Yang-Baxter algebra and identities with application to Gysin maps

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Abstract

We give the cohomological version as well as a summary of the K-theory version of some of the results of our recent works on applications of the Yang-Baxter algebra to identities of partition functions, symmetric functions and formulas for algebraic geometry. We also illustrate the case of complete flag bundles in some detail.

1 Introduction

It has been revealed that same symmetric functions appear in the field of Schubert calculus and integrable models and related algebra. One typical example is the relation between the Grothendieck polynomials [1, 2, 3, 4, 5, 6, 7] and mathematical physics and related algebraic methods, partition functions of integrable lattice models and gauge theory, vertex operators and so on. See [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] for examples for various topics. This fact suggests us to explore further and deepen connections. In our recent works [20, 21, 22], we investigated an integrability approach of deriving identities between partition functions and applications to formulas in algebraic geometry: the so-called K-theoretic Gysin map. In this paper, we discuss the cohomological version, which computations/proofs go pararell with the K-theory version. The difference is the R-matrix which we use. We also illustrate the case of complete flag bundles in some detail, and briefly summarize some of the results of the K-theory version. The details can be found in the papers mentioned above.

2 Cohomological version

2.1 R-matrix, Yang-Baxter algebra, Partition functions

First we introduce the R-matrix, the Yang-Baxter algebra associated with it and the partition functions which we use for the cohomological version.

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Let W be an $(m+1)$-dimensional complex vector space and denote its standard basis as $\{|0\rangle, |1\rangle, \dots, |m\rangle\}$. We denote the dual of $|k\rangle$ as $\langle k|$ $(k = 0, 1, \dots, m)$. The dual vector space is denoted as W^*, which is spanned by $\{\langle 0|, \langle 1|, \dots, \langle m|\}$. Using the bra-ket notation without taking complex conjugation, the orthogonality of standard basis is expressed as $\langle k|\ell\rangle = \delta_{k\ell}$, $k, \ell = 0, 1, \dots, m$.

The tensor product of two m-dimensional vector spaces $W_i \otimes W_j$ has $\{|k\rangle_i \otimes |\ell\rangle_j \mid k, \ell = 0, 1, \dots, m\}$ as the standard basis. The R-matrix $R_{ij}(u, w)$ acting on $W_i \otimes W_j$ is defined by acting on this basis as

$$R_{ij}(u,w)|k\rangle_i \otimes |k\rangle_j = |k\rangle_i \otimes |k\rangle_j, \quad k = 0, 1, \dots, m, \tag{2.1}$$

$$R_{ij}(u,w)|k\rangle_i \otimes |\ell\rangle_j = |\ell\rangle_i \otimes |k\rangle_j, \quad 0 \le k < \ell \le m, \tag{2.2}$$

$$R_{ij}(u,w)|k\rangle_i \otimes |\ell\rangle_j = |\ell\rangle_i \otimes |k\rangle_j + (u-w)|k\rangle_i \otimes |\ell\rangle_j, \quad m \ge k > \ell \ge 0. \tag{2.3}$$

Here, u and w are complex numbers.

More generally, for an integer $p \ge 2$, we define $R_{ij}(u, w)$ $(1 \le i < j \le p)$ as an operator acting on $W_1 \otimes W_2 \otimes \cdots \otimes W_p$, acting on the tensor product of W_i and W_j as (2.1), (2.2), (2.3), and acts as identity on all the other vector spaces W_k, $k \ne i, j$.

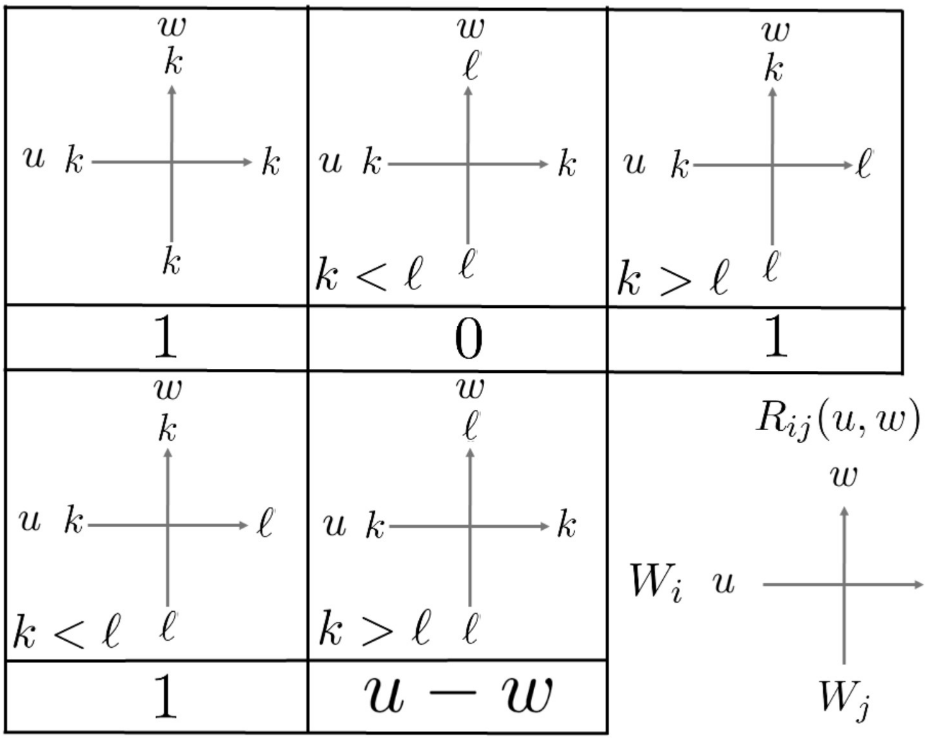

Figure 1: The R-matrix $R_{ij}(u, w)$ (2.1), (2.2), (2.3).

The R-matrix $R_{ij}(u, w)$ satisifies the Yang-Baxter relation

$$R_{ij}(u,v)R_{ik}(u,w)R_{jk}(v,w) = R_{jk}(v,w)R_{ik}(u,w)R_{ij}(u,v). \tag{2.4}$$

We view (2.4) as a relation in $W_1 \otimes \cdots \otimes W_p$ for an integer $p \ge 3$ by regarding that $R_{ij}(u, v)$ acts nontrivially on W_i and W_j, $R_{ik}(u, w)$ nontrivially on W_i and W_k, $R_{jk}(v, w)$ nontrivially on W_j and W_k, and each of them acts as identity on all the other spaces.

