Vanishing integrability for Riesz potentials

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Abstract

Our aim in this note is to establish vanishing Morrey-Sobolev integrability for Riesz potentials of functions in Morrey-Orlicz spaces. We discuss the size of the exceptional sets by using a capacity and Hausdorff measure.

We also give Trudinger-type exponential Morrey integrability for Riesz potentials of functions in Morrey-Orlicz spaces.

1 Introduction: Historical background

For $0 < \alpha < n$ let us define the Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy$$

for functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Sobolev inequality for Riesz potentials says that for $p > 1$

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)|^{p^*} dx \right)^{1/p^*} \le C \left(\int_{\mathbb{R}^n} |f(y)|^p dy \right)^{1/p}, \qquad (1.1)$$

where p^* is the Sobolev exponent defined by

$$1/p^* = 1/p - \alpha/n > 0.$$

In view of Meyers [5, 6], if $f \in L^p(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \frac{1}{|B(z,r)|} \int_{B(z,r)} |I_\alpha f(x) - I_\alpha f(z)|^{p^*} dx = 0$$

holds except for a set of capacity zero.

For this, it is shown that

$$\lim_{r \to 0} \frac{1}{|B(z,r)|} \int_{B(z,r)} |I_\alpha f(x)|^{p^*} dx = 0$$

when $z \in \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$ satisfies

$$\lim_{r \to 0} \frac{r^{\alpha p}}{|B(z,r)|} \int_{B(z,r)} |f(y)|^p dy = 0.$$

In what follows we extend this to the Morrey-Orlicz case.

If $1/p - \alpha/n = 0$, then Trudinger's inequality says that

$$\int_{B(0,r)} \exp(|I_\alpha f(x)|^{p'})dx < \infty$$

for $r > 0$, where $1/p + 1/p' = 1$.

Our final goal is to discuss vanishing exponential integrability.

Throughout this note, let C denote various constants independent of the variables in question and $C(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$. Moreover, $f \sim g$ means that $C^{-1}g(r) \le f(r) \le Cg(r)$ for a constant $C > 0$.

2 Preliminaries

2.1 Orlicz spaces

Consider a positive convex function Φ on $(0, \infty)$ satisfying

(Φ0) $\Phi(0) = \lim_{r\to 0} \Phi(r) = 0$;

(Φ1) there exists a constant $A_1 \ge 1$ such that

$$\Phi(2t) \le A_1\Phi(t) \quad \text{whenever } t > 0.$$

Note here that

(Φ2) $t \mapsto \Phi(t)/t$ is increasing in $(0, \infty)$;

(Φ3) there exists a constant $C > 0$ such that

$$\Phi^{-1}(A^m t) \le CA\Phi^{-1}(t) \quad \text{for } 0 < A < 1 \text{ and } t > 0, \tag{2.1}$$

where m is given by $A_1 = 2^m$;

(Φ4) there exists a constant $C > 0$ such that

$$\Phi^{-1}(2t) \le C\Phi^{-1}(t) \quad \text{for } t > 0.$$

The typical example is :

$$\Phi(r) = r^p(\log(c + r))^q,$$

where $p \ge 1$, q is a real number and $c \ge e$ is chosen so that

$$(1 + \log c)(p - 1) + q \ge 0.$$

If $\Phi_1(r) = r^p(\log(e + r))^q$ with $p > 1$ and a real number q, then it may be replaced by

$$\Phi_2(r) = \int_0^r \{\sup_{0<s<t} s^p(\log(e + s))^q\}t^{-1}dt,$$

which is convex; note further that $\Phi_1 \sim \Phi_2 \sim \Phi$.

2.2 Hardy-Littlewood maximal functions

Given $\Phi(t)$ as above, the Orlicz space

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \int_{\mathbb{R}^n} \Phi\big(|f(y)|\big)\, dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 \,:\, \int_{\mathbb{R}^n} \Phi\big(|f(y)|/\lambda\big)\, dy \le 1 \right\}.$$

In case $\Phi(t) = t^p, p > 1$, $L^{\Phi}(\mathbb{R}^n)$ is denoted by $L^p(\mathbb{R}^n)$.

Consider the Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Our fundamental tool is the boundedness of maximal operator. For this purpose, we need the following condition: there exists a constant $A_2 > 0$ such that

($\Phi5$) $\displaystyle\int_0^t \Phi(s)/s^2 ds \le A_2 \Phi(t)/t$ for $t > 0$.

