Finite-time blow-up of weak solutions to a degenerate chemotaxis system with logistic source

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1. Introduction

In this report we consider the *degenerate* parabolic–elliptic chemotaxis system

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u^\alpha \nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \overline{M_\ell}(t) + u^\ell, & x \in \Omega,\ t > 0, \\ \nabla u^m \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

with

$$\overline{M_\ell}(t) := \frac{1}{|\Omega|} \int_\Omega u^\ell(x,t)\, dx,$$

where $\Omega := B_R(0) \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a ball with some $R > 0$, ν is the outward normal vector to $\partial\Omega$, $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa \geq 1$, $\ell > 0$, and $u_0 \in L^\infty(\Omega)$ is nonnegative, radially symmetric, nonincreasing with respect to $|x|$.

The purpose of this report is to establish finite-time blow-up to the system (1.1). For a pair (u, v) of nonnegative and radially symmetric functions, we regard (u, v) as $(u(r,t), v(r,t))$ with $r := |x|$ if necessary. Given $s_0 \in (0, R^n)$ and $\gamma \in (-\infty, 1)$, we set

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \geq 0,$$
$$\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s) w(s,t)\, ds \quad \text{for } t \geq 0.$$

Definition 1.1 (moment solutions)**.** Let $T \in (0, \infty]$. A pair (u, v) of nonnegative and radially symmetric functions on $\Omega \times (0,T)$ is called a *moment solution* of (1.1) on $[0,T)$ if

(i) $u \in C^0_{\mathrm{w}-\star}([0,T); L^\infty(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,T); L^\infty(\Omega))$, $v \in L^\infty_{\mathrm{loc}}([0,T); H^1(\Omega))$, and, $u^m \in L^2(0,T; H^1(\Omega))$ if $T < \infty$; $u^m \in L^2_{\mathrm{loc}}([0,T); H^1(\Omega))$ if $T = \infty$,

(ii) for $\varphi \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ for a.a. $x \in \Omega$,

$$\int_0^T \int_\Omega (\nabla u^m \cdot \nabla\varphi - \chi u^\alpha \nabla v \cdot \nabla \varphi - (\lambda u - \mu u^\kappa)\varphi - u\varphi_t)\, dxdt = \int_\Omega u_0(x)\varphi(x,0)\, dx,$$
$$\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi\, dxdt + \int_0^T \left(\overline{M_\ell}(t) \int_\Omega \varphi\, dx \right) dt - \int_0^T \int_\Omega u^\ell \varphi\, dxdt = 0,$$

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(iii) (u, v) satisfies the following moment inequality:

$$\phi(t) - \phi(0) \geq K \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in (0, T) \tag{1.2}$$

for some constant $K = K(R, m, \chi, \alpha, \mu, \kappa, \ell, \gamma, s_0) > 0$.

We next define *maximal moment solutions*, which are guaranteed by Zorn's lemma as in the proof of [**4**, Lemma 2.4].

Definition 1.2 (maximal moment solutions)**.** Define the set $\mathcal{S}$ as

$$\mathcal{S} := \{(T, u, v) \mid T \in (0, \infty],\ (u, v) \text{ is a moment solution of (1.1) on } [0, T)\},$$

which is not empty by Proposition 2.1, with the order relation $\preceq$ given by

$$(T_1, u_1, v_1) \preceq (T_2, u_2, v_2) \ :\Longleftrightarrow\ T_1 \leq T_2,\ u_2|_{(0,T_1)} = u_1,\ v_2|_{(0,T_1)} = v_1.$$

Then Zorn's lemma assures some maximal element $(T_{\max}, u, v) \in \mathcal{S}$, and (u, v) is called a *maximal moment solution* of (1.1) on $[0, T_{\max})$.

Definition 1.3 (blow-up)**.** Let (u, v) be a maximal moment solution of (1.1) on $[0, T_{\max})$. If u satisfies

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty,$$

then we say that (u, v) *blows up* at $T_{\max}$.

Now the main theorem reads as follows.

Theorem 1.1 ([**6**])**.** *Let $n \in \mathbb{N}$, $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa \geq 1$ and $\ell > 0$. Assume that*

$$\alpha + \ell > \max\left\{m + \frac{2}{n}\kappa, \kappa\right\}. \tag{1.3}$$

Then for all $M_0 > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ which satisfy the following property: If

$$u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \tag{1.4}$$

and

$$u_0 \text{ is radially symmetric, nonincreasing with respect to } |x| \tag{1.5}$$

as well as

$$\int_\Omega u_0(x)\, dx = M_0 \quad \text{and} \quad \int_{B_{r_\star}(0)} u_0(x)\, dx \geq M_0 - \eta_0, \tag{1.6}$$

then a maximal moment solution of (1.1) *on* $[0, T_{\max})$ *blows up at* $T_{\max} < \infty$.

Remark 1.1. The parameter values appearing in Theorem 1.1 are basically the same as in [**5**, Theorem 1.2], where finite-time blow-up has been obtained for the nondegenerate system. In particular, the condition (1.3) coincides with that in [**5**] in the case that $m \geq 1$ and $\alpha \geq 1$.

2. Local existence of moment solutions

The goal of this section is to show local existence of moment solutions to (1.1) as in the following key proposition, which plays an important role in the proof of blow-up.

Proposition 2.1 (local existence of moment solutions)**.** *Let $n \in \mathbb{N}$, $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa \geq 1$ and $\ell > 0$. Assume that* (1.3) *is satisfied. Then for all $M_0 > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ which satisfy the following property: If u_0 satisfies* (1.4)–(1.6)*, then there exists $T > 0$ such that* (1.1) *admits a moment solution (u, v) on $[0, T)$, i.e.,* (1.1) *has a weak solution (u, v) satisfying the moment inequality* (1.2)*.*

The key to the proof of blow-up is to construct the moment inequality (1.2), which is usually shown via the corresponding differential inequality as in [**1, 5, 7**]. However, we cannot derive it for *weak* solutions of (1.1) due to the lack of the smoothness. Therefore we will obtain it for approximate *smooth* solutions, denoted by u_ε with parameter $\varepsilon > 0$. Here the maximal existence time T_ε depends on ε, and so there is a possibility that T_ε vanishes in the passage to the limit as $\varepsilon \to 0$. This explains the reason for proving *uniform* lower bound of T_ε in Section 2.1.

