ON FORCIBILITY OF Σ_2 SENTENCES OVER $L(V_\delta)$

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Abstract. We prove a reflection property, with respect to forcibility of Σ_2 sentences, for $L(V_\delta)$, where δ is the least ordinal γ which is a Woodin cardinal in $L(V_\gamma)$.

1. Introduction

Given a model M of enough of ZF and given an ordinal $\delta \in M$, let $\mathrm{Coll}(V_\delta, \delta)^M$ denote the partial order, ordered by reverse inclusion, of all functions $f : \alpha \longrightarrow V_\delta^M$ in M, for $\alpha < \delta$. If α is strongly inaccessible, $M \models V = L(V_\delta)$, and for every $\alpha < \delta$ there is some well-order of V_α^M in M, then $\mathrm{Coll}(V_\delta, \delta)^M$ forces ZFC over M and adds no sets to M of rank less than δ. Also, if δ is Woodin in M, then δ remains Woodin in the extension of M by $\mathrm{Coll}(V_\delta, \delta)^M$.

The main purpose of this note is to prove the following theorem.

Theorem 1.1. *Suppose δ is the least ordinal γ such that γ is a Woodin cardinal in $L(V_\gamma)$. Let $\epsilon > \delta$ be such that $L_\epsilon(V_\delta)$ satisfies enough of* ZF *and let M be a countable transitive model for which there is an elementary embedding $\pi : M \longrightarrow L_\epsilon(V_\delta)$. Let σ be a Σ_2 sentence and suppose N is a countable transitive model of a large enough fragment of* ZFC *such that*

(1) *$M \in N$ and M is countable in N,*
(2) *$N[H]$ is $\underset{\sim}{\Sigma}^1_3$-correct in V for every set-generic filter H over N, and*
(3) *there is some ordinal $\alpha \in N$ and some partial order $\mathbb{P} \in V_\alpha^N$ such that $V_\alpha^N \models \mathbb{P}$ forces σ.*

Then there is a $\mathbb{P}$-generic filter G over N, a transitive model $M' \in N[G]$, an elementary embedding $j : M \longrightarrow M'$, $j \in N[G]$, and an ordinal $\alpha^* < \delta^* := j(\pi^{-1}(\delta))$ such that, letting $\mathbb{Q}_0 = \mathrm{Coll}(V_{\delta^*}^{M'}, \delta^*)^{M'}$,

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there is a $\mathbb{Q}_0$-name $\dot{\mathbb{Q}}_1 \in M'$ for a partial order in $V_{\delta^*+1}^{M'[\dot{G}_{\mathbb{Q}_0}]}$ such that

$$M'[\dot{G}_{\mathbb{Q}_0}] \models \dot{\mathbb{Q}}_1 \textit{ has the } \delta^*\textit{-c.c. and forces } V_{\alpha^*}^{M'[\dot{G}_{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}]} \models \sigma.$$

We will be using the following well-known fact (s. for example [4] or [5]).

Lemma 1.2. *Let κ be a cardinal and let $\delta < \kappa$ be a Woodin cardinal. Suppose $X^\sharp$ exists for every $X \in H_\kappa$. Let N be a countable transitive model such that there is an elementary embedding $\pi : N \longrightarrow H_\kappa$ with $\delta \in \mathrm{range}(\pi)$, and let $\overline{\delta} \in N$ be such that $\pi(\overline{\delta}) = \delta$. Let $H \in V$ be a $\mathcal{P}$-generic filter over N for some partial order $\mathcal{P} \in V_{\overline{\delta}}^N$. Then $N[H]$ is $\underset{\sim}{\Sigma}^1_3$-correct in V.*

Theorem 1.1 and Lemma 1.2 have, as an immediate consequence, the following reflection statement, for forcible Σ_2 sentences, at the first ordinal γ which is a Woodin cardinal in $L(V_\gamma)$.

Corollary 1.3. *Suppose there is a proper class of Woodin cardinals and δ is the least ordinal γ such that γ is a Woodin cardinal in $L(V_\gamma)$. Let $\mathbb{Q}_0 = \mathrm{Coll}(V_\delta, \delta)$. Suppose σ is a forcible Σ_2 sentence. Then there is an ordinal $\alpha < \delta$ and a $\mathbb{Q}_0$-name $\dot{\mathbb{Q}}_1 \in L(V_\delta)$ for a partial order on partial order in $V_{\delta+1}^{L(V_\delta)[\dot{G}_{\mathbb{Q}_0}]}$ such that*

$$L(V_\delta)[\dot{G}_{\mathbb{Q}_0}] \models \dot{\mathbb{Q}}_1 \textit{ has the } \delta\textit{-c.c. and forces } V_\alpha^{L(V_\delta)[\dot{G}_{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}]} \models \sigma.$$

Proof. It is enough to prove that if $\epsilon > \delta$ is any ordinal such that $L_\epsilon(V_\delta)$ satisfies enough of ZF, then there is an ordinal $\alpha < \delta$ and a $\mathbb{Q}_0$-name $\dot{\mathbb{Q}}_1 \in L_\epsilon(V_\delta)$ for a partial order on partial order in $V_{\delta+1}^{L_\epsilon(V_\delta)[\dot{G}_{\mathbb{Q}_0}]}$ such that

$$L_\epsilon(V_\delta)[\dot{G}_{\mathbb{Q}_0}] \models \dot{\mathbb{Q}}_1 \text{ has the } \delta\text{-c.c. and forces } V_\alpha^{L(V_\delta)[\dot{G}_{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}]} \models \sigma.$$

Let $\mathbb{P}$ be a partial order forcing σ and let κ a sufficiently high cardinal which is a limit of Woodin cardinals.