The monodromy matrix $T_a(u)$

$$T_a(u) : = R_{a,p}(u,0) \cdots R_{a1}(u,0), \tag{2.5}$$

is an operator acting on $W_a \otimes W_1 \otimes \cdots \otimes W_p$. The vector space W_a is called as the auxiliary space, and the other spaces $W_1, \ldots, W_p$ are called as quantum spaces.

We denote the matrix elements of the monodromy matrix with respect to the auxiliary space W_a as $T_{ij}(u)$:

$$T_{ij}(u) := {}_a\langle j|T_a(u)|i\rangle_a, \quad i,j = 0, \ldots, m. \tag{2.6}$$

Each of the elements $T_{ij}(u)$ act on $W_1 \otimes \cdots \otimes W_p$ and can be regarded as $(m+1)^p \times (m+1)^p$ matrices. Among the elements, we use the following ones (Figure 2)

$$B_k(u) : = T_{mk}(u), \quad k = 0, 1, \ldots, m-1, \tag{2.7}$$

$$D_m(u) : = T_{mm}(u). \tag{2.8}$$

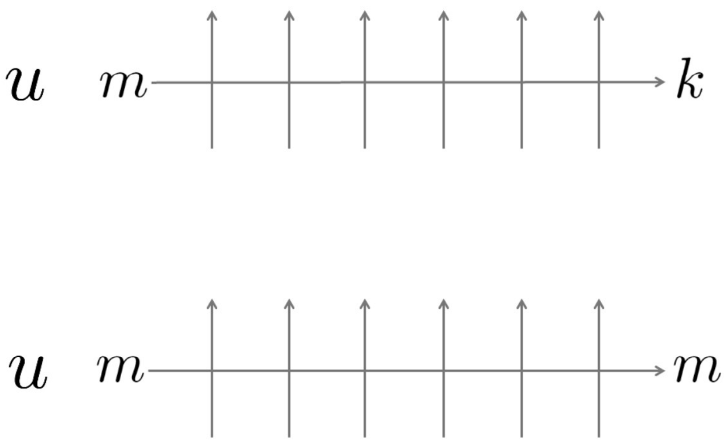

Figure 2: The operators $B_k(u) = T_{mk}(u) = {}_a\langle k|T_a(u)|m\rangle_a$ $(k = 0, 1, \ldots, m-1)$ (2.7) (top) and $D_m(u) = T_{mm}(u) = {}_a\langle m|T_a(u)|m\rangle_a$ (2.8) (bottom).

There are fundamental nontrivial relations between the elements $T_{ij}(u)$ of the monodromy matrix, which are obtained as matrix elements of the following RTT intertwining relation

$$R_{ab}(u_1,u_2)T_a(u_1)T_b(u_2) = T_b(u_2)T_a(u_1)R_{ab}(u_1,u_2). \tag{2.9}$$

The intertwining relation (2.9) follows from the Yang-Baxter relation (2.4), and writing down the matrix elements of (2.9) explicitly, we get commutation relations between the elements

$T_{ij}(u)$, $i, j = 0, 1, \dots, m$. Some of them which we use are (Figure 3)

$$D_m(u_1)B_j(u_2) = \frac{1}{u_1 - u_2} B_j(u_2)D_m(u_1) + \frac{1}{u_2 - u_1} B_j(u_1)D_m(u_2), \quad j = 0, 1, \dots, m-1, \tag{2.10}$$

$$B_j(u_1)B_k(u_2) = \frac{1}{u_1 - u_2} B_k(u_2)B_j(u_1) + \frac{1}{u_2 - u_1} B_k(u_1)B_j(u_2), \quad 0 \le k < j \le m-1, \tag{2.11}$$

$$D_m(u_1)B_j(u_2) = D_m(u_2)B_j(u_1), \quad j = 0, 1, \dots, m-1, \tag{2.12}$$

$$B_j(u_1)B_k(u_2) = B_j(u_2)B_k(u_1), \quad 0 \le k \le j \le m-1, \tag{2.13}$$

$$D_m(u_1)D_m(u_2) = D_m(u_2)D_m(u_1). \tag{2.14}$$

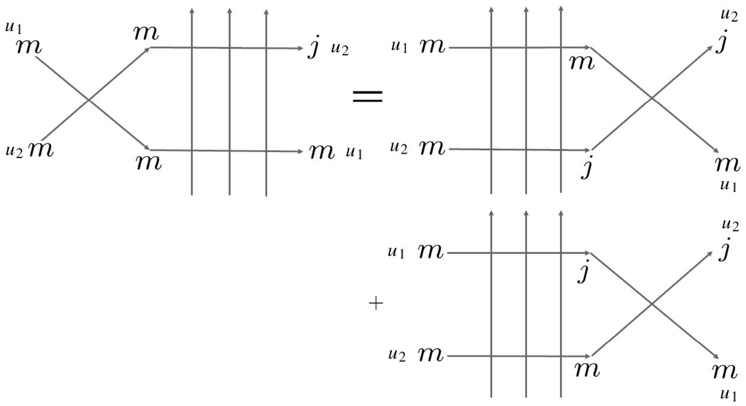

Figure 3: A graphical description of an element of the intertwining relation. The left hand side represents $B_j(u_2)D_m(u_1)$, and the top and bottom panel of the right hand side and represents $(u_1 - u_2)D_m(u_1)B_j(u_2)$ and $B_j(u_1)D_m(u_2)$ respectively. This means $B_j(u_2)D_m(u_1) = (u_1 - u_2)D_m(u_1)B_j(u_2) + B_j(u_1)D_m(u_2)$, which is essentially (2.10).

We introduce a set of integers $q_1, q_2, \dots, q_m$ satisfying $q_0 := 0 < q_1 < q_2 < \dots < q_m < q_{m+1} := n$. Using the argument given in [23], we can derive the following multiple commutation relations from the basic commutation relations (2.10), (2.11), (2.12), (2.13) and (2.14).

