Maximality Principle under a Laver-generic supercompact cardinal

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Abstract

We give a survey on the set-theoretic axioms formulated in terms of existence of a Laver-generic large cardinal.

We show that the Maximality Principle without parameters is independent over ZFC with the axiom asserting the existence of a $\mathcal{P}$-Laver generically supercompact cardinal for an iterable class of posets $\mathcal{P}$ as far as the existence of such a cardinal can be forced naturally starting from a genuine supercompact cardinal.

1 Introduction

In sections 2, 3 of the present note, we give a survey on the axioms formulated in terms of existence of a Laver-generic large cardinal. Most of the results are from [15] and [16] but there are also small improvements and a couple of new results. The extended version of the paper is going to include detailed proofs.

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In Section 5, we prove that the Maximality Principle without parameters is independent over ZFC with the axiom asserting the existence of a $\mathcal{P}$-Laver generically supercompact cardinal for some iterable class $\mathcal{P}$ of posets.

2 Generic large cardinals

Let us begin with recalling the definition of supercompact cardinal: A cardinal κ is *supercompact* if, for any $\lambda > \kappa$, there are classes j, M such that ① $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, ② $j(\kappa) > \lambda$ and ③ ${}^\lambda M \subseteq M$.

Here, "$j : N \overset{\prec}{\rightarrow}_\kappa M$" denotes the set of conditions that N and M are transitive (sets or classes); j is a non-trivial elementary embedding of the structure $(N, \in)$ into the structure $(M, \in)$; $\kappa \in N$, and $crit(j) = \kappa$.

Note that a supercompact cardinal is a large large cardinal which is a normal measure one limit of measurable cardinals, and more. This is not the case with the generic large cardinal version of the notion of supercompactness (e.g. see Examples 2.1, 2.2 below).

For a class $\mathcal{P}$ of posets, a cardinal κ is *$\mathcal{P}$-generically supercompact* (*$\mathcal{P}$-gen. supercompact*, for short) if, for every $\lambda > \kappa$, there is $\mathbb{P} \in \mathcal{P}$ such that, for $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that ① $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, ② $j(\kappa) > \lambda$, and ③' $j''\lambda \in M$.

Example 2.1 *Suppose κ is a supercompact cardinal and $\mathcal{P} = \mathrm{Col}(\aleph_1, \kappa)$ (the standard collapsing of all cardinals strictly between $\aleph_1$ and κ by countable conditions). Then for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\kappa = (\aleph_2)^{\mathsf{V}[\mathbb{G}]}$ and $\mathsf{V}[\mathbb{G}] \models$ "κ is σ-*closed-gen. supercompact*". ❏

Example 2.2 *If MA is forced starting from an supercompact cardinal κ with an ccc-iteration of length κ in finite support along with a supercompact Laver-function, then we obtain a model in which κ is the continuum (though still quite large, e.g. hyper-hyper etc. weakly Mahlo, and more) and it is ccc-gen. supercompact in the generic extension.* ❏

These examples are going to be revisited in Theorem 3.3 below. The situation created in Example 2.1 can be also seen as a strong reflection property.

Theorem 2.3 (B. König [27]) *The following are equivalent:*

(a) *Game Reflection Principle (*GRP*) holds.*

(b) $\aleph_2$ *is σ-closed-gen. supercompact.* ❏

As in [15], what we call the Game Reflection Principle (GRP) is the principle called GRP^+ in [27]. As its name suggests, GRP is actually a reflection statement about the non-existence of winning strategy of certain games of length ω_1 down to subgames of size $< \aleph_2$.

We will not go into the details of the definition of GRP but just note that GRP implies the Continuum Hypothesis (CH) and it implies practically all reflection principles with reflection down to $< \aleph_2$ available under CH:

(2.1) GRP implies Rado's Conjecture (RC) (König, [27]).

(2.2) GRP implies strong downward Löwenheim-Skolem Theorem of $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$ down to $< \aleph_2$ ($\mathsf{SDLS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}, < \aleph_2)$ in the notation of [15]).

(2.3) RC and $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}, < \aleph_2)$ imply Fodor-type Reflection Principle (FRP), see [6].

(2.4) FRP is known to be equivalent to many "mathematical" reflection principles with reflection down to $< \aleph_2$, see [4], [5], [8], [14], [22].

(2.5) GRP implies a "generic" solution to the Hamburger's problem (see Corollary 2.5 below, for the original Hamburger's Problem see [17] and reference given there).

