測地距離空間における不動点近似点列の生成法とその収束性

Generation method and its convergence of an approximation sequence to a fixed point in geodesic spaces

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Abstract

We propose a new approximation method convergent to a fixed point of a given mapping defined on a Hadamard space. It is a modified version of the known CQ projection method.

1 Introduction

Approximation methods of fixed points for nonexpansive mappings have been widely investigated in nonlinear analysis. For example, we know that various generating techniques of iterative schemes converge to a fixed point. In this work, we mainly focus on the projection methods among them. The following result is an example of such a technique, the CQ projection method proved by Nakajo and Takahashi [5].

Theorem 1 (Nakajo and Takahashi [5]). *Let C be a nonempty closed convex subset of a Hilbert space H, and $T\colon C \to C$ a nonexpansive mapping, that is,*

$$\|Tx - Ty\| \leq \|x - y\|$$

for every $x, y \in C$. Suppose that the set $\operatorname{Fix} T$ of all fixed points of T is nonempty. Let $\{\alpha_n\} \subset [0, a]$ be a real sequence, where $a \in [0, 1[$. Generate a sequence $\{x_n\} \subset X$

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as follows: $x_0 = x_1 \in X$, *and*

$$\begin{aligned} y_n &= \alpha_n + (1-\alpha)Tx_n, \\ C_{n+1} &= \{z \in X \mid \|y_n - z\| \le \|x_n - z\|\}, \\ Q_{n+1} &= \{z \in X \mid \langle x_n - z, x_0 - x_n \rangle\}, \\ x_{n+1} &= P_{C_{n+1}\cap Q_{n+1}} x_0 \end{aligned}$$

for every $n \in \mathbb{N}$. *Then* $\{x_n\}$ *is well defined and is convergent to* $P_{\operatorname{Fix} T} x_0 \in \operatorname{Fix} T$, *where* P_K *is the metric projection onto a nonempty closed convex subset* K *of* H.

This method has been generalized in the setting of complete geodesic space; see [2] for instance.

This paper proposes a new iterative scheme Δ-convergent to a fixed point of a given mapping. The method is similar to the abovementioned CQ method; however, our new method does not converge strongly. The notion of Δ-convergence was initially introduced by Lim [4] and was intensively investigated by Kirk and Panyanak [3] in the setting of geodesic spaces. It corresponds to the weak convergence on Hilbert spaces. Therefore, our new result becomes a weak convergence theorem in the setting of Hilbert spaces.

2 Preliminaries

Let X be a metric space. For $x, y \in X$ with $l = d(x,y)$, a geodesic $\gamma_{xy}\colon [0,l] \to X$ is a mapping such that $\gamma_{xy}(0) = x$, $\gamma_{xy}(l) = y$, and that $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s-t|$ for every $s, t \in [0,l]$. If for all $x, y \in X$, a geodesic γ_{xy} exists, then we say X is a geodesic space. In particular, if γ_{xy} is unique for each pair $x, y \in X$, then we say X is uniquely geodesic. In what follows, we only consider uniquely geodesic spaces.

If X is a uniquely geodesic space, then we can define a convex combination between $x, y \in X$ with $t \in [0,1]$ in a natural way. Namely, we define $tx \oplus (1-t)y = \gamma_{xy}((1-t)d(x,y))$. By this definition, $w = tx \oplus (1-t)y$ is a unique point on the geodesic segment $[x,y] = \gamma_{xy}([0, d(x,y)])$ satisfying that

$$\begin{aligned} d(x,w) &= d(\gamma_{xy}(0), \gamma_{xy}((1-t)d(x,y))) = (1-t)d(x,y), \\ d(w,y) &= d(\gamma_{xy}((1-t)d(x,y)), \gamma_{xy}(d(x,y))) = td(x,y) \end{aligned}$$

for every $x, y \in X$. A subset C of X is said to be convex if $tx \oplus (1-t)y \in C$ for every $x, y \in C$ and $t \in [0,1]$.

A uniquely geodesic space X is called a CAT(0) space if the inequality

$$d(tx \oplus (1-t)y, z)^2 \le td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2$$

holds for every $x, y, z \in X$ and $t \in [0,1]$. Further, a complete CAT(0) space is called a Hadamard space.