Proposition 2.1. *The following multiple commutation relations hold:*

$$\begin{aligned} &\prod_{j=1}^{q_1} D_m(u_j) \prod_{j=q_1+1}^{q_2} B_{m-1}(u_j) \cdots \prod_{j=q_{m-1}+1}^{q_m} B_1(u_j) \prod_{j=q_m+1}^{n} B_0(u_j) \\ &= \sum_{\overline{w} \in S_n / S_{q_1} \times S_{q_2 - q_1} \times \cdots \times S_{n-q_m}} w \cdot \left[\frac{1}{\prod_{k=1}^{m} \prod_{q_{k-1} < i \le q_k} \prod_{q_k < j \le n} (u_i - u_j)} \right. \\ &\left. \times \prod_{j=q_m+1}^{n} B_0(u_j) \prod_{j=q_{m-1}+1}^{q_m} B_1(u_j) \cdots \prod_{j=q_1+1}^{q_2} B_{m-1}(u_j) \prod_{j=1}^{q_1} D_m(u_j) \right]. \end{aligned} \tag{2.15}$$

Here, w in the right hand side of (2.15) *acts as permutation on the the spectral parameters* $(u_1, u_2, \dots, u_n)$*, and the sum is over all representatives of elements of the fixed subgroup* $S_n/S_{q_1} \times S_{q_2-q_1} \times \cdots \times S_{n-q_m}$ *of the symmetric group* S_n*.*

We next introduce partition functions. We first introduce notations for the basis of the space $W_1 \otimes W_2 \otimes \cdots \otimes W_p$ and its dual $W_1^* \otimes W_2^* \otimes \cdots \otimes W_p^*$. For an ordered set of integers $I = \{i_1, i_2, \dots, i_p\}$ satisfying $0 \le i_1, i_2, \dots, i_p \le m$, we define $|I\rangle$ and $\langle I|$ as

$$|I\rangle = |i_1\rangle_1 \otimes |i_2\rangle_2 \otimes \cdots \otimes |i_p\rangle_p \in W_1 \otimes W_2 \otimes \cdots \otimes W_p, \tag{2.16}$$

$$\langle I| = {}_1\langle i_1| \otimes {}_2\langle i_2| \otimes \cdots \otimes {}_p\langle i_p| \in W_1^* \otimes W_2^* \otimes \cdots \otimes W_p^*. \tag{2.17}$$

$\{|I\rangle \mid 0 \le i_1, i_2, \dots, i_p \le m\}$ and $\{\langle I| \mid 0 \le i_1, i_2, \dots, i_p \le m\}$ forms a standard basis of $W_1 \otimes W_2 \otimes \cdots \otimes W_p$ and $W_1^* \otimes W_2^* \otimes \cdots \otimes W_p^*$ respectively.

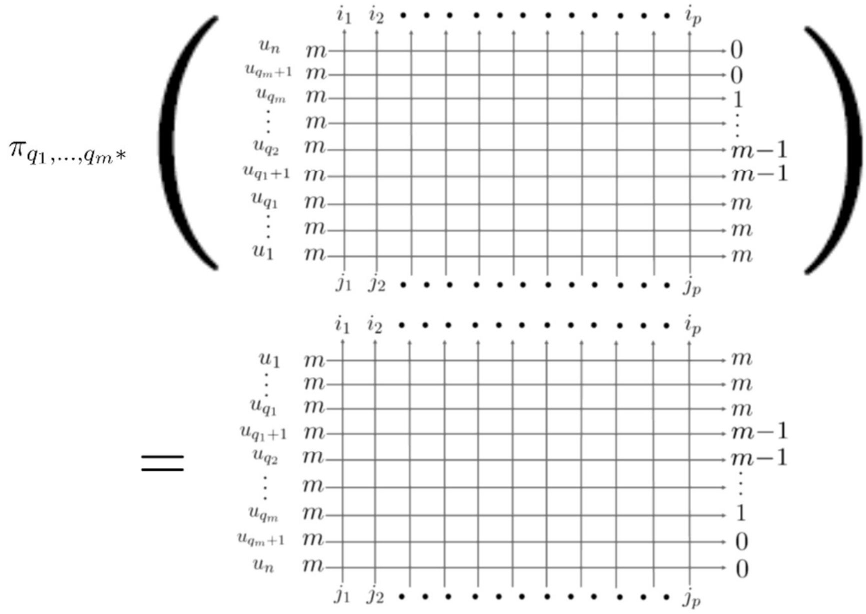

Figure 4: The figure inside $\pi_{q_1,\dots,q_m*}()$ of the top panel represents the partition functions $F_{IJ}(u_1, \dots, u_n)$ (2.18), and the bottom panel represents $G_{IJ}(u_1, \dots, u_n)$ (2.19).

The first type of partition functions $F_{IJ}(u_1, \dots, u_n)$ which we introduce is (Figure 4, top panel)

$$\begin{aligned} &F_{IJ}(u_1, \dots, u_n) \\ =&\langle I| \prod_{j=q_m+1}^{n} B_0(u_j) \prod_{j=q_{m-1}+1}^{q_m} B_1(u_j) \cdots \prod_{j=q_1+1}^{q_2} B_{m-1}(u_j) \prod_{j=1}^{q_1} D_m(u_j) |J\rangle. \end{aligned} \tag{2.18}$$

The second type of partition functions $G_{IJ}(u_1, \dots, u_n)$ is (Figure 4, bottom panel) the following one

$$
\begin{aligned}
&G_{IJ}(u_1,\dots,u_n)\\
=&\langle I|\prod_{j=1}^{q_1}D_m(u_j)\prod_{j=q_1+1}^{q_2}B_{m-1}(u_j)\cdots\prod_{j=q_{m-1}+1}^{q_m}B_1(u_j)\prod_{j=q_m+1}^{n}B_0(u_j)|J\rangle,
\end{aligned}
\tag{2.19}
$$

which the order of operators to define $G_{IJ}(u_1,\dots,u_n)$ in (2.19) is reversed from the one for $F_{IJ}(u_1,\dots,u_n)$ (2.18).

We omit here, but we can also introduce the third type of partition functions which uses only B_0-operators and acts on a larger quantum space and is equivalent to the second type $G_{IJ}(u_1,\dots,u_n)$ in the sense that the partition functions are represented by the same polynomials. The third type of partition functions imply that $G_{IJ}(u_1,\dots,u_n)$ is fully symmetric in $u_1,\dots,u_n$. See [22] for details.