2.1. Lower bound of existence time for approximate solutions

We recall that the system (1.1) includes the degenerate diffusion term Δu^m. Hence, in order to compensate for the lack of regularity of solutions to (1.1), we consider the following approximate problem:

$$\begin{cases} (u_\varepsilon)_t = \Delta(u_\varepsilon + \varepsilon)^m - \chi \nabla \cdot (u_\varepsilon (u_\varepsilon + \varepsilon)^{\alpha - 1} \nabla v_\varepsilon) + \lambda u_\varepsilon - \mu u_\varepsilon^\kappa, & x \in \Omega,\ t > 0, \\ 0 = \Delta v_\varepsilon - \overline{M_{\ell,\varepsilon}}(t) + u_\varepsilon^\ell, & x \in \Omega,\ t > 0, \\ \nabla u_\varepsilon \cdot \nu = \nabla v_\varepsilon \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $\varepsilon \in (0, 1)$ and

$$\overline{M_{\ell,\varepsilon}}(t) := \frac{1}{|\Omega|} \int_\Omega u_\varepsilon^\ell(x, t)\, dx$$

as well as $u_{0\varepsilon} \in C^\infty(\overline{\Omega})$ is given by $u_{0\varepsilon} := (\rho_\varepsilon * \overline{u_0})|_{\overline{\Omega}}$, where $\overline{u_0}$ is the zero-extension of u_0 outside Ω and $\rho_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is the standard mollifier. We know that ρ_ε is nonnegative, radially symmetric and nonincreasing with respect to $|x|$. Additionally, if u_0 is nonnegative, radially symmetric and nonincreasing with respect to $|x|$, then so is $u_{0\varepsilon}$ from the definition of $u_{0\varepsilon}$.

We first recall a well-known result about local existence of classical solutions to (2.1). The proof is based on a standard fixed point argument (see e.g. [**8**]).

Lemma 2.2. *Let $\varepsilon \in (0, 1)$ and let $m \geq 1$, $\chi > 0$, $\alpha \geq 1$, $\lambda > 0$, $\mu > 0$, $\kappa \geq 1$ and $\ell > 0$. Then there exist $T_\varepsilon \in (0, \infty]$ and a unique classical solution $(u_\varepsilon, v_\varepsilon)$ of* (2.1) *satisfying*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_\varepsilon)) \cap C^{2,1}(\overline{\Omega} \times (0, T_\varepsilon)), \\ v_\varepsilon \in \bigcap_{q > n} C^0([0, T_\varepsilon); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_\varepsilon)). \end{cases}$$

Moreover, u_ε and v_ε are nonnegative and radially symmetric.

In the following let $(u_\varepsilon, v_\varepsilon)$ be the solution of (2.1) on $[0, T_\varepsilon)$ as in Lemma 2.2. Next, in order to guarantee that the existence time T_ε does not vanish as $\varepsilon \to 0$, we confirm uniform lower bound of T_ε, that is, we find $T_0 \in (0, \infty)$ such that for any $\varepsilon \in (0,1)$,

$$T_0 \le T_\varepsilon \quad \text{and} \quad \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K_0 \quad \text{for all } t \in [0, T_0), \tag{2.2}$$

where $K_0 > 0$ is a constant independent of ε. In order to prove (2.2) we need the following lemma, which can be proved as in the proof of [**3**, Lemma 2.4].

Lemma 2.3. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa \ge 1$, $\ell > 0$ and*

$$p > \max\left\{1, m - 2(\alpha+\ell) + 1, \frac{n}{2}(\alpha + \ell - m)\right\}.$$

Then there exists $T_p \in (0, \infty]$ such that for any $\varepsilon \in (0,1)$,

$$T_p \le T_\varepsilon \quad \textit{and} \quad \|u_\varepsilon(\cdot,t)\|^p_{L^p(\Omega)} \le (\|u_0\|_{L^p(\Omega)} + |\Omega|^{\frac{1}{p}})^p + 1 \quad \textit{for all } t \in [0, T_p). \tag{2.3}$$

Next, we give an interval ensuring L^∞-estimate for u_ε uniformly with respect to ε.

Lemma 2.4. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa \ge 1$ and $\ell > 0$. Then there exist $T_0 \in (0, \infty)$ and $K_0 = K_0(|\Omega|, \|u_0\|_{L^{p_0}(\Omega)}, \|u_0\|_{L^\infty(\Omega)}, m, \chi, \alpha, \lambda, \mu, \kappa, \ell) > 0$ with some large constant $p_0 = p_0(m, \alpha, \ell) > 1$ such that for any $\varepsilon \in (0,1)$,*

$$T_0 \le T_\varepsilon \quad \textit{and} \quad \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K_0 \quad \textit{for all } t \in (0, T_0). \tag{2.4}$$

Proof. By making use of Lemma 2.3 in conjunction with the Moser iteration we can arrive at (2.4). □

2.2. Convergence of approximate solutions

We first show some estimates for approximate solutions u_ε.