Under this condition, by using weak L^1 estimate in Stein [14, Chapter 1] and [7, Theorem 1.10.2], we have the boundedness of maximal operator as in [9, Lemma 2.5].

LEMMA 2.1. *There exists a constant $C > 1$ such that*

$$\|Mf\|_{L^{\Phi}(\mathbb{R}^n)} \le C \|f\|_{L^{\Phi}(\mathbb{R}^n)}$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

2.3 Almost increasing functions

A positive function $F(r)$ on $(0,\infty)$ is called if

($F0$) there exists a constant $C_1 > 0$ such that

$$F(s) \le C_1 F(t) \quad \text{for } 0 < s < t.$$

A typical example is

$$F(t) = t^a (\log(e+t))^b$$

for $a > 0$ and a real number b.

We say that a function F is almost decreasing if $1/F(\cdot)$ is almost increasing, and F is almost monotone if F is almost increasing or almost decreasing.

For inverse functions, it is convenient to see the following result.

LEMMA 2.2 (cf. [3, Lemma 5.1]). *For a positive continuous function F on $(0,\infty)$ satisfying ($F0$), set*

$$F^{-1}(s) = \sup\{t > 0\,;\, F(t) < s\}$$

for $s > 0$. Then:

($F1$) *$F^{-1}(\cdot)$ is nondecreasing;*

($F2$) *$F(F^{-1}(t)) = t$ for $t > 0$.*

3 Integability for Riesz potentials in Morrey-Orlicz spaces

In this section, we discuss integrability for Riesz potentials of functions in Morrey-Orlicz spaces. For $0 < \alpha < n$, we define the Riesz potential $I_\alpha f$ of order α by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy$$

for $f \in L^1_{\rm loc}(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} (1+|y|)^{\alpha-n}|f(y)|dy < \infty, \tag{3.1}$$

which is equivalent to $I_\alpha |f| \in L^1_{\rm loc}(\mathbb{R}^n)$ (see [7, Theorem 1.1, Chapter 2]).

Let k be a continuous function on $[0,\infty)$ satisfying

($k0$) $k(0) = \lim_{r\to 0} k(r) = 0$;

($k1$) k is almost increasing.

Let us consider a positive convex function Ψ on $(0,\infty)$ satisfying

($\Psi 0$) $\Psi(0) = \lim_{r\to 0} \Psi(r) = 0$;

($\Psi 1$) $\Psi(2r) \le L_1 \Psi(r)$ for $r > 0$;

($\Psi\Phi k$) $\Psi(t(k^{-1}(\Phi(t)^{-1}))^\alpha) \le L_2 \Phi(t)$ for $t > 0$,

where L_1 and L_2 are positive constants.

It is seen that Ψ plays like the Sobolev conjugate of Φ.

Note here that ($\Psi\Phi k$) is equivalent to

($\Psi\Phi k1$) $\Psi(r^\alpha \Phi^{-1}(k(r)^{-1})) \le L_3 k(r)^{-1}$ for $r > 0$,

where L_3 is a positive constant.

Let us consider the norm

$$\|f\|_{L^{\Phi,k}(\mathbb{R}^n)} = \inf\{\lambda > 0 : \sup_{r>0, x\in\mathbb{R}^n} \frac{k(r)}{|B(x,r)|} \int_{B(x,r)} \Phi(|f(y)|/\lambda)dy \le 1\}.$$

Note here that $\|f\|_{L^{\Phi,k}(\mathbb{R}^n)} \le 1$ if and only if

$$\sup_{r>0, x\in\mathbb{R}^n} \frac{k(r)}{|B(x,r)|} \int_{B(x,r)} \Phi(|f(y)|)dy \le 1. \tag{3.2}$$

THEOREM 3.1. *In addition to the conditions on k, Φ and Ψ mentioned above, further suppose*

($\Phi k\alpha+$) $r^{\alpha+\varepsilon}\Phi^{-1}(k(r)^{-1})$ *is almost decreasing in* $(0,\infty)$ *for some* $0 < \varepsilon < 1$.

Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{\Psi,k}(\mathbb{R}^n)} \le C\|f\|_{L^{\Phi,k}(\mathbb{R}^n)}$$

for $f \in L^1_{\rm loc}(\mathbb{R}^n)$ satisfying (3.1).