Let P be a countable elementary submodel of $L_\epsilon(V_\delta)$ and M the Mostowski collapse of P. Let $\pi : M \longrightarrow P$ be the inverse of the collapsing function of P. Let Q be a countable elementary submodel of H_κ such that M, $\mathbb{P} \in Q$ and let N be the Mostowski collapse of Q. Let $\pi^* : N \longrightarrow H_\kappa$ be the inverse of the transitive collapse of Q and let $\overline{\mathbb{P}}$ be such that $\pi^*(\overline{\mathbb{P}}) = \mathbb{P}$. We clearly have that $M \in N$, M is countable in N, and $N \models \overline{\mathbb{P}}$ forces σ. Let $\alpha \in N$ be an ordinal such that $V_\alpha^M \models$ "$\overline{\mathbb{P}}$ forces σ". Since κ is a limit of Woodin cardinals and $Q \preccurlyeq H_\kappa$, we have by Lemma 1.2 that $N[H]$ is $\underset{\sim}{\Sigma}^1_3$-correct in V for every forcing notion $\mathbb{Q} \in N$ and every $\mathbb{Q}$-generic filter H over

N. By Theorem 1.1 there are then a $\overline{\mathbb{P}}$-generic filter G over N, a transitive model $M' \in N[G]$, an elementary embedding $j : M \longrightarrow M'$, $j \in N[G]$, and an ordinal $\alpha^* < \delta^* := j(\pi^{-1}(\delta))$ such that, letting $\mathbb{Q}_0 = \mathrm{Coll}(V_{\delta^*}^{M'}, \delta^*)^{M'}$, there is a $\mathbb{Q}_0$-name $\dot{\mathbb{Q}}_1 \in M'$ for a partial order in $V_{\delta^*+1}^{M'[\dot{G}_{\mathbb{Q}_0}]}$ such that

$$M'[\dot{G}_{\mathbb{Q}_0}] \models \dot{\mathbb{Q}}_1 \text{ has the } \delta^*\text{-c.c. and forces } V_{\alpha^*}^{M'[\dot{G}_{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}]} \models \sigma.$$

But then the desired conclusion holds by elementarity of $j \circ \pi^{-1}$. □

Remark 1.4. As will be immediate from the proof, assuming there is a proper class of Woodin cardinals, the conclusions of Theorem 1.1 and Corollary 1.3 extend to any ordinal γ such that γ is Woodin in $L(V_\gamma)$ and the set of $L(V_\gamma)$-Woodin cardinals is bounded in γ.

Before proceeding to the proof of Theorem 1.1, we will point out that Hugh Woodin has proved similar results.

2. Proving Theorem 1.1

Throughout this section, a premouse is meant to be simply a transitive structure $(M, \in, \delta)$, with M satisfying enough of ZFC and $\delta \in \mathsf{Ord}^M$, as given by [3]. We will consider iteration trees in the sense of [3], Definition 1.4.

The following is Definition 1.9 from [3].

Definition 2.1. An iteration tree $\mathcal{T}$ is *normal* iff there are ordinals ρ_α, for $\alpha < \mathrm{lh}(\mathcal{T})$, such that for all α, β with $\alpha + 1$, $\beta + 1 < \mathrm{lh}(\mathcal{T})$,

(1) $\rho_\alpha + 2 \leq \mathrm{strength}^{\mathcal{M}_\alpha^\mathcal{T}}(E_\alpha)$,
(2) $\rho_\alpha < \rho_\beta$ for all $\alpha < \beta < \mathrm{lh}(\mathcal{T})$, and
(3) for every α such that $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, $\mathcal{T}$-pred$(\alpha + 1)$ is the least $\gamma \leq \alpha$ such that $\mathrm{crit}(E_\alpha) \leq \rho_\gamma$.

If $\mathcal{T}$ is an iteration tree of length λ and $\alpha < \beta \leq \lambda$, then

$$\rho^\mathcal{T}(\alpha, \beta) = \min\{\mathrm{strength}^{\mathcal{M}_\gamma^\mathcal{T}}(E_\gamma) \, : \, \alpha \leq \gamma < \beta\}$$

Theorems 2.2 and 2.3 below are, respectively, Theorems 2.2 and Theorem 4.3 from [3].

Theorem 2.2. *Let $\mathcal{T}$ be a iteration tree of limit length λ, and let b and c be distinct cofinal branches of $\mathcal{T}$. Let $\theta = \sup\{\rho^\mathcal{T}(\alpha, \lambda) \, : \, \alpha < \lambda\}$, and suppose $\theta \in \mathrm{wfp}(\mathcal{M}_b^\mathcal{T}) \cap \mathrm{wfp}(\mathcal{M}_c^\mathcal{T})$. Let $f : \theta \longrightarrow \theta$, $f \in \mathcal{M}_b^\mathcal{T} \cap \mathcal{M}_c^\mathcal{T}$. Then $\mathcal{M}_b^\mathcal{T} \models$ "θ is Woodin with respect to f"; in other words, $\mathcal{M}_b^\mathcal{T}$ satisfies that there is some $\kappa < \theta$ such that $f``\kappa \subseteq \kappa$ and there is an extender E with* $\mathrm{crit}(E) = \kappa$ *and* $\mathrm{strength}(E) > i_E(f)(\kappa)$.