Lemma 2.5. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa \ge 1$ and $\ell > 0$. Moreover, assume that there exist $T_0 \in (0, \infty)$ and $K_0 > 0$ such that for any $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le K_0 \quad \textit{for all } t \in (0, T_0). \tag{2.5}$$

Then there exists $C = C(|\Omega|, \|u_0\|_{L^2(\Omega)}, m, \chi, \alpha, \lambda, \ell, T_0, K_0) > 0$ such that

$$\|\nabla(u_\varepsilon + \varepsilon)^m\|_{L^2(0,T_0;L^2(\Omega))} \le C.$$

Proof. Multiplying the first equation in (2.1) by u_ε and integrating it over Ω, we have

$$\begin{aligned} \frac{1}{2} \cdot \frac{d}{dt}\|u_\varepsilon\|^2_{L^2(\Omega)} \le &-\frac{4m}{(m+1)^2}\|\nabla(u_\varepsilon+\varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(\Omega)} \\ &+ \chi \int_\Omega u_\varepsilon(u_\varepsilon+\varepsilon)^{\alpha-1}\nabla v_\varepsilon \cdot \nabla u_\varepsilon \, dx + \lambda\|u_\varepsilon\|^2_{L^2(\Omega)} \end{aligned} \tag{2.6}$$

for all $t \in (0, T_0)$. Then it follows from (2.5) that

$$\chi \int_\Omega u_\varepsilon (u_\varepsilon + \varepsilon)^{\alpha-1} \nabla v_\varepsilon \cdot \nabla u_\varepsilon \, dx \le \frac{\chi}{\alpha+1} \int_\Omega (u_\varepsilon + \varepsilon)^{\alpha+\ell+1} \, dx \le \frac{\chi}{\alpha+1} (K_0+1)^{\alpha+\ell+1} |\Omega|.$$

Combining this inequality with (2.6) and integrating it over $(0, T_0)$, we obtain

$$\begin{aligned} \frac{1}{2} \|u_\varepsilon(\cdot, T_0)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_{0\varepsilon}\|^2_{L^2(\Omega)} \le & -\frac{4m}{(m+1)^2} \|\nabla (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,T_0;L^2(\Omega))} \\ & + \frac{\chi}{\alpha+1} (K_0+1)^{\alpha+\ell+1} |\Omega| T_0 + \lambda \|u_\varepsilon\|^2_{L^2(0,T_0;L^2(\Omega))}. \end{aligned}$$

Hence, noting $\|u_{0\varepsilon}\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)}$ and (2.5), we can show that

$$\|\nabla (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,T_0;L^2(\Omega))} \le c_1,$$

where $c_1 := \frac{(m+1)^2}{4m} \left(\frac{1}{2} \|u_0\|^2_{L^2(\Omega)} + \frac{\chi}{\alpha+1} (K_0+1)^{\alpha+\ell+1} |\Omega| T_0 + \lambda K_0^2 |\Omega| T_0 \right) > 0$. This entails

$$\begin{aligned} \|\nabla (u_\varepsilon + \varepsilon)^m\|^2_{L^2(0,T_0;L^2(\Omega))} &= \frac{4m^2}{(m+1)^2} \|(u_\varepsilon + \varepsilon)^{\frac{m-1}{2}} \nabla (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}}\|^2_{L^2(0,T_0;L^2(\Omega))} \\ &\le \frac{4m^2}{(m+1)^2} (K_0+1)^{m-1} c_1 \end{aligned}$$

for any $\varepsilon \in (0, 1)$, which implies the end of the proof. □

The next lemma can be shown as in the proof of [**2**, Lemma 5.2].

Lemma 2.6. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa \ge 1$ and $\ell > 0$. Moreover, assume that there exist $T_0 \in (0, \infty)$ and $K_0 > 0$ such that (2.5) holds for any $\varepsilon \in (0, 1)$. Then there exists $C = C(|\Omega|, \|u_0\|_{L^2(\Omega)}, m, \chi, \alpha, \lambda, \mu, \kappa, \ell, T_0, K_0) > 0$ such that*

$$\left\| \sqrt{t} \frac{\partial}{\partial t} u_\varepsilon^m \right\|^2_{L^2(0,T_0;L^2(\Omega))} + \sup_{t \in (0,T_0)} \|\sqrt{t} \nabla u_\varepsilon^m(\cdot, t)\|^2_{L^2(\Omega)} \le C.$$

Finally we shall establish convergence of approximate solutions $(u_\varepsilon, v_\varepsilon)$.

Lemma 2.7. *Let $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa \ge 1$ and $\ell > 0$. Moreover, assume that there exist $T_0 \in (0, \infty)$ and $K_0 > 0$ such that (2.5) holds for any $\varepsilon \in (0, 1)$. Then there exist subsequences $\{u_{\varepsilon_k}\}$, $\{v_{\varepsilon_k}\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) and nonnegative functions u, v such that*

- $u \in L^\infty(0, T_0; L^\infty(\Omega))$, $u^m \in L^2(0, T_0; H^1(\Omega))$,
- $v \in L^\infty(0, T_0; W^{1,\infty}(\Omega))$,

and as $k \to \infty$,

$$u_{\varepsilon_k} \to u \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega)), \tag{2.7}$$

$$u_{\varepsilon_k} \to u \quad \text{strongly in } C^0([\delta, T_0]; L^p(\Omega)) \quad \text{for all } \delta \in (0, T_0) \text{ and } p \in [1, \infty), \tag{2.8}$$

$$\nabla (u_{\varepsilon_k} + \varepsilon)^m \to \nabla u^m \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), \tag{2.9}$$

$$\nabla v_{\varepsilon_k} \to \nabla v \quad \text{weakly}^\star \text{ in } L^\infty(0, T_0; L^\infty(\Omega)). \tag{2.10}$$

Proof. Applying the elliptic regularity theory to the second equation in (2.1), from (2.5) and the Sobolev embedding theorem we obtain $c_1 > 0$ and $c_2 > 0$ such that

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_1 \quad \text{and} \quad \|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_2$$

for all $t \in (0, T_0)$. Therefore we can show that there exist a subsequence $\{v_{\varepsilon_k}\}$ and a function $v \in L^\infty(0, T_0; W^{1,\infty}(\Omega))$ satisfying (2.10). Moreover, thanks to Lemmas 2.5 and 2.6, as in the proof of [**2**, Lemma 5.3] we can extract a subsequence $\{u_{\varepsilon_k}\}$ and a function $u \in L^\infty(0, T_0; L^\infty(\Omega))$ with $u^m \in L^2(0, T_0; H^1(\Omega))$ such that (2.7)–(2.9) holds. □