EXAMPLE 3.2. *Let $\Phi(r) \sim r^p(\log(e+r))^q$ for $1 < p < \nu/\alpha$, $\nu \le n$ and a real number q, and $k(r) = r^\nu$. Set $\Psi(r) \sim \left(r(\log(e+r))^{q/p}\right)^{p_\nu}$, where $1/p_\nu = 1/p - \alpha/\nu > 0$.*

4 Vanishing integrability for modified Riesz potentials in Morrey-Orlicz spaces

Suppose there exist positive functions H, h_1 and h_2 on $(0,\infty)$ satisfying

$(H1)$ $\lim_{r\to 0} H(r) = 0$;

$(H2)$ $\lim_{r\to 0} h(r)h_2(r) = 0$;

$(H3)$ $\Phi(h_1(r)t) \le C\dfrac{h(r)}{k(r)}\{\Phi(t) + h_2(r)\}$ for $t > 0$ and $0 < r < 1$;

$(H4)$ $h_1(r)\Phi^{-1}\left(h(r)^{-1}\right) \le C\Phi^{-1}\left(k(r)^{-1}\right)$ for $0 < r < 1$;

$(H5)$ $\dfrac{H(r)}{k(r)}\Psi(t) \le C\Psi(h_1(r)t)$ for $t > 0$ and $0 < r < 1$;

$(\Phi h\alpha)$ $t^{\alpha}\Phi^{-1}\left(h(t)^{-1}\right)$ is almost decreasing in $(0,\infty)$.

THEOREM 4.1. *Let H, h_1 and h_2 be as above, and suppose further $(\Phi k\alpha+)$ and $(\Phi h\alpha)$ hold. Let f be a function in $L^{\Phi,k}(\mathbb{R}^n)$ satisfying* (3.1)*. For $x_0 \in \mathbb{R}^n$, if*

$$\lim_{r\to 0} \frac{h(r)}{|B(x_0,r)|}\left(\sup_{0<t<r, z\in B(x_0,r)} \frac{k(t)}{|B(z,t)|}\int_{B(z,t)} \Phi(|f(y)|)dy\right) = 0,$$

then

$$\lim_{r\to 0} \frac{H(r)}{|B(x_0,r)|}\int_{B(x_0,r)} \Psi(|I_\alpha f(z)|)dz = 0.$$

4.1 Example

EXAMPLE 4.2. *Let $\Phi(r) \sim r^p(\log(e+r))^q$ for $p > 1$ and $q \ge 0$, $k(r) = r^{\nu}$ with $\alpha p < \nu \le n$ and $h(r) = r^{\nu_1}(\log(e+1/r))^{-q}$ with $\alpha p < \nu_1 \le \nu \le n$. Set $\Psi(r) \sim r^{p_\nu}(\log(e+r))^{qp_\nu/p}$, $h_1(r) = r^{(\nu_1-\nu)/p}(\log(e+r))^{-q/p}$ and $H(r) = r^{(\nu_1/p-\alpha)p_\nu}(\log(e+r))^{-qp_\nu/p}$.*

5 Trudinger exponential inequality for modified Riesz potentials

We give Trudinger type exponential inequality for $I_\alpha f$ when f is a function in $L^1_{\rm loc}(\mathbb{R}^n)$ satisfying (3.1) and

$$\sup_{r>0} \frac{k(r)}{|B(x,r)|}\int_{B(x,r)} \Phi(|f(y)|)dy \le 1. \tag{5.1}$$

For simplicity set

$$(I_\alpha f)_{B(x,R)} = \frac{1}{|B(x,R)|}\int_{B(x,R)} I_\alpha f(z)dz.$$

5.1 Log type functions

A function k on $(0,\infty)$ is called log type if there exists a constant $C > 1$ such that

$$C^{-1}k(r) \le k(r^2) \le Ck(r) \quad \text{for } r > 0.$$

It follows from the log type condition that k satisfies the doubling condition, that is,

$$C^{-1}k(r) \le k(2r) \le Ck(r) \qquad \text{for } r > 0,$$

where C is a positive constant. If $\delta > 0$, then, in view of [7], we can find a positive constant $C = C(\delta)$ for which

$$s^{\delta}k(s) \le Ct^{\delta}k(t) \qquad \text{whenever } t > s > 0.$$

For the properties of the functions of log type, see e.g. [7, Section 5.3] and [11, Section 2].

In what follows, suppose

(K) $r^{\alpha}\Phi^{-1}(k(r)^{-1})$ is almost monotone and of log type in $(0,\infty)$.