Given a model M, an elementary embedding $\pi : (M, \in) \longrightarrow (V_\alpha, \in)$, an iteration tree $\mathcal{T}$ on M, and a branch b through $\mathcal{T}$, we say that b *is π-realizable* if there is an elementary embedding

$$k : (M_b^{\mathcal{T}}, \in) \longrightarrow (V_\alpha, \in)$$

such that $\pi = k \circ j_{0,b}^{\mathcal{T}}$. Also, given any $\beta < \mathrm{lh}(\mathcal{T})$ and an extender E on $M_\beta^{\mathcal{T}}$, we say that $\mathrm{Ult}(M_\beta^{\mathcal{T}}, E)$ *is π-realizable* in case there is an elementary embedding

$$k : \mathrm{Ult}(M_\beta^{\mathcal{T}}, E) \longrightarrow (V_\alpha, \in)$$

such that $\pi = k \circ i_E^{M_b^{\mathcal{T}}} \circ j_{0,\beta}^{\mathcal{T}}$, where

$$i_E^{M_b^{\mathcal{T}}} : M_\beta^{\mathcal{T}} \longrightarrow \mathrm{Ult}(M_\beta^{\mathcal{T}}, E)$$

is the canonical extender embedding.

Theorem 2.3. *Let $\mathcal{T}$ be a normal*[1] *iteration tree on a countable model M, and let $\pi : (M, \in) \longrightarrow (V_\alpha, \in)$ be an elementary embedding for some ordinal α. Suppose there is no maximal branch b of $\mathcal{T}$ such that* $\sup(b) < \mathrm{lh}(\mathcal{T})$ *and b is π-realizable.*

(1) *If* $\mathrm{lh}(\mathcal{T})$ *is a limit ordinal, then $\mathcal{T}$ has a cofinal branch which is π-realizable.*
(2) *If $\beta < \gamma < \mathrm{lh}(\mathcal{T})$, $\mathcal{M}_\gamma^{\mathcal{T}} \models$ "E is an extender", and* $\mathrm{crit}(E) + 1 < \rho^{\mathcal{T}}(\beta, \gamma)$*, then* $\mathrm{Ult}(M_\beta^{\mathcal{T}}, E)$ *is π-realizable.*

We will now start with the proof of Theorem 1.1.

Let δ be the least ordinal γ such that γ is a Woodin cardinal in $L(V_\gamma)$, let $\epsilon > \delta$ be such that $L_\epsilon(V_\delta)$ satisfies enough of ZF, and let M be a countable transitive model for which there is an elementary embedding $\pi : M \longrightarrow L_\epsilon(V_\delta)$. We also fix a Σ_2 sentence σ and suppose N is a countable transitive model of a large enough fragment of ZFC such that

(1) $M \in N$ and M is countable in N,
(2) $N[H]$ is $\utilde{\Sigma}^1_3$-correct in V for every set-generic filter H over N, and
(3) there is some ordinal $\alpha \in N$ and some partial order $\mathbb{P} \in V_\alpha^N$ such that $V_\alpha^N \models \mathbb{P}$ forces σ.

We need to prove that there is a $\mathbb{P}$-generic filter G over N, a transitive model $M' \in N[G]$, an elementary embedding $j : M \longrightarrow M'$, $j \in N[G]$, and an ordinal $\alpha^* < \delta^* := j(\pi^{-1}(\delta))$ such that, letting $\mathbb{Q}_0 =$

[1]The conclusion holds actually with 'normal' replaced by 'plus two', which is more general and is in fact how Theorem 4.3 in [3] is stated. However, we will not be using the notion of plus two iteration tree and therefore we are not defining it.

$\mathrm{Coll}(V_{\delta^*}^{M'}, \delta^*)^{M'}$, there is a $\mathbb{Q}_0$-name $\dot{\mathbb{Q}}_1 \in M'$ for a partial order in $V_{\delta^*+1}^{M'[\dot{G}_{\mathbb{Q}_0}]}$ such that

$$M'[\dot{G}_{\mathbb{Q}_0}] \models \dot{\mathbb{Q}}_1 \text{ has the } \delta^*\text{-c.c. and forces } V_{\alpha^*}^{M'[\dot{G}_{\mathbb{Q}_0 * \dot{\mathbb{Q}}_1}]} \models \sigma.$$

The basic strategy for achieving this is standard (s. [2]). Let $\bar{\delta} \in M$ be such that $\pi(\bar{\delta}) = \delta$ and let $\mathcal{E}$ be the collection of all extenders in $V_{\bar{\delta}}^M$. Let $g_0 \in N$ be a $\mathrm{Coll}(V_{\bar{\delta}}, \bar{\delta})^M$-generic filter over M (which exist since M is countable in M). Then $M[g_0]$ satisfies (enough of) ZFC and $\mathcal{E}$ is a collection of extenders still witnessing the Wodinness of $\bar{\delta}$ in $M[g_0]$. Hence, in what follows we will write M for $M[g_0]$.