2.3. Moment inequality for approximate solutions

In this subsection we derive the moment inequality for $(u_\varepsilon, v_\varepsilon)$. To this end, introducing $r := |x|$, we denote by $(u_\varepsilon, v_\varepsilon) = (u_\varepsilon(r,t), v_\varepsilon(r,t))$ the radially symmetric local solution of (2.1) on $[0, T_\varepsilon)$. Moreover, we define the function w_ε and the moment-type functional ϕ_ε for the approximate solution u_ε as

$$\begin{aligned} w_\varepsilon(s,t) &:= \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_\varepsilon(\rho,t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_\varepsilon), \\ \phi_\varepsilon(t) &:= \int_0^{s_0} s^{-\gamma}(s_0 - s) w_\varepsilon(s,t)\, ds \quad \text{for } t \in [0, T_\varepsilon). \end{aligned}$$

Here we know that $\phi_\varepsilon \in C^0([0, T_\varepsilon)) \cap C^1((0, T_\varepsilon))$.

Now we state the following lemma on the moment inequality for approximate solutions.

Proposition 2.8. *Let $n \in \mathbb{N}$, $m \ge 1$, $\chi > 0$, $\alpha \ge 1$, $\lambda > 0$, $\mu > 0$, $\kappa \ge 1$ and $\ell > 0$. Assume that* (1.3) *is satisfied. Then for all $M_0 > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ which satisfy the following property: If u_0 satisfies* (1.4)–(1.6), *then there exist $T_0 \in (0, \infty)$ and $K_0 > 0$ such that* (2.4) *holds. Moreover, one can find $K = K(R, m, \chi, \alpha, \mu, \kappa, \ell) > 0$ and $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \ge K \int_0^t \phi_\varepsilon^{\alpha+\ell}(\tau)\, d\tau \tag{2.11}$$

for all $t \in (0, T_0)$.

As to the proof of Proposition 2.8, we use arguments of [**5**, Lemmas 3.4–3.10].

Lemma 2.9. *Assume that u_0 satisfies* (1.4). *Then for any $\varepsilon \in (0, 1)$, $(u_\varepsilon)_r(r,t) \le 0$ for all $r \in (0, R)$ and $t \in (0, T_\varepsilon)$, that is, $(w_\varepsilon)_{ss} \le 0$ for all $s \in (0, R^n)$ and $t \in (0, T_\varepsilon)$.*

Proof. By virtue of (1.4) and the definition of $u_{0\varepsilon}$, we see that $u_{0\varepsilon}$ is also nonincreasing with respect to $|x|$. Therefore the claim can be proved by an argument similar to that in the proof of [**7**, Lemma 2.2]. □

In the proof of [**5**, Lemma 3.4] we know that [**5**, Lemmas 3.2 and 3.3] have been used with the assumption such that $u_{0\varepsilon}$ fulfills $\int_\Omega u_{0\varepsilon} = M_0$. However, in our case this may not be satisfied. Indeed, since $\int_\Omega u_{0\varepsilon} \le \int_\Omega u_0 = M_0$ and $u_{0\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$, we can pick $\xi_0 > 0$ so small and find some $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$M_0 - \xi_0 \le \int_\Omega u_{0\varepsilon} \le M_0. \tag{2.12}$$

Moreover, in order to prove Proposition 2.8, we need to use arguments similar to those in the proofs of these lemmas. Thus we should give minor changes to [**5**, Lemmas 3.2 and 3.3]. In the following, we fix ξ_0 and take ε_0 such that (2.12) holds. Furthermore, let $T_0 \in (0,\infty)$ and $K_0 > 0$ fulfill (2.4) and we define the set S_{ϕ_ε} as

$$S_{\phi_\varepsilon} := \left\{ t \in (0, T_0) \;\middle|\; \phi_\varepsilon(t) \ge \frac{M_0 - \xi_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} \right\}. \tag{2.13}$$

The following two lemmas (see Lemmas 2.10 and 2.11) are able to be proved with minor changes in the proofs of [**5**, Lemmas 3.2 and 3.3], respectively.

Lemma 2.10. *Assume that u_0 satisfies* (1.4) *and let $s_0 \in (0, R^n)$ and $\gamma \in (-\infty, 1)$. Then for any $\varepsilon \in (0, \varepsilon_0)$,*

$$w_\varepsilon\left(\frac{s_0}{2}, t\right) \ge \frac{M_0 - \delta_0}{\omega_n} \quad \textit{for all } t \in S_{\phi_\varepsilon},$$

where $\delta_0 := \frac{4(\xi_0 + s_0)}{2^\gamma(3-\gamma)}$.

Proof. The proof of this lemma is similar to that of [**7**, Lemma 3.1]. □

We next establish the estimate for $\overline{M_{\ell,\varepsilon}}(t)$.

Lemma 2.11. *Assume that u_0 satisfies* (1.4) *and let $s_0 \in \left(0, \frac{R^n}{4}\right]$ and $\gamma \in (-\infty, 1)$. Then for any $\varepsilon \in (0, \varepsilon_0)$,*

$$\overline{M_{\ell,\varepsilon}}(t) \le L + \frac{1}{2s}\int_0^s \left[n(w_\varepsilon)_s(\sigma, t)\right]^\ell d\sigma \quad \textit{for all } s \in (0, s_0) \textit{ and } t \in S_{\phi_\varepsilon}, \tag{2.14}$$

where $L := \left(\frac{2n\delta_0}{\omega_n s_0}\right)^\ell$.

Proof. The proof is similar to that of [**7**, Lemma 3.2]. □

Now we prove Proposition 2.8.