2.2 Partition functions and cohomological Gysin map for partial flag bundles

The multiple commutation relations of the Yang-Baxter algebra (2.15) have similarities with the description of the cohomological pushforward (Gysin map) for partial flag bundles using symmetrizing operators. See [24, 25, 26, 27, 28, 29, 30, 31, 32] and [33] in the same volume of the RIMS Kokyuroku for examples for literature on the Gysin maps from various point of views besides the symmetrizing operator description and applications such as geometric derivation of identities and formulas. Let us first recall the symmetrizing operator description.

Let $E\longrightarrow X$ be a complex vector bundle of rank n. We denote the bundles of flags of subspaces of dimensions $q_1,\dots,q_m$ ($q_0:=0<q_1<q_2<\cdots<q_m<q_{m+1}:=n$) as $\pi_{q_1,\dots,q_m}:\mathcal{F}\ell_{q_1,\dots,q_m}(E)\longrightarrow X$. There exists a universal flag of subbundles of the pullback $\pi^*_{q_1,\dots,q_m}(E)$ of E on $\mathcal{F}\ell_{q_1,\dots,q_m}(E)$,

$$
U_{q_0}:=0\subsetneq U_{q_1}\subsetneq U_{q_2}\subsetneq\cdots\subsetneq U_{q_m}\subsetneq U_{q_{m+1}}:=\pi^*_{q_1,\dots,q_m}(E),
\tag{2.20}
$$

where the rank of subbundle U_{q_i} is q_i for $i=0,1,\dots,m+1$.

The special case $m=n-1$, $q_j=j$ ($j=1,2,\dots,n-1$) of the flag bundle $\pi_{1,2,\dots,n-1}:\mathcal{F}\ell_{1,2,\dots,n-1}(E)\longrightarrow X$ is called the complete flag bundle, on which there exists the universal flag of subbundles

$$
0=U_0\subsetneq U_1\subsetneq\cdots\subsetneq U_{n-1}\subsetneq\pi^*_{1,2,\dots,n-1}(E).
\tag{2.21}
$$

We denote the first Chern class of the dual line bundle $(U_i/U_{i-1})^\vee$ as u_i, i.e., we set $u_i:=c_1((U_i/U_{i-1})^\vee)$.

One of the latest results of the Gysin map is the extension to the generalized cohomology theory by Nakagawa-Naruse ([30] Thm 4.10, [32] Remark 3.9). The cohomology case is the following.

Theorem 2.2. *Let $\pi_{q_1,\dots,q_m}$ be the partial flag bundle $\pi_{q_1,\dots,q_m}:\mathcal{F}\ell_{q_1,\dots,q_m}(E)\longrightarrow X$. The pushforward $\pi_{q_1,\dots,q_m*}:H^*(\mathcal{F}\ell_{q_1,\dots,q_m}(E))\longrightarrow H^*(X)$ of a symmetric polynomial $f(t_1,\dots,t_n)\in$*

$H^*(X)[t_1,\ldots,t_n]^{S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}}$ *is given by*

$$
\begin{aligned}
&\pi_{q_1,\ldots,q_m *}(f(u_1,\ldots,u_n))\\
&=\sum_{\overline{w}\in S_n/S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}} w\cdot\left[\frac{f(u_1,\ldots,u_n)}{\prod_{k=1}^{m}\prod_{q_{k-1}<i\le q_k}\prod_{q_k<j\le n}(u_i-u_j)}\right].
\end{aligned}
\tag{2.22}
$$

Here, w in the right hand side of (2.22) *acts as permutation on the Chern roots $(u_1,u_2,\ldots,u_n)$, and the sum is over all representatives of elements of the fixed subgroup $S_n/S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}$ of the symmetric group S_n.*

Combining and (2.22), we have the following formula for the cohomological pushforward (Figure 4).

Theorem 2.3. *Let $\pi_{q_1,\ldots,q_m}$ be the partial flag bundle $\pi_{q_1,\ldots,q_m}:\mathcal{F}\ell_{q_1,\ldots,q_m}(E)\longrightarrow X$. The pushforward $\pi_{q_1,\ldots,q_m *}:H^*(\mathcal{F}\ell_{q_1,\ldots,q_m}(E))\longrightarrow H^*(X)$ of $F_{IJ}(u_1,\ldots,u_n)$ is given by*

$$
\pi_{q_1,\ldots,q_m *}(F_{IJ}(u_1,\ldots,u_n))=G_{IJ}(u_1,\ldots,u_n). \tag{2.23}
$$

Theorem 2.3 follows from or is essentially equivalent to the following identities between partition functions

$$
\begin{aligned}
&G_{IJ}(u_1,\ldots,u_n)\\
&=\sum_{\overline{w}\in S_n/S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}} w\cdot\left[\frac{F_{IJ}(u_1,\ldots,u_n)}{\prod_{k=1}^{m}\prod_{q_{k-1}<i\le q_k}\prod_{q_k<j\le n}(u_i-u_j)}\right],
\end{aligned}
\tag{2.24}
$$

which follows from taking matrix elements of (2.15).

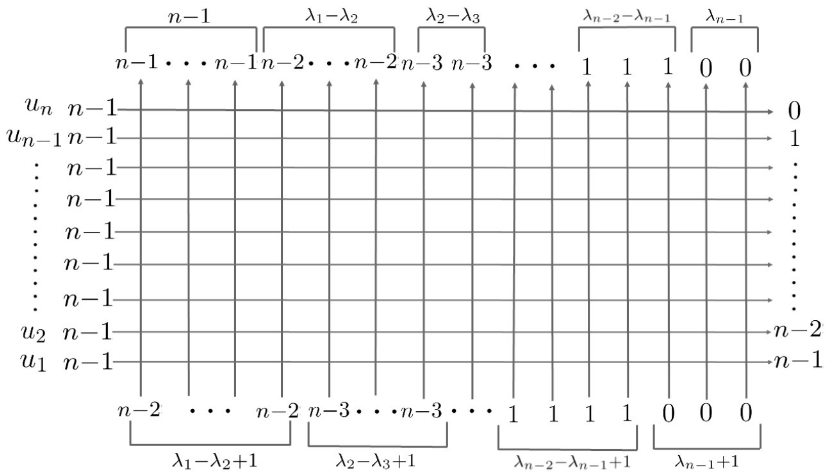

Figure 5: The partition functions $F_{IJ}(u_1,\ldots,u_n)$ corresponding to the case $m=n-1$, $p=\lambda_1+n-1$, $I=((n-1)^{n-1},(n-2)^{\lambda_1-\lambda_2},(n-3)^{\lambda_2-\lambda_3},\ldots,1^{\lambda_{n-2}-\lambda_{n-1}},0^{\lambda_{n-1}})$, $J=((n-2)^{\lambda_1-\lambda_2+1},(n-3)^{\lambda_2-\lambda_3+1},\ldots,1^{\lambda_{n-2}-\lambda_{n-1}+1},0^{\lambda_{n-1}+1})$ of complete flag bundles.