Recall the definition of Woodin's extender algebra on M corresponding to $\mathcal{E}$ with $\bar{\delta}$ generators, which we will refer to by $\mathcal{W}_{\bar{\delta},\bar{\delta}}(\mathcal{E})$, or simply $\mathcal{W}$: $\mathcal{W}_{\bar{\delta},\bar{\delta}}(\mathcal{E})$ is the quotient Boolean algebra $(\mathcal{B}_{\bar{\delta},\bar{\delta}}/\mathcal{T}_{\bar{\delta},\bar{\delta}}(\mathcal{E}))^M$, where $\mathcal{B}_{\bar{\delta},\bar{\delta}}$ is the propositional algebra of $\mathcal{L}_{\bar{\delta},\bar{\delta}}$-formulas (i.e., the infinitary formulas obtained from variables a_ξ, for $\xi < \bar{\delta}$, by closing under the usual propositional connectives, together with infinite conjunctions $\bigwedge_{\xi<\kappa} \phi_\xi$ and disjunctions $\bigvee_{\xi<\kappa} \phi_\xi$ for $\kappa < \bar{\delta}$), and $\mathcal{T}_{\bar{\delta},\bar{\delta}}(\mathcal{E})$ is the deductive closure in $\mathcal{L}_{\bar{\delta},\bar{\delta}}$ of all sentences

$$\Psi(\vec{\phi}, \kappa, \eta) : \bigvee_{\xi<\kappa} \phi_\xi \leftrightarrow \bigvee_{\xi<\eta} \phi_\xi,$$

for measurable cardinals $\kappa < \eta < \bar{\delta}$, a sequence $\vec{\phi} = (\phi_\xi : \xi < \bar{\delta})$ of $\mathcal{L}_{\bar{\delta},\bar{\delta}}$-formulas with $\phi_\xi \in V_\kappa$ for all $\xi < \kappa$, and a $(\vec{\phi}, \eta + 2)$-strong extender $E \in \mathcal{E}$ such that $\mathrm{crit}(E) = \kappa$ and such that E has length η^*, where η^* is the least inaccessible above η. In M, $\mathcal{W}$ has the $\bar{\delta}$-c.c.[2]

Let $G \in V$ be a $\mathbb{P}$-generic filter over N (which exists since N is countable). For the remainder of the proof we will be working mostly in $N[G]$.

Let $\tau = |V_\alpha|^N$ and let $a \in N[G]$ be a subset of τ coding $V_\alpha^{N[G]}$. Let $H \in V$ be a $\mathrm{Coll}(\omega, \tau)$-generic filter over $N[G]$. Working in $N[G]$, we will build a certain normal iteration tree $\mathcal{T}$ on $(M, \in, \bar{\delta})$ of length $\bar{\tau}$, for some $\bar{\tau} < (\tau^+)^N$, together with a sequence $(\rho_\alpha : \alpha < \bar{\tau})$ of ordinals witnessing its normality. The construction will be arranged in such a way that the following holds.

(1) For every $\alpha < \bar{\tau}$, $M_\alpha^{\mathcal{T}}$ is correct about sharps of sets in $V_{j_{0,\alpha}^{\mathcal{T}}(\bar{\delta})} \cap M_\alpha^{\mathcal{T}}$.

[2]See e.g. [1]. $\mathcal{W}_{\bar{\delta},\bar{\delta}}(\mathcal{E})$ is actually a mild variant of the original extender algebra. We refer to [1] for the relevant facts on the theory of $\mathcal{W}_{\bar{\delta},\bar{\delta}}(\mathcal{E})$ (whose proof is the same as for the original extender algebra).

(2) For every nonzero limit ordinal $\gamma < \bar{\tau}$, $j^{\mathcal{T}}_{0,\gamma}(\bar{\delta})$ is the minimum ordinal μ with the property that there is, in V, a cofinal well-founded branch b through $\mathcal{T} \restriction \gamma$ such that
   (a) $j^{\mathcal{T}}_{0,b}(\bar{\delta}) = \mu$ and
   (b) $M^{\mathcal{T}}_b$ is correct about sharps of sets in $V_{j^{\mathcal{T}}_{0,b}(\bar{\delta})} \cap M^{\mathcal{T}}_b$.

(3) $\sup\{\rho^{\mathcal{T}}(\alpha,\gamma) : \alpha < \gamma\} < j^{\mathcal{T}}_{0,\gamma}(\bar{\delta})$ for every limit ordinal γ such that $\gamma + 1 < \bar{\tau}$.

If $\bar{\tau} = \gamma_0 + 1$, we will get a $j^{\mathcal{T}}_{0,\gamma_0}(\mathcal{W})$-generic filter $g \in N[G]$ over $M^{\mathcal{T}}_{\gamma_0}$ such that $a \in M^{\mathcal{T}}_{\gamma_0}[g]$, which will yield the desired conclusion since then $V_\alpha^{M^{\mathcal{T}}_{\gamma_0}[g]} = V_\alpha^{N[G]}$ as $a \in N[G]$ codes $V_\alpha^{N[G]}$ and $M^{\mathcal{T}}_{\gamma_0} \in N[G]$. If $\bar{\tau}$ is a limit ordinal, we will obtain a cofinal branch $c \in N[G]$ through $\mathcal{T}$ such that $M^{\mathcal{T}}_c$ is well-founded up to $j^{\mathcal{T}}_{0,c}(\bar{\delta})$, together with a $j^{\mathcal{T}}_{0,c}(\mathcal{W})$-generic filter $g \in N[G]$ over $M^{\mathcal{T}}_c$ such that $a \in M^{\mathcal{T}}_c[g]$. This will again yield the desired conclusion for the same reason as in the previous case.

We start out by iterating linearly in length τ. From stage τ onwards, the construction proceeds as follows. Let $\gamma < (\tau^+)^N$, $\gamma \geq \tau$, and suppose $\mathcal{T} \restriction \gamma$ has been defined.