Proof of Proposition 2.8. We first show (2.11). By means of Lemma 2.4, for any initial data u_0 with the properties (1.4) and (1.5), we can find $T_0 \in (0, \infty)$ and $K_0 > 0$ satisfying (2.4). Now let $\xi_0 > 0$ and $\varepsilon_0 \in (0,1)$ fulfill (2.12). In view of Lemmas 2.9–2.11, we see from an argument similar to that in the proof of [**5**, Lemma 3.4] that

$$\begin{aligned} \phi_\varepsilon'(t) \ge{}& \frac{n^\ell}{2}\int_0^{s_0} s^{1-\gamma}(s_0 - s)\left(n(w_\varepsilon)_s + \varepsilon\right)^{\alpha-1}(w_\varepsilon)_s^{\ell+1}\,ds \\ &- L\int_0^{s_0} s^{1-\gamma}(s_0 - s)\left(n(w_\varepsilon)_s + \varepsilon\right)^{\alpha-1}(w_\varepsilon)_s\,ds \\ &+ mn^2\int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)\left(n(w_\varepsilon)_s + \varepsilon\right)^{m-1}(w_\varepsilon)_{ss}\,ds \\ &- n^{\kappa-1}\mu\int_0^{s_0} s^{-\gamma}(s_0 - s)\left\{\int_0^{s_0} (w_\varepsilon)_s^\kappa\,d\sigma\right\}ds \end{aligned} \tag{2.15}$$

for all $s_0 \in \big(0, \frac{R^n}{4}\big]$ and $t \in S_{\phi_\varepsilon}$. Since we can apply [**5**, Lemmas 3.5–3.10] with S_ϕ replaced by S_{ϕ_ε} to (2.15), there are $\gamma \in (-\infty, 1)$ and $c_1 = c_1(R, m, \chi, \alpha, \mu, \kappa, \ell, \gamma) > 0$ as well as $c_2 = c_2(R, m, \chi, \alpha, \mu, \kappa, \ell, \gamma) > 0$ such that

$$\phi_\varepsilon'(t) \geq c_1 s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi_\varepsilon^{\alpha+\ell}(t) - c_2 s_0^{3-\gamma-\frac{2}{n}\cdot\frac{\alpha+\ell}{\alpha+\ell-m}} \tag{2.16}$$

for all $s_0 \in \big(0, \frac{R^n}{4}\big]$ and $t \in S_{\phi_\varepsilon}$. Here we note from [**5**, Remark 3.1] that c_1 and c_2 are independent of ε. We fix $s_0 > 0$ such that

$$s_0 \leq \min\left\{ \frac{R^n}{4}, \frac{M_0 - \xi_0}{2} \right\} \tag{2.17}$$

and

$$s_0^{(\alpha+\ell)\left(1-\frac{1}{\alpha+\ell-m}\right)} \leq \frac{c_1}{2c_2} \left(\frac{M_0 - \xi_0}{2(1-\gamma)(2-\gamma)\omega_n} \right)^{\alpha+\ell}. \tag{2.18}$$

We additionally pick $\eta_0 \in \big(0, \frac{s_0}{4}\big)$ so small and take $s_\star \in (0, s_0)$ satisfying

$$\frac{M_0 - \xi_0 - \eta_0}{\omega_n} \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s)\, ds > \frac{M_0 - \xi_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma}.$$

Moreover, in the following we suppose that u_0 fulfills (1.4)–(1.6) with $r_\star := s_\star^{\frac{1}{n}}$. In order to derive (2.11), we define the set

$$\widetilde{S}_\varepsilon := \left\{ \tau \in (0, T_0) \;\middle|\; \phi_\varepsilon(t) > \frac{M_0 - \xi_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} \quad \text{for all } t \in [0, \tau] \right\}.$$

Here we can see that $\widetilde{S}_\varepsilon \neq \emptyset$ for sufficiently small ε. Indeed, from the second condition of (1.6) and $u_{0\varepsilon} \to u_0$ in $L^1(\Omega)$ as $\varepsilon \to 0$, we observe that $\int_{B_{r_\star}(0)} u_{0\varepsilon}\, dx \geq M_0 - \xi_0 - \eta_0$ for all $\varepsilon \in (0, \varepsilon_0)$. This inequality yields that $w_\varepsilon(s, 0) \geq w_\varepsilon(s_\star, 0) \geq \frac{M_0 - \xi_0 - \eta_0}{\omega_n}$ for all $s \in (s_\star, s_0)$. Hence we obtain that

$$\phi_\varepsilon(0) \geq \int_{s_\star}^{s_0} s^{-\gamma}(s_0 - s) w_\varepsilon(s, 0)\, ds > \frac{M_0 - \xi_0 - s_0}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma},$$

which together with the continuity of ϕ_ε implies that $\widetilde{S}_\varepsilon$ is not empty for any $\varepsilon \in (0, \varepsilon_0)$. Now let $\widetilde{T}_\varepsilon := \sup \widetilde{S}_\varepsilon \in (0, T_0]$. Then from (2.13) we can confirm that $(0, \widetilde{T}_\varepsilon) \subset S_{\phi_\varepsilon}$. Thanks to (2.16)–(2.18), as in the proof of [**5**, Theorem 1.2], we have

$$\phi_\varepsilon'(t) \geq \frac{c_1}{2} s_0^{-(3-\gamma)(\alpha+\ell-1)} \phi_\varepsilon^{\alpha+\ell}(t) > 0$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in (0, \widetilde{T}_\varepsilon)$. This ensures that $\widetilde{T}_\varepsilon = T_0$. Choosing an arbitrary $t \in (0, T_0)$ and integrating the above inequality over $(0, t)$, we attain (2.11). □

2.4. Proof of Proposition 2.1

We establish local existence of moment solutions to the system (1.1) by virtue of the passage to the limit as $\varepsilon \to 0$ in (2.11).