Consider positive functions k_1, k_2 and k_3 on $(0,\infty)$ of log type such that

$(K0)$ k_1 is continuous in $(0,\infty)$, and k_2 is increasing in $(0,\infty)$ and

$$k_2(0) = \lim_{r\to 0} k_2(r) = 0;$$

$(K1)$ either $k_1(t)\Phi(t^{-\alpha}k_1(t))^{-1} \le Ck(t)k_1(t)$ for $0 < t < 1$ or $t^{-n}k_1(t)\Phi(t^{-\alpha}k_1(t))^{-1} \le Ck_2(1/t)$ for $0 < t < 1$;

$(K2)$ $\displaystyle\int_r^2 k_1(t)t^{-1}dt \le Ck_2(1/r)$ for $0 < r < 1$;

$(K3)$ $k_3(r)^{-1} \le Ck_2(1/r)$ for $0 < r < 1$ and $k_3(r) \le C$ for $r > 1$;

$(K4)$ $\Phi(r^{\alpha}k_3(r)t) \le Ck(r)\Phi(t)$ for $r, t > 0$.

5.2 Trudinger exponential inequality for Riesz potentials

THEOREM 5.1. *Let k_1, k_2 and k_3 be as above. Then there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\frac{1}{|B(x,R)|}\int_{B(x,R)} k_2^{-1}\left(|I_\alpha f(z) - (I_\alpha f)_{B(x,R)}|/c_1\right) dz \le c_2$$

for $0 < R < 1$ and

$$\frac{1}{|B(x,R)|}\int_{B(x,R)} k_2^{-1}\left(k_3(R)|I_\alpha f(z) - (I_\alpha f)_{B(x,R)}|/c_1\right) dz \le c_2$$

for $R \ge 1$, when $x \in \mathbb{R}^n$ and f is a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying (5.1) *and* (3.1)*.*

5.3 Examples and Corollaries

EXAMPLE 5.2. *Let* $\Phi(r) \sim r^p(\log(e+r))^q$ *and* $k(r) = r^\nu$ *for* $p = \nu/\alpha > 1$, $0 < \nu \le n$ *and a real number* q. *Let*

$$k_1(r) = \begin{cases} (\log(e+1/r))^{-q/p} & \text{when } \nu < n \text{ and } q < p, \\ (\log(e+1/r))^{-1/p-q/p} & \text{when } \nu = n \text{ and } q < p-1; \end{cases}$$

$$k_2(r) = \begin{cases} (\log(e+r))^{1-q/p} & \text{when } \nu < n \text{ and } q < p, \\ (\log(e+r))^{1-1/p-q/p} & \text{when } \nu = n \text{ and } q < p-1; \end{cases}$$

$$k_3(r) = \begin{cases} (\log(e+r))^{-q/p} & \text{when } q > 0, \\ (\log(e+1/r))^{q/p} & \text{when } q \le 0. \end{cases}$$

COROLLARY 5.3 (cf. [10, Theorem A], [12, Theorem A]). *Let* $\Phi(r) \sim r^p(\log(e+r))^q$ *for* $p = \nu/\alpha > 1$, $0 < \nu \le n$ *and* $q \in \mathbb{R}$. *If*

$$\beta = \begin{cases} p/(p-q) & \text{when } \nu < n \text{ and } q < p, \\ p/(p-1-q) & \text{when } \nu = n \text{ and } q < p-1, \end{cases}$$

then there exist constants $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\sup_{0<R<1} \frac{1}{|B(x,R)|} \int_{B(x,R)} \exp\left(|I_\alpha f(z) - (I_\alpha f)_{B(x,R)}|/c_1)^\beta\right) dz \le c_2$$

for all $x \in \mathbb{R}^n$ *and* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *satisfying* (5.1) *and* (3.1).

EXAMPLE 5.4. *Let*

$$\Phi(r) \sim \begin{cases} r^p(\log(e+r))^p\,(\log(e+\log(e+r)))^q & \text{when } \nu < n \text{ and } q < p \\ r^p(\log(e+r))^{p-1}\,(\log(e+\log(e+r)))^q & \text{when } \nu = n \text{ and } q < p-1, \end{cases} \quad \text{and}$$