If $\gamma = \gamma_0 + 1$, then $\mathcal{T} \restriction \gamma$ is given by the following specification.

Suppose there is some extender $E \in j^{\mathcal{T}}_{0,\gamma_0}(\mathcal{E})$ which, in $M^{\mathcal{T}}_{\gamma_0}$, is a witness to the existence of some $\Psi(\vec{\phi}, \kappa, \eta) \in j^{\mathcal{T}}_{0,\gamma_0}(\mathcal{T}_{\bar{\delta},\bar{\delta}}(\mathcal{E}))$ such that $a \not\models \Psi(\vec{\phi}, \kappa, \eta)$ and $\eta > \rho_{\bar{\gamma}}$ for all $\bar{\gamma} < \gamma_0$. Let $\mathcal{F}$ be the set of all extenders $E \in M^{\mathcal{T}}_{\gamma_0}$ as above with η minimal and let ρ_{γ_0} be that minimal value of η. Note that all extenders in $\mathcal{F}$ have strength, in $M^{\mathcal{T}}_{\gamma_0}$, at least $\eta + 2$. We then pick E_{γ_0} to be a member of $\mathcal{F}$ of minimal Mitchell rank in $M^{\mathcal{T}}_{\gamma_0}$, which is possible as the Michell order on the class of short extenders is well-founded (s. [6]). We also extend $\mathcal{T} \restriction \gamma$ to a tree order on $\gamma + 1$ by setting the $\mathcal{T}$-predecessor of γ to be the least $\bar{\gamma}$ with $\mathrm{crit}(E_{\gamma_0}) \leq \rho_{\bar{\gamma}}$. We then have, thanks to Theorem 2.3 (2), that $M^{\mathcal{T}}_\gamma = \mathrm{Ult}(M^{\mathcal{T}}_{\bar{\gamma}}, E_{\gamma_0})$ is well-founded and correct about sharps of sets in $V_{i(j^{\mathcal{T}}_{0,\bar{\gamma}}(\bar{\delta}))} \cap \mathrm{Ult}(M^{\mathcal{T}}_{\bar{\gamma}}, E_{\gamma_0})$, where $i : M^{\mathcal{T}}_{\bar{\gamma}} \longrightarrow \mathrm{Ult}(M^{\mathcal{T}}_{\bar{\gamma}}, E_{\gamma_0})$ is the canonical extender embedding, so we preserve condition (1) of our construction.

If there is no E as above, then we set $\bar{\tau} = \gamma$ and stop the construction.

Now suppose γ is a limit ordinal.

Claim 2.4. *There is a cofinal π-realizable branch through $\mathcal{T} \restriction \gamma$.*

Proof. This is essentially the proof of Corollary 5.11 from [3]. If the conclusion fails, then by Theorem 2.3 (1) there is a maximal branch b through $\mathcal{T} \restriction \gamma$ such that $\lambda := \sup(b) < \gamma$ and b is π-realizable. In particular, $M^{\mathcal{T}}_b$ is correct about sharps of sets in $V_{j^{\mathcal{T}}_{0,b}(\bar{\delta})} \cap M^{\mathcal{T}}_b$. Let

$\mathcal{T}' = \mathcal{T} \restriction \lambda$ and let $c = \{\alpha < \lambda : \alpha <_{\mathcal{T}} \lambda\}$. Since b is a maximal branch through $\mathcal{T} \restriction \gamma$, $b \neq c$. Let

$$\theta = \sup\{\rho^{\mathcal{T}}(\alpha,\lambda) : \alpha < \lambda\} < j^{\mathcal{T}}_{0,c}(\bar{\delta}) \leq j^{\mathcal{T}}_{0,b}(\bar{\delta}),$$

where the first inequality holds by condition (3) in the construction since it did not stop at stage $\lambda+1$ and the second inequality follows from condition (2) in the construction.

By Lemma 2.2 we know that for every function $f : \theta \longrightarrow \theta$, if $f \in \mathcal{M}^{\mathcal{T}}_b \cap \mathcal{M}^{\mathcal{T}}_c$, then $\mathcal{M}^{\mathcal{T}}_b \models$ "θ is Woodin with respect to f". In order to finish the proof it suffices to show that θ is Woodin in $L(V_\theta)^{M^{\mathcal{T}}_c}$ (this of course yields a contradiction since it holds in $M^{\mathcal{T}}_c$ that $j^{\mathcal{T}}_{0,c}(\bar{\delta})$ is the least ordinal μ such that μ is Woodin in $L(V_\mu)$). The Woodinness of θ in $L(V_\theta)^{M^{\mathcal{T}}_c}$ will be established if we show that $(X^\sharp)^{M^{\mathcal{T}}_b} = (X^\sharp)^{M^{\mathcal{T}}_c}$, where $X = V_\theta^{M^{\mathcal{T}}_b} = V_\theta^{M^{\mathcal{T}}_c} \in M^{\mathcal{T}}_b \cap M^{\mathcal{T}}_c$.[3] But $(X^\sharp)^{M^{\mathcal{T}}_b} = X^\sharp = (X^\sharp)^{M^{\mathcal{T}}_c}$ since $M^{\mathcal{T}}_b$ and $M^{\mathcal{T}}_c$ are both correct about the sharp of X. □