Proof of Proposition 2.1. Let $M_0 > 0$ and let $\eta_0 \in (0, M_0)$ and $r_\star \in (0, R)$ given by Proposition 2.8. Also, we pick u_0 fulfilling (1.4)–(1.6). Then, thanks to Lemma 2.2 and Proposition 2.8, we can obtain the approximate solution $(u_\varepsilon, v_\varepsilon)$ of (2.1) and find $T_0 \in (0, \infty)$ and $K_0 > 0$ such that (2.4) holds, and we have

$$\phi_\varepsilon(t) - \phi_\varepsilon(0) \geq K \int_0^t \phi_\varepsilon^{\alpha+\ell}(\tau)\, d\tau \tag{2.19}$$

for all $t \in (0, T_0)$ with some $K > 0$. By virtue of (2.4), we can apply Lemma 2.7. Hence there exist $\{u_{\varepsilon_k}\}$, $\{v_{\varepsilon_k}\}$ ($\varepsilon_k \to 0$ as $k \to \infty$) and nonnegative functions u, v such that $(u, v) = \lim_{k\to\infty}(u_{\varepsilon_k}, v_{\varepsilon_k})$ satisfies (i) and (ii) in Definition 1.1 except for the condition $u \in C^0_{\mathrm{w}-\star}([0, T); L^\infty(\Omega))$. We next show that $u \in C^0_{\mathrm{w}-\star}([0, T); L^\infty(\Omega))$. Let us pick $\psi \in L^1(\Omega)$. Then for all $\xi > 0$ there is $\psi_0 \in C_{\mathrm{c}}(\Omega)$ such that

$$\|\psi - \psi_0\|_{L^1(\Omega)} < \xi. \tag{2.20}$$

Moreover, noting from (2.8) that $u \in C^0([\delta, T_0]; L^1(\Omega))$ for all $\delta \in (0, T_0)$, we see that for all $t_0 > 0$,

$$\int_\Omega u(\cdot, t)\psi_0\, dx \to \int_\Omega u(\cdot . t_0)\psi_0\, dx \quad \text{as } t \to t_0, \tag{2.21}$$

and from (2.7) it follows that

$$\|u\|_{L^\infty(0,T_0;L^\infty(\Omega))} \leq \liminf_{k\to\infty} \|u_{\varepsilon_k}\|_{L^\infty(0,T_0;L^\infty(\Omega))} < \infty. \tag{2.22}$$

In light of (2.20)–(2.22) we can confirm that $u \in C^0_{\mathrm{w}-\star}((0, T_0); L^\infty(\Omega))$. Furthermore, by relying on the fact that $u_{\varepsilon_k} \in C^0(\overline{\Omega} \times [0, T_0))$ and $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$ as $k \to \infty$, it follows that $u \in C^0_{\mathrm{w}-\star}([0, T_0); L^\infty(\Omega))$. Next we make sure that the moment inequality (1.2) holds. Invoking $u_{0\varepsilon_k} \to u_0$ in $L^1(\Omega)$ as $k \to \infty$, we see that $\phi_{\varepsilon_k}(0) \to \phi(0) \quad$ as $k \to \infty$. Furthermore, due to (2.8) it follows that $u_{\varepsilon_k} \to u$ in $C^0((0, T_0]; L^1(\Omega))$ as $k \to \infty$, which ensures that $\phi_{\varepsilon_k}(t) \to \phi(t)$ as $k \to \infty$ for all $t \in (0, T_0)$. Additionally, noticing that $w_{\varepsilon_k}(s, t) \leq \frac{K_0|\Omega|}{\omega_n}$, we can observe that

$$\phi_{\varepsilon_k}^{\alpha+\ell}(t) \leq \left(\frac{K_0|\Omega|}{(1-\gamma)(2-\gamma)\omega_n} s_0^{2-\gamma} \right)^{\alpha+\ell}$$

for all $t \in (0, T_0)$. In view of the Lebesgue dominated convergence theorem, we infer that

$$\int_0^t \phi_{\varepsilon_k}^{\alpha+\ell}(\tau)\, d\tau \to \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \quad \text{as } k \to \infty$$

for all $t \in (0, T_0)$, and so letting $k \to \infty$ in (2.19), we see that (iii) in Definition 1.1 holds. This implies the end of the proof. $\square$

3. Finite-time blow-up

In this section we prove finite-time blow-up of maximal moment solutions to (1.1). Before proceeding to the proof, we confirm the following standard equivalence.

Lemma 3.1. *Let $T \in (0,\infty)$. Assume that nonnegative functions u, v satisfy*

$$u \in L^\infty(0,T;L^\infty(\Omega)), \quad u^m, v \in L^2(0,T;H^1(\Omega)), \quad u \in C^0_{\mathrm{w}-\star}([0,T);L^\infty(\Omega)). \tag{3.1}$$

Then the following two conditions are equivalent.

(a) *For $\varphi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ for a.a. $x \in \Omega$,*

$$\int_0^T \int_\Omega (\nabla u^m \cdot \nabla\varphi - \chi u^\alpha \nabla v \cdot \nabla\varphi - (\lambda u - \mu u^\kappa)\varphi - u\varphi_t)\, dxdt = \int_\Omega u_0(x)\varphi(x,0)\, dx,$$
$$\int_0^T \int_\Omega \nabla v \cdot \nabla\varphi\, dxdt + \int_0^T \left(\overline{M_\ell}(t)\int_\Omega \varphi\, dx\right) dt - \int_0^T \int_\Omega u^\ell \varphi\, dxdt = 0;$$

(b) *$u_t \in L^2(0,T;(H^1(\Omega))^\star)$, and for all $\psi \in H^1(\Omega)$,*

$$\int_\Omega u_t\psi\, dx = -\int_\Omega (\nabla u^m \cdot \nabla\psi - \chi u^\alpha \nabla v \cdot \nabla\psi - (\lambda u - \mu u^\kappa)\psi)\, dx, \tag{3.2}$$
$$\int_\Omega \nabla v \cdot \nabla\psi\, dx + \overline{M_\ell}(t)\int_\Omega \psi\, dx - \int_\Omega u^\ell\psi\, dx = 0 \tag{3.3}$$

for a.a. $t \in [0,T)$ with $u(\cdot,0) = u_0$.

We finally prove Theorem 1.1.