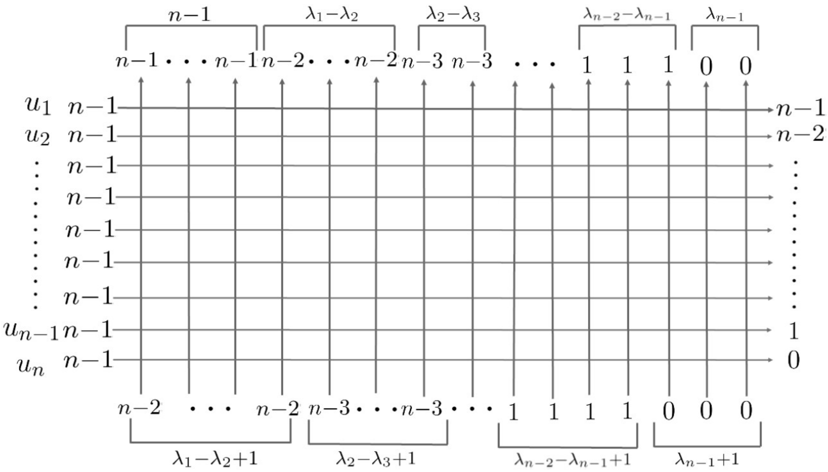

Figure 6: The partition functions $G_{IJ}(u_1,\dots,u_n)$ corresponding to the case $m = n-1$, $p = \lambda_1+n-1$, $I = ((n-1)^{n-1},(n-2)^{\lambda_1-\lambda_2},(n-3)^{\lambda_2-\lambda_3},\dots,1^{\lambda_{n-2}-\lambda_{n-1}},0^{\lambda_{n-1}})$, $J = ((n-2)^{\lambda_1-\lambda_2+1},(n-3)^{\lambda_2-\lambda_3+1},\dots,1^{\lambda_{n-2}-\lambda_{n-1}+1},0^{\lambda_{n-1}+1})$ of complete flag bundles. (2.27) means this partition function is expressed as $s_{(\lambda_1,\dots,\lambda_{n-1})}(u_1,\dots,u_n)$.

2.3 Complete flag bundles

As for the case of complete flag bundles, which corresponds to the case $m = n-1$ of partial flag bundles, the following is known (see [27, 30] for example).

Theorem 2.4. *Let $\pi_{1,2,\dots,n-1}$ be the complete flag bundle $\pi_{1,2,\dots,n-1} : \mathcal{F}\ell_{1,2,\dots,n-1}(E) \longrightarrow X$. The pushforward $\pi_{1,2,\dots,n-1*} : H^*(\mathcal{F}\ell_{1,2,\dots,n-1}(E)) \longrightarrow H^*(X)$ of $\prod_{j=1}^n u_j^{\lambda_j+n-j}$ is given by*

$$\pi_{1,2,\dots,n-1*}\left(\prod_{j=1}^n u_j^{\lambda_j+n-j}\right) = s_\lambda(E), \tag{2.25}$$

where $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_n)$ is a set of nonnegative integers satisfying $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$, and $s_\lambda(E)$ is $\pi^*_{1,2,\dots,n-1}s_\lambda(E) := s_\lambda(u_1,\dots,u_n) \in H^{2|\lambda|}(\mathcal{F}\ell_{1,2,\dots,n-1}(E))$.

We can derive a realization of Schur polynomials using higher rank vertex models by combining Theorem 2.3 and Theorem 2.4. We consider the case $m = n-1$, $p = \lambda_1+n-1$, $I = ((n-1)^{n-1},(n-2)^{\lambda_1-\lambda_2},(n-3)^{\lambda_2-\lambda_3},\dots,1^{\lambda_{n-2}-\lambda_{n-1}},0^{\lambda_{n-1}})$, $J = ((n-2)^{\lambda_1-\lambda_2+1},(n-3)^{\lambda_2-\lambda_3+1},\dots,1^{\lambda_{n-2}-\lambda_{n-1}+1},0^{\lambda_{n-1}+1})$ of Theorem 2.3. In this case, the partition function $F_{IJ}(u_1,\dots,u_n) = \langle I|B_0(u_n)B_1(u_{n-1})\cdots B_{n-2}(u_2)D_{n-1}(u_1)|J\rangle$ (2.18) is graphically represented as Figure 5. By graphical consideration, we can see that actually only one configuration is allowed. Each of the other configurations has at least one local configuration whose corresponding weight (matrix element of the R-matrix) is 0 and therefore does not contribute to the partition function. The process of freezing corresponding to the case $n = 3$, $I = (2,2,1,0)$, $J = (1,1,0,0)$ is illustrated in Figure 7, and Figure 8 is the final result from which we conclude $F_{IJ}(u_1,\dots,u_n) = u_1^4u_2^2$ in that case. In general, we find the product of all the R-matrix elements appearing in the unique configuration is given by $\prod_{j=1}^{n-1} u_j^{\lambda_j+n-j}$, i.e.,