Let μ be minimal such that, in V, there is a cofinal well-founded branch b through $\mathcal{T} \restriction \gamma$ such that $j^{\mathcal{T}}_{0,b}(\bar{\delta}) = \mu$ and such that $M^{\mathcal{T}}_b$ is correct about sharps of sets in $V_{j^{\mathcal{T}}_{0,b}(\bar{\delta})} \cap M^{\mathcal{T}}_b$. Using the $\underset{\sim}{\Sigma}^1_3$-correctness in V of $N[G][H]$, we have that in $N[G][H]$ there is a cofinal well-founded branch b trough $\mathcal{T} \restriction \gamma$ such that $j^{\mathcal{T}}_{0,b}(\bar{\delta}) = \mu$ and such that $M^{\mathcal{T}}_b$ is correct about sharps of sets in $V_{j^{\mathcal{T}}_{0,b}(\bar{\delta})} \cap M^{\mathcal{T}}_b$.

If $\sup\{\rho^{\mathcal{T}}(\alpha,\gamma) : \alpha < \gamma\} = \mu = j^{\mathcal{T}}_{0,b}(\bar{\delta})$, then the construction of $\mathcal{T}$ stops and we set $\bar{\tau} = \gamma$.

Now suppose that $\theta := \sup\{\rho^{\mathcal{T}}(\alpha,\gamma) : \alpha < \gamma\} < j^{\mathcal{T}}_{0,b}(\bar{\delta})$.

Claim 2.5. *In $N[G][H]$ there is exactly one cofinal well-founded branch b through $\mathcal{T} \restriction \gamma$ such that $j^{\mathcal{T}}_{0,b}(\bar{\delta}) = \mu$ and such that $M^{\mathcal{T}}_b$ is correct about sharps of sets in $V_\mu^{M^{\mathcal{T}}_b}$.*

Proof. Assume, towards a contradiction, that in $N[G][H]$ there are two distinct cofinal well-founded branches b_0 and b_1 through $\mathcal{T} \restriction \gamma$ such that $j^{\mathcal{T}}_{0,b_0}(\bar{\delta}) = j^{\mathcal{T}}_{0,b_1}(\bar{\delta}) = \mu$ and such that for each i, $M^{\mathcal{T}}_{b_i}$ is correct about sharps of sets in $V_\mu^{M^{\mathcal{T}}_{b_i}}$. Since $\theta < \mu$, by Lemma 2.2 we have that θ is Woodin with respect to f for every function $f : \theta \longrightarrow \theta$ in $M^{\mathcal{T}}_{b_0} \cap M^{\mathcal{T}}_{b_1}$. As in the proof of Claim 2.4, and using the correctness about the sharp of $V_\theta^{M^{\mathcal{T}}_{b_0}} = V_\theta^{M^{\mathcal{T}}_{b_1}}$ of both $M^{\mathcal{T}}_{b_0}$ and $M^{\mathcal{T}}_{b_1}$, it follows that θ is Woodin in $L(V_\theta)^{M^{\mathcal{T}}_{b_0}}$. But this is a contradiction since $\mu > \theta$ is the least ordinal $\mu^* \in M^{\mathcal{T}}_{b_0}$ which is Woodin in $L(V_{\mu^*})^{M^{\mathcal{T}}_{b_0}}$. □

[3] $V_\theta^{M^{\mathcal{T}}_b} = V_\theta^{M^{\mathcal{T}}_c}$ follows from the definition of θ as $\sup\{\rho^{\mathcal{T}}(\alpha,\lambda) : \alpha < \lambda\}$.

By the homogeneity of $\text{Coll}(\omega, \tau)$, the unique branch b given by Claim 2.5 is an actual member of $N[G]$. We then extend $\mathcal{T} \restriction \gamma$ to an iteration tree of length $\gamma + 1$ by letting $\alpha <_{\mathcal{T}} \gamma$ if and only if $\alpha \in b$.

A standard reflection argument shows that the construction cannot run in length $(\tau^+)^N + 1$ (s. for example the proofs of Lemma 3.7 and Theorem 4.1 in [1]). Hence $\bar{\tau}$ exists and is at most $(\tau^+)^N$.

Suppose first that $\bar{\tau}$ is a successor ordinal, $\bar{\tau} = \gamma_0 + 1$. Let us see that, letting $\delta^* = j^{\mathcal{T}}_{0,\gamma_0}(\bar{\delta})$,

$$g = \{\phi \in \mathcal{L}_{\delta^*,\delta^*} \cap M^{\mathcal{T}}_{\gamma_0} \,:\, a \models \phi\}$$

is a $\mathcal{W}_{\delta^*,\delta^*}(j^{\mathcal{T}}_{0,\gamma_0}(\mathcal{E}))$-generic filter over $M^{\mathcal{T}}_{\gamma_0}$. That will finish the proof of the theorem in this case as then of course $a \in M^{\mathcal{T}}_{\gamma_0}[g]$.

Assuming otherwise, by the general theory of the extender algebra, there is some extender $E \in j^{\mathcal{T}}_{0,\gamma_0}(\mathcal{E})$ which, in $M^{\mathcal{T}}_{\gamma_0}$, is a witness to the existence of some $\Psi(\vec{\phi}, \kappa, \eta_0) \in j^{\mathcal{T}}_{0,\gamma_0}(\mathcal{T}_{\bar{\delta},\bar{\delta}}(\mathcal{E}))$ such that $a \not\models \Psi(\vec{\phi}, \kappa, \eta_0)$ (s. [1]).