Proof of Theorem 1.1. Let $M_0 > 0$ and let $\eta_0 \in (0, M_0)$ and $r_\star \in (0,R)$ given by Proposition 2.8. We pick u_0 as in (1.4)–(1.6). From Proposition 2.1 and Definition 1.2, there is a maximal moment solution (u,v) of (1.1) on $[0,T_{\max})$. We first show that $T_{\max} < \infty$ by contradiction. To this end, we assume that $T_{\max} = \infty$. Then we have

$$\phi(t) - \phi(0) \ge K \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \tag{3.4}$$

for all $t \in (0,\infty)$ with some $K > 0$, and put the function Φ as

$$\Phi(t) := \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau + \frac{\phi(0)}{K} \quad \text{for } t \in (0,\infty).$$

Also, we infer that ϕ is bounded on $[0,T')$ for all $T' < \infty$ and continuous on $[0,\infty)$ because u belongs to $L^\infty_{\mathrm{loc}}(0,\infty;L^\infty(\Omega))$ and $C^0_{\mathrm{w}-\star}([0,\infty);L^\infty(\Omega))$ due to (i). Hence we note that $\Phi \in C^0([0,\infty)) \cap C^1((0,\infty))$. From (3.4) we obtain that

$$\Phi'(t) \ge K^{\alpha+\ell}\Phi^{\alpha+\ell}(t) \quad \text{for all } t \in (0,\infty),$$

and thereby we can derive that $-\frac{1}{(\alpha+\ell-1)\Phi^{\alpha+\ell-1}(t)} + \frac{1}{(\alpha+\ell-1)\Phi^{\alpha+\ell-1}(0)} \ge K^{\alpha+\ell}t$ for all $t \in (0,\infty)$. Thus it follows that $t \le \frac{1}{(\alpha+\ell-1)K^{\alpha+\ell}\Phi^{\alpha+\ell-1}(0)}$ for all $t \in (0,\infty)$, which is a contradiction. Therefore we see that $T_{\max} < \infty$.

Next, we prove that

$$\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty \tag{3.5}$$

by contradiction. To this end, we assume that $\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$, that is, $u \in L^\infty(0,T_{\max};L^\infty(\Omega))$. By this assumption and (i) in Definition 1.1, it follows that (3.1) and (a) in Lemma 3.1 with $T = T_{\max}$ hold. Hence, noting from (b) in Lemma 3.1 that $u_t \in L^2(0,T_{\max};(H^1(\Omega))^\star)$, we have $\|u(\cdot,t)-u(\cdot,s)\|_{(H^1(\Omega))^\star} \le \|u_t\|_{L^2(0,T_{\max};(H^1(\Omega))^\star)}|t-s|^{\frac{1}{2}}$ for all $t,s \in [0,T_{\max})$, so that u is uniformly continuous on $[0,T_{\max})$ in $(H^1(\Omega))^\star$. This continuity provides $\widetilde{u}_{T_{\max}} \in (H^1(\Omega))^\star$ such that

$$\widetilde{u}_{T_{\max}} = \lim_{t\nearrow T_{\max}} u(\cdot,t) \quad \text{in } (H^1(\Omega))^\star.$$

Moreover, the condition (i) in Definition 1.1 with $T = T_{\max}$ guarantees that $\widetilde{u}_{T_{\max}} \in L^\infty(\Omega)$. Indeed, by virtue of the condition (i) in Definition 1.1 and the assumption $\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$, we see that there exist $\{t_n\} \subset [0,T_{\max})$ and $g \in L^\infty(\Omega)$ such that $t_n \nearrow T_{\max}$ and $u(\cdot,t_n) \to g$ weakly* in $L^\infty(\Omega)$ as $n\to\infty$. Since we observe that $u(\cdot,t_n) \to \widetilde{u}_{T_{\max}}$ in $(H^1(\Omega))^\star$ as $n\to\infty$, it follows that $g = \widetilde{u}_{T_{\max}}$ in $(H^1(\Omega))^\star$. Noting that $L^\infty(\Omega) \subset L^2(\Omega) = (L^2(\Omega))^\star \subset (H^1(\Omega))^\star$, we arrive at the desired fact that $\widetilde{u}_{T_{\max}} \in L^\infty(\Omega)$. Choosing the initial data as $\widetilde{u}_{T_{\max}}$, by an argument similar to those in the proofs of Lemmas 2.3–2.7, we can find $T_1 > 0$ and construct a weak solution $(\widetilde{u},\widetilde{v})$ on $[T_{\max},T_{\max}+T_1)$. Now, we put

$$(\overline{u},\overline{v}) := \begin{cases} (u,v) & \text{for a.a. } t \in [0,T_{\max}), \\ (\widetilde{u},\widetilde{v}) & \text{for a.a. } t \in [T_{\max},T_{\max}+T_1), \end{cases}$$

and confirm that $(\overline{u},\overline{v})$ is a weak solution of (1.1) on $[0,T_{\max}+T_1)$. The definition of $\widetilde{u}_{T_{\max}}$ implies that $\int_\Omega u(\cdot,t)\psi_0\,dx \to \int_\Omega \widetilde{u}_{T_{\max}}\psi_0\,dx$ as $t \nearrow T_{\max}$ for all $\psi_0 \in C_{\mathrm{c}}^\infty(\Omega)$, and $u \in L^\infty(0,T_{\max};L^\infty(\Omega))$, and hence we see that $u \in C^0_{\mathrm{w}-\star}([0,T_{\max}];L^\infty(\Omega))$. On the other hand, the condition corresponding to (i) in Definition 1.1 says that $\widetilde{u} \in C^0_{\mathrm{w}-\star}([T_{\max},T_{\max}+T_1);L^\infty(\Omega))$. Consequently, we deduce that

$$\overline{u} \in C^0_{\mathrm{w}-\star}([0,T_{\max}+T_1);L^\infty(\Omega)). \tag{3.6}$$