These and some other implications are put together in the following diagram:

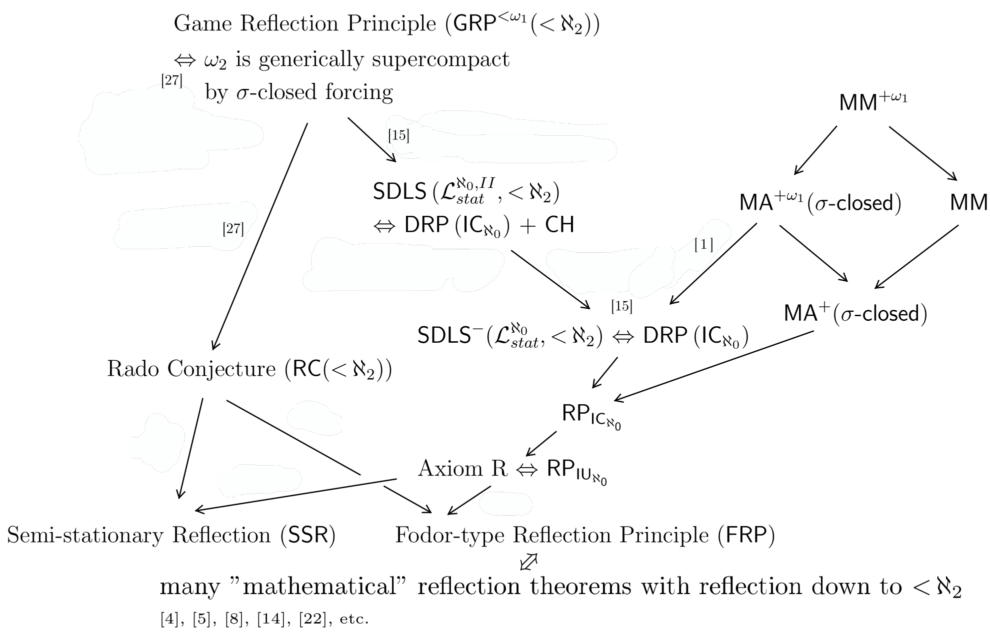

Figure 1.

Proposition 2.4 *Suppose that κ is $\mathcal{P}$-gen. supercompact. Then, for any topological space X of character $< \kappa$, if $\Vdash_{\mathbb{P}}$ "X is not metrizable" for any $\mathbb{P} \in \mathcal{P}$ then there is a non metrizable subspace Y of X of cardinality $< \kappa$.* ❑

Corollary 2.5 *Suppose that* GRP *holds. Then, for any topological space X of character $\leq \aleph_1$ such that $\Vdash_{\mathbb{P}}$ "X is not metrizable" for any σ-closed poset $\mathbb{P}$, there is a non-metrizable subspace of X of cardinality $\leq \aleph_1$.*

Proof. By Theorem 2.3 and Proposition 2.4. □ (Corollary 2.5)

3 Laver-generic large cardinals

The axioms claiming the existence of Laver-generic large cardinals defined below for respective classes of posets complete the picture of reflection and absoluteness in terms of double plused versions of forcing axioms.

A (definable) class $\mathcal{P}$ of posets is said to be *iterable* if (a) $\mathcal{P}$ is closed with respect to forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$), (b) closed with respect to restriction (i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{p} \in \mathbb{P}$), and (c) for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

For an iterable class $\mathcal{P}$ of posets, a cardinal κ is said to be *$\mathcal{P}$-Laver-gen. supercompact* [1)] if, for any $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$, and ③' $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H}$, $j''\lambda \in M$.

Recall that a cardinal κ is *superhuge* (*super-almost-huge* resp.) if, for any $\lambda > \kappa$, there are classes j, M such that ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$ and ③ ${}^{j(\kappa)}M \subseteq M$ (${}^{j(\kappa)>}M \subseteq M$ resp.).

These notions of large cardinals can be straightforwardly translated into their Laver-generic versions: For an iterable class $\mathcal{P}$ of posets, κ is *$\mathcal{P}$-Laver-gen. superhuge* (*$\mathcal{P}$-Laver-gen. super-almost-huge*, resp.) if, for any $\lambda \geq \kappa$, $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$, and ③' $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, and $j''j(\kappa) \in M$ ($j''\mu \in M$ for all $\mu < j(\kappa)$, resp.).

Sometimes it is more convenient to consider the following additional property which we called the tightness of Laver-genericity: For an iterable $\mathcal{P}$, a $\mathcal{P}$-Laver-gen. supercompact cardinal ($\mathcal{P}$-Laver-gen. huge cardinal, etc., resp.) is *tightly $\mathcal{P}$-Laver-gen. supercompact* (*tightly $\mathcal{P}$-Laver-gen. huge*, etc., resp.) if the condition

④ $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of cardinality $\leq j(\kappa)$.

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1) The definition of Laver-generic large cardinals given here is slightly stronger than the one given in [16]. The Laver-generic large cardinals in the sense of present subsection is called *strongly* Laver-generic large cardinals in [16].

additionally holds for the elementary embedding j in the definition.