Let $\{x_n\}$ be a bounded sequence in a metric space X. Define a function $g\colon X \to \mathbb{R}$ by

$$g(x) = \limsup_{n\to\infty} d(x_n, x)$$

for $x \in X$. An asymptotic center of $\{x_n\}$ is defined by a point $x_0 \in X$ such that $g(x_0) = \inf_{x\in X} g(x)$. That is, it is a minimizer of g on X. We know that if X is a Hadamard space, then there exists a unique asymptotic center of $\{x_n\}$. We say a bounded sequence $\{x_n\}$ is Δ-convergent to $x_0 \in X$ if x_0 is a unique asymptotic center of any subsequence of $\{x_n\}$. It is known that any bounded sequence in a Hadamard space has a Δ-convergent subsequence. For more detail, see [3].

Let X be a metric space and $T\colon X \to X$. The set of all fixed points of T is denoted by $\operatorname{Fix} T$, that is,

$$\operatorname{Fix} T = \{z \in X \mid z = Tz\}.$$

A mapping $T\colon X \to X$ is said to be quasinonexpansive if $\operatorname{Fix} T \neq \emptyset$ and

$$d(Tx, z) \leq d(x, z)$$

for every $x \in X$ and $z \in \operatorname{Fix} T$. We say T is Δ-demiclosed if $x_0 \in \operatorname{Fix} T$ whenever a sequence $\{x_n\}$ is Δ-convergent to $x_0 \in X$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let C be a nonempty closed convex subset of a Hadamard space X. Then, we know that, for each $x \in X$, there exists a unique $y_x \in C$ such that

$$d(x, y_x) = \inf_{y\in C} d(x, y).$$

Using this fact, we can define a metric projection $P_C\colon X \to C$ by $P_C(x) = y_x$ for $x \in C$. In a Hadamard space, a metric projection is quasinonexpansive and Δ-demiclosed with $\operatorname{Fix} P_C = C$.

3 Main result

In this section, we prove a Δ-convergence theorem of an iterative scheme generated by a new type of projection methods similar to the CQ projection method.

Theorem 2. *Let X be a Hadamard space and suppose that subsets $\{z \in X \mid d(x, z) \leq d(y, z)\}$ and $\{z \in X \mid d(x, y)^2 + d(y, z)^2 \leq d(x, z)^2\}$ are convex for every $x, y \in X$. Let $T\colon X \to X$ be a quasinonexpansive and Δ-demiclosed mapping such that $\operatorname{Fix} T \neq \emptyset$. Generate a sequence $\{x_n\} \subset X$ as follows: $x_0 = x_1 \in X$ and*

$$\begin{aligned}
&C_{n+1} = \{z \in X \mid d(Tx_n, z) \leq d(x_n, z)\},\\
&Q_{n+1} = \left\{z \in X \mid d(x_{n-1}, x_n)^2 + d(x_n, z)^2 \leq d(x_{n-1}, z)^2\right\},\\
&x_{n+1} = P_{C_{n+1}\cap Q_{n+1}} x_n
\end{aligned}$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ is well defined and is Δ-convergent to $z \in \operatorname{Fix} T$.

Proof. We begin with showing that $\{x_n\}$ is well defined. To prove it, we define $C_1 = Q_1 = X$, and we show that x_n is well defined and $\operatorname{Fix} T \subset C_n \cap Q_n$ for every $n \in \mathbb{N}$ by induction. Since x_1 is given and $\operatorname{Fix} T \subset X = C_1 \cap Q_1$, it holds for the case where $n = 1$. Next, suppose that x_k is well defined and $\operatorname{Fix} T \subset C_k \cap Q_k$ for fixed $k \in \mathbb{N}$. Let $z \in \operatorname{Fix} T$. Then, since T is quasinonexpansive, we have

$$d(Tx_k, z) \leq d(x_k, z).$$

It implies that $z \in C_{k+1}$ and hence $\operatorname{Fix} T \subset C_{k+1}$ To show $\operatorname{Fix} T \subset Q_{k+1}$, let $z \in \operatorname{Fix} T$. Then we have $z \in C_k \cap Q_k$ by the assumption of induction. Since $C_k \cap Q_k$ is convex and $x_k = P_{C_k \cap Q_k} x_{k-1}$, we have $tz \oplus (1-t)x_k \in C_k \cap Q_k$ for $t \in]0, 1[$. It follows that

$$\begin{aligned} d(x_{k-1}, x_k)^2 &\leq d(x_{k-1}, tz \oplus (1-t)x_k)^2 - d(x_k, tz \oplus (1-t)x_k)^2 \\ &= d(x_{k-1}, tz \oplus (1-t)x_k)^2 - t^2 d(z, x_k)^2 \\ &\leq td(x_{k-1}, z)^2 + (1-t)d(x_{k-1}, x_k)^2 - t(1-t)d(z, x_k)^2 - t^2 d(z, x_k)^2 \\ &= td(x_{k-1}, z)^2 + (1-t)d(x_{k-1}, x_k)^2 - td(z, x_k)^2, \end{aligned}$$

which implies that

$$d(x_{k-1}, x_k)^2 + d(x_k, z)^2 \leq d(x_{k-1}, z)^2.$$

Thus we have $z \in Q_{k+1}$ and thus $\operatorname{Fix} T \subset Q_{k+1}$. Therefore, the sequence $\{x_n\}$ is well defined and $\operatorname{Fix} T \subset C_n \cap Q_n$ for every $n \in \mathbb{N}$.