Recalling that $u_t \in L^2(0,T_{\max};(H^1(\Omega))^\star)$ and $\widetilde{u}_t \in L^2([T_{\max},T_{\max}+T_1);(H^1(\Omega))^\star)$ with $\widetilde{u}(\cdot,T_{\max}) = \widetilde{u}_{T_{\max}}$, we can show that $\overline{u}_t \in L^2([0,T_{\max}+T_1);(H^1(\Omega))^\star)$. Indeed, it follows from (3.6) that for any $\varphi \in H^1([0,T_{\max}+T_1);H^1(\Omega))$,

$$\begin{aligned}
&-\int_0^{T_{\max}+T_1} \langle \overline{u}(\cdot,t),\varphi_t(\cdot,t)\rangle_{(H^1(\Omega))^\star,H^1(\Omega)}\,dt \\
&= -\int_0^{T_{\max}}\int_\Omega u\varphi_t\,dxdt - \int_{T_{\max}}^{T_{\max}+T_1}\int_\Omega \widetilde{u}\varphi_t\,dxdt \\
&= \int_\Omega \overline{u}(\cdot,T_{\max})\varphi(\cdot,T_{\max})\,dx + \int_0^{T_{\max}} \langle u_t(\cdot,t),\varphi(\cdot,t)\rangle_{(H^1(\Omega))^\star,H^1(\Omega)}\,dt \\
&\quad - \int_\Omega \overline{u}(\cdot,T_{\max})\varphi(\cdot,T_{\max})\,dx + \int_{T_{\max}}^{T_{\max}+T_1} \langle \widetilde{u}_t(\cdot,t),\varphi(\cdot,t)\rangle_{(H^1(\Omega))^\star,H^1(\Omega)}\,dt \\
&= \int_0^{T_{\max}+T_1} \langle g(\cdot,t),\varphi(\cdot,t)\rangle_{(H^1(\Omega))^\star,H^1(\Omega)}\,dt,
\end{aligned}$$

where

$$g := \begin{cases} u_t & \text{for a.a. } t \in [0, T_{\max}), \\ \widetilde{u}_t & \text{for a.a. } t \in [T_{\max}, T_{\max} + T_1), \end{cases}$$

which means that $\overline{u}_t = g \in L^2([0, T_{\max} + T_1); (H^1(\Omega))^\star)$. Moreover, since (u, v) and $(\widetilde{u}, \widetilde{v})$ satisfy (3.2), (3.3) for a.a. $t \in [0, T_{\max})$ and for a.a. $t \in [T_{\max}, T_{\max} + T_1)$, respectively, $(\overline{u}, \overline{v})$ fulfills (3.2), (3.3) for a.a. $t \in [0, T_{\max} + T_1)$, and hence, by means of Lemma 3.1, $(\overline{u}, \overline{v})$ is a weak solution of (1.1) on $[0, T_{\max} + T_1)$. We shall show that the weak solution $(\overline{u}, \overline{v})$ fulfills the moment inequality on $[0, T_{\max} + \sigma_1)$ with some $\sigma_1 > 0$. For this purpose, defining $\overline{w}$ and $\overline{\phi}$ as

$$\overline{w}(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} \overline{u}(\rho, t)\, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, T_{\max} + T_1),$$
$$\overline{\phi}(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)\overline{w}(s, t)\, ds \quad \text{for } t \in [0, T_{\max} + T_1),$$

we have only to prove that there exists $\overline{K} > 0$ such that

$$\overline{\phi}(t) - \overline{\phi}(0) \geq \overline{K} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in [0, T_{\max} + \sigma_1). \tag{3.7}$$

We know that

$$\phi(t) - \phi(0) \geq K \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in [0, T_{\max}). \tag{3.8}$$

In order to construct the moment inequality beyond $T_{\max}$, we make sure that

$$\overline{\phi}(T_{\max}) - \overline{\phi}(0) \geq K \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau. \tag{3.9}$$

To this end, we confirm that $\overline{\phi} \in C^0([0, T_{\max} + T_1))$. Letting $t \to t_0 \in [0, T_{\max} + T_1)$ and noting from (3.6) that for any $s \in (0, R]$, $\overline{w}(s, \cdot)$ is continuous on $[0, T_{\max} + T_1)$ and $s^{-\gamma}(s_0 - s)\overline{w}(s, t) \leq c_1 s^{-\gamma}(s_0 - s)$ with some $c_1 > 0$, we see from the Lebesgue dominated convergence theorem that $\overline{\phi}(t) \to \overline{\phi}(t_0)$, and so $\overline{\phi} \in C^0([0, T_{\max}+T_1))$. Thus the inequality (3.9) is derived by the passage to the limit in (3.8) as $t \nearrow T_{\max}$. Next, by setting

$$\varepsilon_K := \frac{K}{2} \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau > 0, \tag{3.10}$$

the continuity of $\overline{\phi}$ and $\int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau$ at $t = T_{\max}$ provides $\sigma_1 \in (0, T_1)$ such that for all $t \in [T_{\max}, T_{\max} + \sigma_1)$,

$$\left| \overline{\phi}(t) - \frac{K}{2} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau - \left(\overline{\phi}(T_{\max}) - \frac{K}{2} \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \right) \right| \leq \varepsilon_K,$$

which together with (3.10) implies

$$\overline{\phi}(t) - \frac{K}{2} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \geq \overline{\phi}(T_{\max}) - K \int_0^{T_{\max}} \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \geq \overline{\phi}(0)$$

for all $t \in [T_{\max}, T_{\max} + \sigma_1)$, that is,

$$\overline{\phi}(t) - \overline{\phi}(0) \geq \frac{K}{2} \int_0^t \overline{\phi}^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in [T_{\max}, T_{\max} + \sigma_1). \tag{3.11}$$

On the other hand, in light of (3.8), (u, v) satisfies that

$$\phi(t) - \phi(0) \geq \frac{K}{2} \int_0^t \phi^{\alpha+\ell}(\tau)\, d\tau \quad \text{for all } t \in [0, T_{\max}).$$

Noting that $\overline{\phi} = \phi$ on $[0, T_{\max})$ and combining this inequality and (3.11), we obtain the moment inequality (3.7) on $[0, T_{\max} + \sigma_1)$ with $\overline{K} = \frac{K}{2}$, which contradicts the definition of maximal moment solutions. Therefore we conclude that (3.5) holds. □

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