The strongest notion of large cardinal we are studying in connection with its Laver-generic version at the moment is that of ultrahuge cardinal introduced by Tsaprounis [30]. A cardinal κ is *ultrahuge* if for any $\lambda > \kappa$ there is $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ such that $j(\kappa) > \lambda$ and ${}^{j(\kappa)}M, V_{j(\lambda)} \subseteq M$. In terms of consistency strength ultrahuge cardinal is placed between super huge and 2-almost-huge (Theorem 3.4 in [30]).

For an iterable class $\mathcal{P}$ of posets, a cardinal κ is (*tightly*) *$\mathcal{P}$-Laver gen. ultrahuge*, if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, (V_{j(\lambda)})^{V[\mathbb{H}]} \in M$ (and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of size $j(\kappa)$).

By definition it is obvious that we have the following implications:

tightly $\mathcal{P}$-Laver-gen. ultrahuge	$\Rightarrow$	tightly $\mathcal{P}$-Laver-gen. superhuge	$\Rightarrow$	tightly $\mathcal{P}$-Laver-gen. super-almost-huge	$\Rightarrow$	tightly $\mathcal{P}$-Laver-gen. supercompact	$\Rightarrow$	tightly $\mathcal{P}$-Laver-gen. measurable
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$\mathcal{P}$-Laver-gen. ultrahuge	$\Rightarrow$	$\mathcal{P}$-Laver-gen. superhuge	$\Rightarrow$	$\mathcal{P}$-Laver-gen. super-almost-huge	$\Rightarrow$	$\mathcal{P}$-Laver-gen. supercompact	$\Rightarrow$	$\mathcal{P}$-Laver-gen. measurable
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$\mathcal{P}$-gen. ultrahuge	$\Rightarrow$	$\mathcal{P}$-gen. superhuge	$\Rightarrow$	$\mathcal{P}$-gen. super-almost-huge	$\Rightarrow$	$\mathcal{P}$-gen. supercompact	$\Rightarrow$	$\mathcal{P}$-gen. measurable

Figure 2.

Some of the horizontal implications should be irreversible. At the moment however we can only prove the reversibility of the implication from (tightly) $\mathcal{P}$-(Laver)-gen. ultrahugeness to (tightly) $\mathcal{P}$-(Laver)-gen. supercompactness.

Proposition 3.1 *Suppose that $\mathbb{P}$ is a class of posets such that there is a construction of a model with a tightly $\mathcal{P}$-Laver gen. supercompact cardinal starting from an arbitrary model with an supercompact cardinal κ by a poset of cardinality κ.*[2)] *Then tightly $\mathcal{P}$-Laver gen. supercompactness of κ does not necessarily imply the $\mathcal{P}$-gen. super-almost-hugeness.*

For the proof of Proposition 3.1 we use the following observation:

Lemma 3.2 *Suppose that κ is $\mathcal{P}$-gen. ultrahuge for an arbitrary class $\mathcal{P}$ of posets. If there is an inaccessible $\lambda_0 > \kappa$ then there are cofinally many inaccessible in* V.

Proof. Let $\lambda > \lambda_0$ be an arbitrary cardinal. Then there is $\mathbb{P} \in \mathcal{P}$ such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are $j, M \subseteq \mathsf{V}[\mathbb{G}]$ such that

(3.1) $\quad j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $\qquad$ (3.2): $j(\kappa) > \lambda$, and

(3.3) $\quad (V_{j(\lambda)})^{\mathsf{V}[\mathbb{G}]} \in M$.

2) By the following Theorem 3.3, the class of all σ-closed posets and the class of all ccc posets satisfy this condition.

By (3.2) and elementarity (3.1), we have $j(\lambda_0) > \lambda$. By elementarity (3.1), $M \models$ "$j(\lambda_0)$ is inaccessible". By (3.3), $V[\mathbb{G}] \models$ "$j(\lambda_0)$ is inaccessible", and hence $\mathsf{V} \models$ "$j(\lambda_0)$ is inaccessible". ❑ (Lemma 3.2)

Proof of Proposition 3.1: Suppose that κ is a supercompact cardinal and $\lambda_0 > \kappa$ is an inaccessible cardinal.

We may assume that λ_0 is the largest inaccessible cardinal: If there is inaccessible cardinal larger than λ_0, then let λ_1 be the least such inaccessible cardinal. Then, in V_{λ_1}, λ_0 is the largest inaccessible cardinal and κ is supercompact (see e.g. Exercise 22.8, (a) in [26]).

Let $\mathbb{P}$ poset of size κ such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\mathsf{V}[\mathbb{G}] \models$ "κ is tightly $\mathbb{P}$-Laver gen. supercompact". Note that $\mathsf{V}[\mathbb{G}] \models$ "λ_0 is the largest inaccessible cardinal". Thus, by Lemma 3.2, it follows that $\mathsf{V}[\mathbb{G}] \models$ "κ is not $\mathcal{P}$-gen. ultrahuge".