Claim 2.6. *$\eta_0 > \rho_\gamma$ for all $\gamma < \gamma_0$.*

Proof. Let us assume, towards a contradiction, that this is not the case. Let us suppose first that there is some $\gamma < \gamma_0$ such that $\eta_0 < \rho_\gamma$. We then have that $E \in M^{\mathcal{T}}_\gamma$ since $\eta^*_0 < \rho_\gamma < \rho^*_\gamma$ and since $M^{\mathcal{T}}_\gamma$ and $M^{\mathcal{T}}_{\gamma_0}$ agree below ρ^*_γ. But this contradicts the minimality in the choice of ρ_γ at stage $\gamma + 1$ of the construction.

Since $\eta_0 \leq \rho_\gamma$ for some γ, $(\rho_\gamma \,:\, \gamma < \gamma_0)$ is strictly increasing, and there is no γ such that $\eta_0 < \rho_\gamma$, it follows that $\gamma_0 = \bar{\gamma}_0 + 1$ and $\eta_0 = \rho_{\bar{\gamma}_0}$. Let $\bar{\gamma}$ be the $\mathcal{T}$-predecessor of γ_0, so that $M^{\mathcal{T}}_{\gamma_0} = \text{Ult}(M^{\mathcal{T}}_{\bar{\gamma}}, E_{\bar{\gamma}_0})$. But $\text{Ult}(M^{\mathcal{T}}_{\bar{\gamma}}, E_{\bar{\gamma}_0})$ and $\text{Ult}(M^{\mathcal{T}}_{\bar{\gamma}_0}, E_{\bar{\gamma}_0})$ agree below $i(\text{crit}(E_{\bar{\gamma}_0})) + 1 > \eta^*_0 + 2$, where $i \,:\, M^{\mathcal{T}}_{\bar{\gamma}} \longrightarrow \text{Ult}(M^{\mathcal{T}}_{\bar{\gamma}}, E_{\bar{\gamma}_0})$ is the canonical extender embedding (since necessarily $i(\text{crit}(E_{\bar{\gamma}_0})) > \eta^*_0 + 1$). In particular $E \in \text{Ult}(M^{\mathcal{T}}_{\bar{\gamma}_0}, E_{\bar{\gamma}_0})$, which violates the minimality of $E_{\bar{\gamma}_0}$ in the Mitchell order. □

But now, by the claim, we are in a position to extend $\mathcal{T}$ one more step as given by the successor step of the construction, which contradicts the fact that the construction already stopped.

It remains to consider the case that $\bar{\tau}$ is a limit ordinal. In this case, we know in particular that in $N[G][H]$ there is a cofinal well-founded branch b through $\mathcal{T}$ such that $\sup\{\rho^{\mathcal{T}}(\beta, \bar{\tau}) \,:\, \beta < \bar{\tau}\} = \mu$ for $\mu = j^{\mathcal{T}}_{0,b}(\bar{\delta})$. Let $\dot{b} \in N[G]$ be a $\text{Coll}(\omega, \tau)$-name for b and let $Q \in N[G]$ be a countable elementary submodel of some large enough $H^{N[G]}_\theta$ such that $\dot{b} \in Q$. Let $h \subseteq Q$, $h \in N[G]$, be a $\mathrm{Coll}(\omega, \tau)^Q$-generic filter, and let $b^* = \dot{b}_h$. Let $\alpha = \sup(Q \cap \bar{\tau})$.

Claim 2.7. $\alpha = \bar{\tau}$

Proof. Suppose, towards a contradiction, that $\alpha < \bar{\tau}$.[4] We will prove that $\sup\{\rho^{\mathcal{T}}(\beta, \alpha) : \beta < \alpha\} = j^{\mathcal{T}}_{0,\alpha}(\bar{\delta})$, which is a contradiction as then the construction has stopped at stage α.

We note that $(j^{\mathcal{T}}_{0,\beta}(\bar{\delta}) : \beta \in b^*)$ is not eventually constant. It follows that

$$\sup(j^{\mathcal{T}}_{\beta,b^*}``j^{\mathcal{T}}_{0,\beta}(\bar{\delta})) < j^{\mathcal{T}}_{0,b^*}(\bar{\delta})$$

for every $\beta \in b^*$. Let us fix $\beta \in b^* \cap Q$. There is then some $\gamma \in \alpha \cap Q$ above β such that

$$\sup(j^{\mathcal{T}}_{\beta,b^*}``j^{\mathcal{T}}_{0,\beta}(\bar{\delta})) < \rho^{\mathcal{T}}(\gamma, \alpha) = \rho^{\mathcal{T}}(\gamma, \bar{\tau}) \in Q,$$

where the equality holds by the fact that $\rho^{\mathcal{T}}(\gamma, \tau_1) \leq \rho^{\mathcal{T}}(\gamma, \tau_0)$ for all $\tau_0 < \tau_1 \leq \bar{\tau}$, the correctness of Q, and the fact that $\gamma \in Q$. We then of course have that also

$$\sup(j^{\mathcal{T}}_{\beta,\alpha}``j^{\mathcal{T}}_{0,\beta}(\bar{\delta})) \leq \sup(j^{\mathcal{T}}_{\beta,b^*}``j^{\mathcal{T}}_{0,\beta}(\bar{\delta})) < \rho^{\mathcal{T}}(\gamma, \alpha)$$