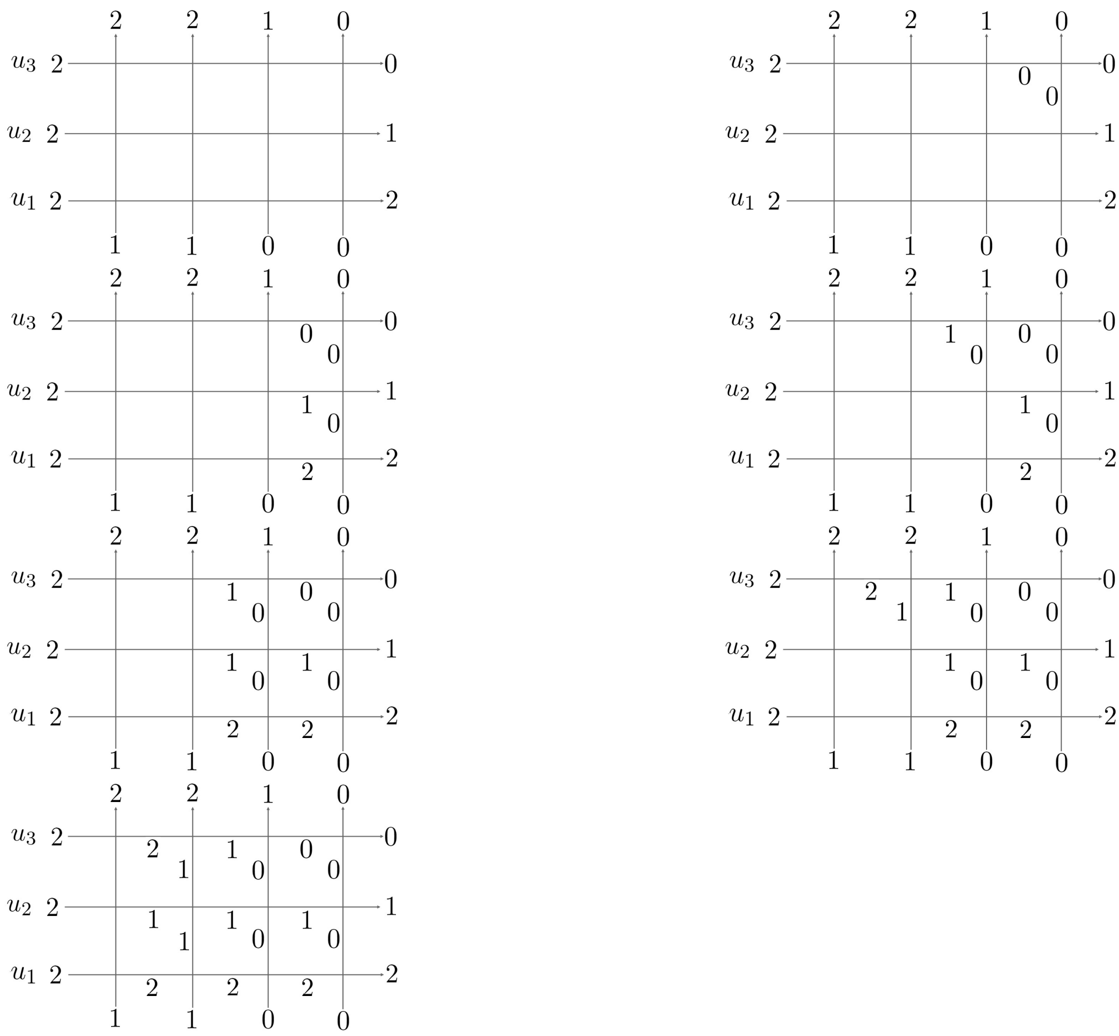

Figure 7: The freezing process of $F_{IJ}(u_1,\dots,u_n)$ for complete flag bundles corresponding to the case $n=3$, $I=(2,2,1,0)$, $J=(1,1,0,0)$.

we have

$$F_{IJ}(u_1,\dots,u_n)=\prod_{j=1}^{n-1}u_j^{\lambda_j+n-j}. \tag{2.26}$$

Theorem 2.3, Theorem 2.4 and (2.26) implies that the partition function $G_{IJ}(u_1,\dots,u_n)=\langle I|D_{n-1}(u_1)B_{n-2}(u_2)\cdots B_1(u_{n-1})B_0(u_n)|J\rangle$ (2.19), which is graphically represented as Figure 6, is nothing but a higher rank vertex model representation of Schur polynomials.

Theorem 2.5. *For $m=n-1$, $p=\lambda_1+n-1$, $I=((n-1)^{n-1},(n-2)^{\lambda_1-\lambda_2},(n-3)^{\lambda_2-\lambda_3},\dots,1^{\lambda_{n-2}-\lambda_{n-1}},0^{\lambda_{n-1}})$, $J=((n-2)^{\lambda_1-\lambda_2+1},(n-3)^{\lambda_2-\lambda_3+1},\dots,1^{\lambda_{n-2}-\lambda_{n-1}+1},0^{\lambda_{n-1}+1})$, we have the following partition function representation of the Schur polynomials*

$$s_{(\lambda_1,\dots,\lambda_{n-1})}(u_1,\dots,u_n)=\langle I|D_{n-1}(u_1)B_{n-2}(u_2)\cdots B_1(u_{n-1})B_0(u_n)|J\rangle. \tag{2.27}$$

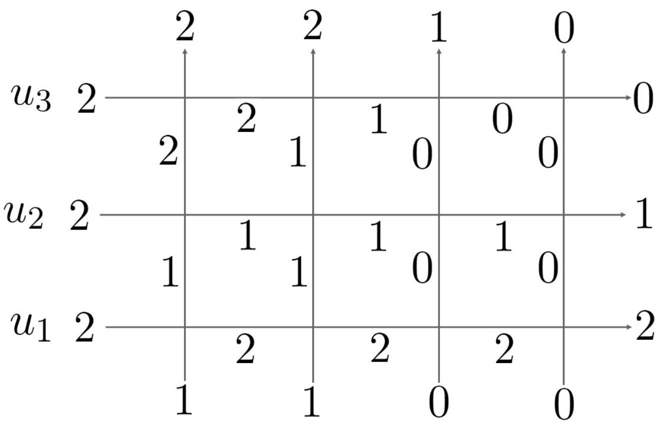

Figure 8: The final result of the freezing process of $F_{IJ}(u_1, \ldots, u_n)$ for complete flag bundles corresponding to the case $n = 3$, $I = (2, 2, 1, 0)$, $J = (1, 1, 0, 0)$. Multiplying all the R-matrix elements which appear in this configuration, one finds $F_{IJ}(u_1, \ldots, u_n) = u_1^4 u_2^2$.

