representation category of the $\mathcal{W}_p$ triplet algebra", *J. Phys. A* **46** (2013), no. 44, 445203, 40 pp.

[11] T. Creutzig, R. McRae. and J. Yang, "On ribbon categories for singlet vertex algebras", *Communications in Mathematical Physics*, 387(2), 865-925, arXiv:2007.12735.

[12] R. McRae and J. Yang, "Structure of Virasoro tensor categories at central charge $13 - 6p - 6p^{-1}$ for integers $p > 1$", arXiv:2011.02170 (2020).