Since

$$j^{\mathcal{T}}_{0,\alpha}(\bar{\delta}) = \sup\{\sup(j^{\mathcal{T}}_{\beta,\alpha}``j^{\mathcal{T}}_{0,\beta}(\bar{\delta})) : \beta \in b^* \cap Q\}$$

and $\rho^{\mathcal{T}}(\gamma, \alpha) < j^{\mathcal{T}}_{0,\alpha}(\bar{\delta})$ for all $\gamma < \alpha$, it follows that

$$j^{\mathcal{T}}_{0,\alpha}(\bar{\delta}) = \sup\{\rho^{\mathcal{T}}(\beta, \alpha) : \beta < \alpha\}$$

□

By the same argument as in the proof of Claim 2.7, it follows that $\sup\{\rho^{\mathcal{T}}(\beta, \bar{\tau}) : \beta < \tau\} = j^{\mathcal{T}}_{0,b^*}(\bar{\delta})$. We note that $M^{\mathcal{T}}_{b^*}$ is well-founded up to $j^{\mathcal{T}}_{0,b^*}(\bar{\delta})$. Since

$$\sup\{\rho^{\mathcal{T}}(\beta, \bar{\tau}) : \beta < \bar{\tau}\} = \mu = j^{\mathcal{T}}_{0,b^*}(\bar{\delta}),$$

by the same argument as in the previous case we have that

$$g = \{\phi \in \mathcal{L}_{\mu,\mu} \cap M^{\mathcal{T}}_{b^*} : a \models \phi\}$$

is a $j^{\mathcal{T}}_{0,b^*}(\mathcal{W})$-generic filter over $M^{\mathcal{T}}_{b^*}$: otherwise there is some extender $E \in j^{\mathcal{T}}_{0,b^*}(\mathcal{E})$ which is a witness to the existence of some $\Psi(\vec{\phi}, \kappa, \eta) \in j^{\mathcal{T}}_{0,b^*}(\mathcal{T}_{\bar{\delta},\bar{\delta}}(\mathcal{E}))$ such that $a \not\models \Psi(\vec{\phi}, \kappa, \eta)$; but $\eta > \rho_\gamma$ for all $\gamma < \bar{\tau}$ by the same argument as in the proof of Claim 2.6, which is impossible as then $\eta \geq \mu$ whereas $E \in V_\mu^{M^{\mathcal{T}}_{b^*}}$. This finishes the proof in this case, and hence the proof of the theorem, since $a \in M^{\mathcal{T}}_{b^*}[g]$.

[4] Equivalently, $\mathrm{cf}(\bar{\tau})^{N[G]} > \omega$.

3. A local form of Ω-logic

Corollary 1.3 motivates a local version of Woodin's Ω-logic ([7]) for which we can prove a reasonable completeness theorem.[5]

Definition 3.1. Let W and M we models of set theory.

(1) M is a 1-*step local forcing extension of* W in case there is some ordinal $\delta \in W$ such that M is a set-forcing extension of $L(V_\delta)^W$.
(2) Given $n \geq 1$, M is a $n+1$-*step local forcing extension of* W in case there is an n-step local forcing extension M_0 of W and there is an ordinal $\delta \in M_0$ such that M is a 1-step local forcing extension of M_0.

M is an *iterated local forcing extension of* W if there is some $n \geq 1$ such that M is an n-step local forcing extension of W.

Our local version of Ω-logic is the following.

Definition 3.2. Given a set T of sentences in the language of set theory and a sentence σ in the language of set theory, we write $T \models_{\Omega^\ell} \sigma$ in case for every iterated local forcing extension M of V and every ordinal α, if $V_\alpha^M \models T$, then $V_\alpha^M \models \sigma$.[6]

Thus, $\models_{\Omega^\ell}$ is a weak version of Ω-logic. We refer to $\models_{\Omega^\ell}$ as *local* Ω-*logic*.

A simple variation of the proofs of Theorem 1.1 and Corollary 1.3 establishes the following.

Theorem 3.3. *Suppose there is a proper class of Woodin cardinals. Let σ be a sentence. Then the following are equivalent.*

(1) $\emptyset \models_{\Omega^\ell} \sigma$
(2) *Suppose δ is an ordinal such that δ is Woodin in $L(V_\delta)$ and the set of $L(V_\delta)$-Woodin cardinals $\gamma < \delta$ is bounded in δ. Then* $L(V_\delta) \models$ *"* $\emptyset \models_{\Omega^\ell} \sigma$*".*
(3) *There is a real r such that for every countable transitive model N of* ZFC*, if $r \in N$ and $N[H] \prec_{\Sigma^1_3} V$ for every set-generic filter $H \in V$ over N, then N models "* $\emptyset \models_{\Omega^\ell} \sigma$*".*

The equivalence between (1) and (3) can be seen as a completeness theorem for local Ω-logic in the spirit of the Ω-conjecture for the original

[5]We recall that the Ω-conjecture is the completeness theorem for Ω-logic relative to the calculus in the definition of $\vdash_\Omega$ in terms of A-closed models M for fixed universally Baire sets $A \subseteq \mathbb{R}$ ([7]).

[6]The 'ℓ' superscript in Ω^ℓ is for 'local'.

Ω-logic. This equivalence also yields the following corollary on the complexity of Ω^ℓ-validity.

Corollary 3.4. *Suppose there is a proper class of Woodin cardinals. Then* $\{\sharp\sigma : \emptyset \models_{\Omega^\ell} \sigma\}$ *is a* Σ^1_5*-definable real.*

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