Figure 9 is the illustration for the case $n = 3$, $(\lambda_1, \lambda_2) = (2, 1)$. The bottom panel is $\langle (2, 2, 1, 0)|D_2(u_1)B_1(u_2)B_0(u_3)|(1, 1, 0, 0)\rangle$, the right hand side of (2.27), and the eight panels above are the eight configurations which give nontrivial contributions to the partition function. This means that $\langle (2, 2, 1, 0)|D_2(u_1)B_1(u_2)B_0(u_3)|(1, 1, 0, 0)\rangle$ is expressed as the sum of products of the matrix elements of the R-matrices appearing in each configuration, which is $u_1^2 u_2$, $u_1^2 u_3$, $u_1 u_2 u_3$, $u_1 u_3^2$, $u_1 u_2^2$, $u_1 u_2 u_3$, $u_2^2 u_3$, $u_2 u_3^2$ respectively. Summing all the monomials, we get $u_1^2 u_2 + u_1^2 u_3 + 2u_1 u_2 u_3 + u_1 u_3^2 + u_1 u_2^2 + u_2^2 u_3 + u_2 u_3^2$ which is nothing but $s_{(2,1)}(u_1, u_2, u_3)$, the left hand side of (2.27).

3 K-theory version

The details for the case of K-theory can be seen in detail in [20, 21, 22]. The point is that instead of the R-matrix used in the previous section for the cohomological case, we use the following R-matrix

$$\tilde{R}_{ij}(u, w)|k\rangle_i \otimes |k\rangle_j = |k\rangle_i \otimes |k\rangle_j, \quad k = 0, 1, \ldots, m, \tag{3.1}$$

$$\tilde{R}_{ij}(u, w)|k\rangle_i \otimes |\ell\rangle_j = w/u|\ell\rangle_i \otimes |k\rangle_j, \quad 0 \leq k < \ell \leq m, \tag{3.2}$$

$$\tilde{R}_{ij}(u, w)|k\rangle_i \otimes |\ell\rangle_j = |\ell\rangle_i \otimes |k\rangle_j + (1 - w/u)|k\rangle_i \otimes |\ell\rangle_j, \quad m \geq k > \ell \geq 0. \tag{3.3}$$

Let us denote the operators and partition functions in the previous section with the R-matrix constructing them replaced by the one defined by (3.1), (3.2), (3.3) as $\tilde{D}_m(u)$, $\tilde{B}_j(u)$, $j = 0, 1, \ldots, m-1$ and

$$\begin{aligned} &\tilde{F}_{IJ}(u_1, \ldots, u_n) \\ =&\langle I| \prod_{j=q_m+1}^{n} \tilde{B}_0(u_j) \prod_{j=q_{m-1}+1}^{q_m} \tilde{B}_1(u_j) \cdots \prod_{j=q_1+1}^{q_2} \tilde{B}_{m-1}(u_j) \prod_{j=1}^{q_1} \tilde{D}_m(u_j)|J\rangle, \end{aligned} \tag{3.4}$$