Let $p \in \operatorname{Fix} T$. Then, since the metric projection $P_{C_{n+1} \cap Q_{n+1}}$ is quasinonexpansive and p is its fixed point, we have

$$d(x_{n+1}, p) = d(P_{C_{n+1} \cap Q_{n+1}} x_n, p) \leq d(x_n, p)$$

for every $n \in \mathbb{N}$. It implies that a sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below, and thus it has a limit $c_p \in [0, \infty[$. We also get that $\{x_n\}$ is a bounded sequence in X.

Since $x_{n+1} = P_{C_{n+1} \cap Q_{n+1}} x_n \in C_{n+1} \cap Q_{n+1}$ and, $C_{n+1} \cap Q_{n+1}$ is convex, for $t \in]0, 1[$ we have

$$\begin{aligned} d(x_n, x_{n+1})^2 &= d(x_n, P_{C_{n+1} \cap Q_{n+1}} x_n)^2 \\ &\leq d(x_n, tx_{n+1} \oplus (1-t)p)^2 \\ &\leq td(x_n, x_{n+1})^2 + (1-t)d(x_n, p)^2 - t(1-t)d(x_{n+1}, p)^2. \end{aligned}$$

It follows that $d(x_n, x_{n+1})^2 \leq d(x_n, p)^2 - td(x_{n+1}, p)^2$. Letting $t \to 1$, we have

$$d(x_n, x_{n+1})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2$$

for every $n \in \mathbb{N}$. Letting $n \to \infty$, we get

$$0 \leq \lim_{n \to \infty} d(x_n, x_{n+1})^2 \leq c_p^2 - c_p^2 = 0$$

and hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Since $x_{n+1} \in C_{n+1}$, we have $d(Tx_n, x_{n+1}) \leq d(x_n, x_{n+1})$, and therefore

$$0 \leq d(Tx_n, x_n) \leq d(Tx_n, x_{n+1}) + d(x_{n+1}, x_n) \leq 2d(x_n, x_{n+1}) \to 0,$$

which implies $d(Tx_n, x_n) \to 0$ as $n \to \infty$.

Let x_0 be a unique asymptotic center of $\{x_n\}$. Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ and we show that its asymptotic center y_0 is identical to x_0. Since $\{x_{n_k}\}$ is bounded, it has a Δ-convergent subsequence $\{w_j\}$ to $w_0 \in X$. Since $\lim_{j\to\infty} d(Tw_j, w_j) = 0$ and T is Δ-demiclosed, we have $w_0 \in \operatorname{Fix} T$. Using the properties of the asymptotic center, we have

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, w_0) = c_{w_0} &= \lim_{j\to\infty} d(w_j, w_0) \\
&\leq \limsup_{j\to\infty} d(w_j, y_0) \leq \limsup_{k\to\infty} d(x_{n_k}, y_0) \\
&\leq \limsup_{k\to\infty} d(x_{n_k}, x_0) \leq \limsup_{n\to\infty} d(x_n, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, w_0) = \lim_{n\to\infty} d(x_n, w_0).
\end{aligned}$$

This implies

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, y_0) &= \limsup_{n\to\infty} d(x_n, x_0), \\
\limsup_{k\to\infty} d(x_{n_k}, w_0) &= \limsup_{k\to\infty} d(x_{n_k}, y_0).
\end{aligned}$$

and hence $x_0 = y_0 = w_0 \in \operatorname{Fix} T$. Since any subsequences of $\{x_n\}$ have the same asymptotic center x_0, the sequence $\{x_n\}$ is Δ-convergent to $x_0 \in \operatorname{Fix} T$, which is the desired result. □

In this result, we need to assume the convexity of two kinds of subsets of X; $\{z \in X \mid d(x, z) \leq d(y, z)\}$ and $\{z \in X \mid d(x, y)^2 + d(y, z)^2 \leq d(x, z)^2\}$ for each $x, y \in X$. Notice that this assumption obviously holds for Hilbert spaces, however it does not in general; see [1, 6].

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