$k(r) = r^\nu$ *for* $p = \nu/\alpha > 1$, $0 < \nu \le n$ *and* $q \in \mathbb{R}$. *Then*

$$\Phi^{-1}(r) \sim \begin{cases} r^{1/p}(\log(e+r))^{-1}\,(\log(e+\log(e+r)))^{-q/p} & \text{when } \nu < n \text{ and } q < p, \\ r^{1/p}(\log(e+r))^{-(p-1)/p}\,(\log(e+\log(e+r)))^{-q/p} & \text{when } \nu = n,\ q < p-1. \end{cases}$$

Further let

$$k_1(r) = \begin{cases} (\log(e+r^{-1}))^{-1}(\log(e+\log(e+r^{-1})))^{-q/p} & \text{when } \nu < n \text{ and } q < p; \\ (\log(e+r^{-1}))^{-1}(\log(e+\log(e+r^{-1})))^{-1/p-q/p} & \text{when } \nu = n \text{ and } q < p-1, \end{cases}$$

$$k_2(r) = \begin{cases} (\log\log(e+r))^{1-q/p} & \text{when } \nu < n \text{ and } q < p; \\ (\log\log(e+r))^{1-1/p-q/p} & \text{when } \nu = n \text{ and } q < p-1, \end{cases}$$

and $k_3(r) = (\log(e+r))^{1/p-1} \begin{cases} (\log(e+\log(e+r)))^{-q/p} & \text{when } q > 0,; \\ (\log(e+\log(e+1/r)))^{q/p} & \text{when } q \le 0. \end{cases}$

The double exponential integrability is derived in the following manner.

COROLLARY 5.5 (cf. [10, Theorem B], [12, Theorem B]). *Let*

$$\Phi(r) \sim \begin{cases} r^p(\log(e+r))^p\,(\log(e+\log(e+r)))^q & \text{when } \nu < n \text{ and } q < p; \\ r^p(\log(e+r))^{p-1}\,(\log(e+\log(e+r)))^q & \text{when } \nu = n \text{ and } q < p-1 \end{cases}$$

for $p = \nu/\alpha > 1$, $0 < \nu \le n$ *and* $q \in \mathbb{R}$. *Then there exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\sup_{0<R<1} \frac{1}{|B(x,R)|} \int_{B(x,R)} \exp\exp\left(|I_\alpha f(z) - (I_\alpha f)_{B(x,R)}|/c_1)^\beta\right) dz \le c_2$$

for all $x \in \mathbb{R}^n$ *and* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *satisfying* (5.1) *and* (3.1), *where* β *is given as in Corollary 5.3.*

5.4 Vanishing Trudinger exponential inequality for modified Riesz potentials

Let k_1, k_2 and k_3 be as above. Suppose further

(1) there exist $K_1, a > 0$ such that

$$k_2^{-1}(r) \le K_1 r^a \quad \text{for } 0 < r < 1;$$

(2) $k(r) \le K_2 h(r)$ for $0 < r < 1$;

(3) H is an almost decreasing function on $(0, \infty)$ of log type such that

$$H(r) \le K_3 \{r^\alpha k_3(r) \Phi^{-1}(h(r)^{-1})\}^{-1} \quad \text{for } 0 < r < 1.$$

Under those conditions we establish vanishing exponential integrability.

THEOREM 5.6. *Let k_1, k_2, k_3, h and H be as above. Let $x \in \mathbb{R}^n$ and f be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying* (5.1) *and* (3.1). *If*

$$\lim_{R \to 0} \frac{h(r)}{|B(x,R)|} \int_{B(x,R)} \Phi(|f(y)|) dy = 0, \tag{5.2}$$

then

$$\lim_{R \to 0} \frac{H(r)}{|B(x,R)|} \int_{B(x,R)} k_2^{-1} \left(|I_\alpha f(z) - (I_\alpha f)_{B(x,R)}| \right) dz = 0.$$

For an example, let $h(r) = r^\nu (\log(e+1/r))^\varepsilon$ and H be an almost increasing function on $(0, \infty)$ of log type such that when $\varepsilon > q \ge 0$,

$$\begin{aligned} H(r) &\le C\{r^\alpha k_3(r) \Phi^{-1}(h(r)^{-1})\}^{-1} \\ &\le C(\log(e+1/r))^{-\varepsilon/p+q/p} \end{aligned}$$

when $\varepsilon > 0 \ge q$,

$$\begin{aligned} H(r) &\le C\{r^\alpha k_3(r) \Phi^{-1}(h(r)^{-1})\}^{-1} \\ &\le C(\log(e+1/r))^{-\varepsilon/p} \end{aligned}$$

for $0 < r < 1$.

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