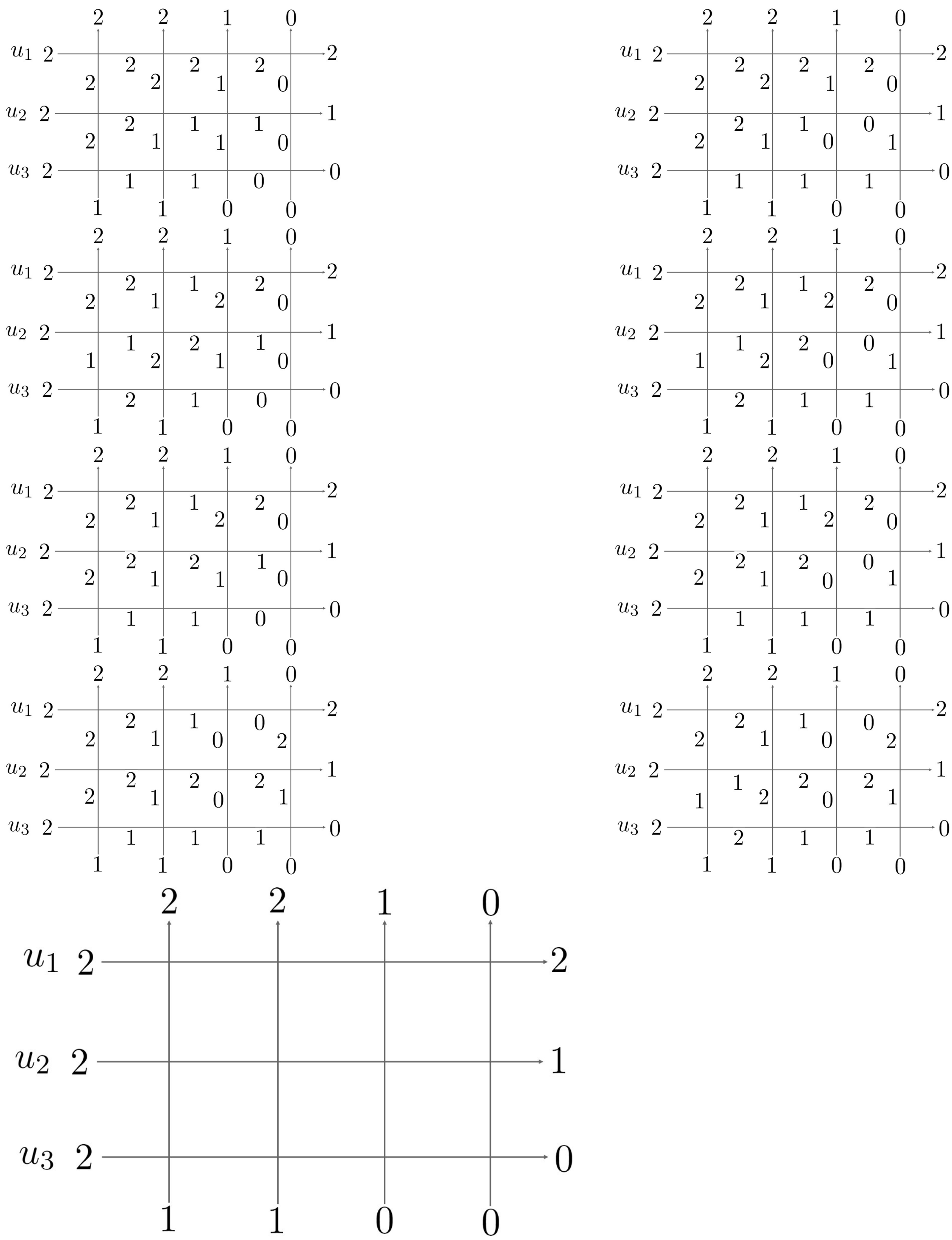

Figure 9: The bottom panel is $G_{IJ}(u_1, \ldots, u_n)$ for complete flag bundles corresponding to the case $n = 3$, $I = (2, 2, 1, 0)$, $J = (1, 1, 0, 0)$. The other eight panels are configurations which give nonzero contributions to the partition functions. Multiplying all the matrix elements of the R-matrix for each configuration, we get $u_1^2 u_2$, $u_1^2 u_3$, $u_1 u_2 u_3$, $u_1 u_3^2$, $u_1 u_2^2$, $u_1 u_2 u_3$, $u_2^2 u_3$, $u_2 u_3^2$ respectively. Summing all the monomials give $u_1^2 u_2 + u_1^2 u_3 + 2u_1 u_2 u_3 + u_1 u_3^2 + u_1 u_2^2 + u_2^2 u_3 + u_2 u_3^2$ which is nothing but $s_{(2,1)}(u_1, u_2, u_3)$.

$$
\begin{aligned}
&\tilde{G}_{IJ}(u_1,\dots,u_n)\\
=&\langle I|\prod_{j=1}^{q_1}\tilde{D}_m(u_j)\prod_{j=q_1+1}^{q_2}\tilde{B}_{m-1}(u_j)\cdots\prod_{j=q_{m-1}+1}^{q_m}\tilde{B}_1(u_j)\prod_{j=q_m+1}^{n}\tilde{B}_0(u_j)|J\rangle.
\end{aligned}\tag{3.5}
$$

The multiple commutation relations become the following form

$$
\begin{aligned}
&\prod_{j=1}^{q_1}\tilde{D}_m(u_j)\prod_{j=q_1+1}^{q_2}\tilde{B}_{m-1}(u_j)\cdots\prod_{j=q_{m-1}+1}^{q_m}\tilde{B}_1(u_j)\prod_{j=q_m+1}^{n}\tilde{B}_0(u_j)\\
=&\sum_{\overline{w}\in S_n/S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}} w\cdot\Bigg[\frac{1}{\prod_{k=1}^{m}\prod_{q_{k-1}<i\le q_k}\prod_{q_k<j\le n}(1-u_j/u_i)}\\
&\times\prod_{j=q_m+1}^{n}\tilde{B}_0(u_j)\prod_{j=q_{m-1}+1}^{q_m}\tilde{B}_1(u_j)\cdots\prod_{j=q_1+1}^{q_2}\tilde{B}_{m-1}(u_j)\prod_{j=1}^{q_1}\tilde{D}_m(u_j)\Bigg].
\end{aligned}\tag{3.6}
$$

For a smooth scheme X, $K^0(X)$ denotes the Grothendieck group of locally free coherent sheaves on X. For $\mathcal{E}$ a locally free sheaf on X, we denote its class in $K^0(X)$ by $[\mathcal{E}]$. We introduce the flag bundles as before, and denote the class of the dual line bundle $(U_i/U_{i-1})^\vee$ as u_i. The following is the symmetrizing operator description for the case of K-theory (see [30] Thm 4.10, [32] Remark 3.9 for example).

Theorem 3.1. *Let $\pi_{q_1,\dots,q_m}$ be the partial flag bundle $\pi_{q_1,\dots,q_m}:\mathcal{F}\ell_{q_1,\dots,q_m}(E)\longrightarrow X$. The pushforward $\pi_{q_1,\dots,q_m *}:K^0(\mathcal{F}\ell_{q_1,\dots,q_m}(E))\longrightarrow K^0(X)$ of a symmetric polynomial $f(1-t_1^{-1},\dots,1-t_n^{-1})\in K^0(X)[t_1^{-1},\dots,t_n^{-1}]^{S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}}$ is given by*

$$
\begin{aligned}
&\pi_{q_1,\dots,q_m *}(f(1-u_1^{-1},\dots,1-u_n^{-1}))\\
=&\sum_{\overline{w}\in S_n/S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}} w\cdot\left[\frac{f(1-u_1^{-1},\dots,1-u_n^{-1})}{\prod_{k=1}^{m}\prod_{q_{k-1}<i\le q_k}\prod_{q_k<j\le n}(1-u_j/u_i)}\right].
\end{aligned}\tag{3.7}
$$

Here, w in the right hand side of (3.7) *acts as permutation on the Grothendieck roots $(u_1,u_2,\dots,u_n)$, and the sum is over all representatives of elements of the fixed subgroup $S_n/S_{q_1}\times S_{q_2-q_1}\times\cdots\times S_{n-q_m}$ of the symmetric group S_n.*

From (3.6) and (3.7), we have the following.

Theorem 3.2. *Let $\pi_{q_1,\dots,q_m}$ be the partial flag bundle $\pi_{q_1,\dots,q_m}:\mathcal{F}\ell_{q_1,\dots,q_m}(E)\longrightarrow X$. The pushforward $\pi_{q_1,\dots,q_m *}:K^0(\mathcal{F}\ell_{q_1,\dots,q_m}(E))\longrightarrow K^0(X)$ of $\tilde{F}_{IJ}(u_1,\dots,u_n)$ is given by*

$$
\pi_{q_1,\dots,q_m *}(\tilde{F}_{IJ}(u_1,\dots,u_n))=\tilde{G}_{IJ}(u_1,\dots,u_n).\tag{3.8}
$$

See [20, 21, 22] for details where the case of the Grassmann bundles is written down explicitly using skew Grothendieck polynomials, and have overlap with Buch's pushforward formulas [34]. The simplest case of (3.6) gives the skew generalization of the identity by Guo-Sun [35] which is the generalization of the one for Schur polynomials by Fehér-Némethi-Rimányi [36].

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