Applications of the class $A^{\rm loc}$

Yoshihiro Sawano

Abstract

This paper is an announcement of the series of recent research of the author jointly done with Izuki, Nogayama and Noi. Actually, the author considered the class $A_{p(\cdot)}^{\rm loc}$ with them. But this note is oriented to general applications of the local class of weights. This is why the author omitted $p(\cdot)$ in the title.

1 Introduction

Let $1 < p < \infty$. A locally integrable weight w is an A_p-*weight* or belongs to A_p-*class*, if $0 < w < \infty$ almost everywhere, and $[w]_{A_p} \equiv \sup_{Q \in \mathcal{Q}} m_Q(w) m_Q^{(\frac{1}{p-1})}(w^{-1}) < \infty$. The quantity $[w]_{A_p}$ is referred to as the A_p-*constant* or the A_p-*characteristic*. The class A_p collects all A_p-weights. Write $A_\infty = \bigcup_{1<p<\infty} A_p$.

Here and below we write

$$m_Q^{(u)}(w) = \left(\frac{1}{|Q|} \int_Q |w(x)|^u {\rm d}x \right)^{\frac{1}{u}}$$

for a cube Q and a function $w \in L^0(\mathbb{R}^n)$.

We know:

- The Hardy–Littlewood maximal operator M is bounded on $L^p(w)$ if and only if $w \in A_p$.
- It is easy to check $A_1 \subset A_p \subset A_q \subset L^1_{\rm loc}(\mathbb{R}^n)$ by Hölder's inequality whenever $1 \le p \le q < \infty$. But any weight in A_q is never integrable.
- Let $r > 1$. $|\cdot|^\alpha \in A_r$ if and only if $\alpha \in (-n, (r-1)n)$.

Among other things, it is sometimes inconvenient that $M\chi_{B(0,1)}$ is not in A_1. The goal of this series of research is to overcome this issue by using the local class.

With this in mind, let

$$M^{\rm loc}f(x) \equiv \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)\chi_{[0,1]}(|Q|)}{|Q|} \int_Q |f(y)|{\rm d}y$$

for a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ or $f : \mathbb{R}^n \to [0,\infty]$.

For a weight w, define its *local A_1-characteristic* by

$$[w]_{A_1^{\rm loc}} \equiv \left\| \frac{M^{\rm loc}w}{w} \right\|_{L^\infty}.$$

The quantity is also called the *local A_1-constant.* A weight w is a *local A_1-weight* if $[w]_{A_1^{\rm loc}}$ is finite. The class $A_1^{\rm loc}$ collects all local A_1-weights.

Let $1 < p < \infty$. A locally integrable weight w is a local *A_p-weight* or belongs to *local A_p-class,* if $0 < w < \infty$ almost everywhere, and $[w]_{A_p^{\rm loc}} \equiv \sup\limits_{Q \in \mathcal{Q}} m_Q(w) m_Q^{(\frac{1}{p-1})}(w^{-1}) < \infty$. The quantity $[w]_{A_p^{\rm loc}}$ is referred to as the $A_p^{\rm loc}$-*constant* or the $A_p^{\rm loc}$-*characteristic.* The class $A_p^{\rm loc}$ collects all $A_p^{\rm loc}$-weights. Write $A_\infty^{\rm loc} = \bigcup_{1<p<\infty} A_p^{\rm loc}$.

It is known that $M_{\rm loc}$ is bounded on $L^p(w)$ if and only if $w \in A_p^{\rm loc}$.

Here we present examples. Let $A \in \mathbb{R}$, $0 \le \kappa \le 1$ and $1 < p < \infty$.

1. As is seen from the fact that $w(x) \sim w(y)$ if $|x-y| \le 1$, $w(x) = (1+|x|)^A \in A_p^{\rm loc}$.
2. Likewise, $w(x) = \exp(A|x|^\kappa) \in A_p^{\rm loc}$.
3. Unfortunately, $w(x) = \exp(-\pi|x|^2) \notin A_\infty^{\rm loc}$.

I would like to present some applications to this local class. Here I can consider the following function spaces:

1. Periodic function spaces: Denote by $L^p(\mathbb{T}^n)$ the p-locally integrable functions with the period $\mathbb{Z}^n$. Then we can embed $L^p(\mathbb{T}^n)$ into $L^p((M\chi_{B(1)})^\alpha)$ for any $\alpha > 1$.
2. Amalgam spaces: Let $1 \le p, q \le \infty$. Let f be a measurable function. Define

$$\|f\|_{\ell^q(L^p)} = \left(\sum_{m \in \mathbb{Z}^n} (\|f\|_{L^p(m+[0,1]^n)})^q \right)^{\frac{1}{q}}.$$

The amalgam space $\ell^q(L^p)$ collects all measurable functions f for which $\|f\|_{\ell^q(L^p)} < \infty$. The space $\ell^\infty(L^p)$ is referred to as the uniformly p-local function space.

As we mentioned in the abstract, what we did is to replace L^p with the variable exponent. The starting point where we characterized the class for variable Lebesgue spaces can be located as the extension of the result by Rychkov [5]. However, his technique does not work (see [1]). We considered the dyadic counterpart (global/local) and we managed to characterize the local class for variable exponents. As further applications, we considered the wavelet characterization [2] and Sobolev spaces [3]. As is well known, the class A_p is monotone. We established the counterpart in [4]. In [4] we considered local Hardy spaces.

References

[1] T. Nogayama and Y. Sawano, Local Muckenhoupt class for variable exponents. J. Inequal. Appl. 2021, Paper No. 70, 27 pp.

[2] M. Izuki, T. Nogayama, T. Noi and Y. Sawano, Wavelet characterization of local Muckenhoupt weighted Lebesgue spaces with variable exponent. Nonlinear Anal. 198 (2020), 111930, 14 pp.

[3] M. Izuki, T. Nogayama, T. Noi and Y. Sawano, Wavelet Characterization of Local Muckenhoupt Weighted Sobolev Spaces with Variable Exponents. Constr. Approx. 57 (2023), no. 1, 161–234.

[4] M. Izuki, T. Nogayama, T. Noi and Y. Sawano, Weighted local Hardy spaces with variable exponents. In preparation.

[5] V.S. Rychkov, Littlewood–Paley theory and function spaces with A_p^{loc} weights. Math. Nachr. 224, 145–180