❑ (Proposition 3.1)

Actually (tightly) Laver-generic large cardinal is first-order definable (i.e. it has a characterization formalizable in the language of ZFC), cf. [20]. Thus "Forcing Theorems" are available for arguments with Laver-genericity. Because of this and because an iterable $\mathcal{P}$ is closed under restriction to a condition, by definition, we may be lazy about the quantification on generic filters like in the context of "for a/any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ ..."

The Examples 2.1, 2.2 are actually examples of the construction of models with a Laver-generic large cardinal.

Theorem 3.3 (Theorem 5.2, [16]) (1) *Suppose κ is supercompact (superhuge, etc., resp.) and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$, for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly σ-closed-Laver-gen. supercompact (superhuge, etc., resp.) and* CH *holds.*

(2) *Suppose κ is super-almost-huge (superhuge, resp.) with a Laver-function $f : \kappa \to V_\kappa$ for super-almost-hugeness (superhugeness, resp.), and $\mathbb{P}$ is the CS-iteration for forcing* PFA *along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly proper-Laver-gen. super-almost-huge (superhuge, resp.) and $2^{\aleph_0} = \aleph_2$ holds.*[3)]

(2′) *Suppose κ is super-almost-huge (superhuge, resp.) with a Laver-function $f : \kappa \to V_\kappa$ for super-almost-hugeness (superhugeness, resp.), and $\mathbb{P}$ is the RCS-iteration for forcing* MM *along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly semi-proper-Laver-gen. super-almost-huge (superhuge, resp.) and $2^{\aleph_0} = \aleph_2$ holds.*[3)]

(3) *Suppose that κ is supercompact (superhuge, etc. resp.) with a Laver-function*

$f : \kappa \to V_\kappa$ for supercompactness (superhugeness, etc. resp.), and $\mathbb{P}$ is a FS-iteration for forcing MA *along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly ccc-Laver-gen. supercompact (superhuge, etc. resp.). $\kappa = 2^{\aleph_0}$, and κ is very large.* ❑

That three possibility of the cardinality of the continuum namely $\aleph_1$, $\aleph_2$, or very large are highlighted in Theorem 3.3, have an explanation in terms of Laver-genericity:

Theorem 3.4 (The Trichotomy Theorem [16], see also [10]) (A) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of posets such that* (a) *all $\mathbb{P} \in \mathcal{P}$ are ω_1 preserving,* (b) *all $\mathbb{P} \in \mathcal{P}$ do not add reals, and* (c) *there is a $\mathbb{P}_1 \in \mathcal{P}$ which collapses ω_2, then $\kappa = \aleph_2$ and* CH *holds.*

(B) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of posets such that* (a) *all $\mathbb{P} \in \mathcal{P}$ are ω_1-preserving,* (b′) *there is a $\mathbb{P}_0 \in \mathcal{P}$ which add a real, and* (c) *there is a $\mathbb{P}_1$ which collapses ω_2, then $\kappa = \aleph_2 \leq 2^{\aleph_0}$. If $\mathcal{P}$ contains enough many proper posets then $\kappa = \aleph_2 = 2^{\aleph_0}$ (For the last assertion see Theorem 3.5 below).*

(Γ) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of posets such that* (a′) *all $\mathbb{P} \in \mathcal{P}$ preserve cardinals, and* (b′) *there is a $\mathbb{P}_0 \in \mathcal{P}$ which adds a real, then κ is "very large" and $\kappa \leq 2^{\aleph_0}$. If κ is tightly $\mathcal{P}$-Laver-gen. superhuge then $\kappa = 2^{\aleph_0}$.* ❑

Laver-generic supercompactness also implies double plused versions of forcing axioms. For a class $\mathcal{P}$ of posets and cardinals κ, μ, let $\mathsf{MA}^{+\mu}(\mathcal{P}, <\kappa)$ and $\mathsf{MA}^{++<\mu}(\mathcal{P}, <\kappa)$ denote the following versions of Martin's Axiom:

$\mathsf{MA}^{+\mu}(\mathcal{P}, <\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of ω_1" for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.

$\mathsf{MA}^{++<\mu}(\mathcal{P}, <\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| < \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$" for some $\omega < \eta_{\underset{\sim}{S}} \leq \theta_{\underset{\sim}{S}} < \mu$ with $\eta_{\underset{\sim}{S}}$ regular, for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is stationary in $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S} \in \mathcal{S}$.

Clearly $\mathsf{MA}^{++<\omega_2}(\mathcal{P}, <\kappa)$ is equivalent to $\mathsf{MA}^{+\omega_1}(\mathcal{P}, <\kappa)$.

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3) It seems that the construction does not work with supercompact κ here.

Theorem 3.5 Theorem 5.7 in [16] (1) *For an iterable class $\mathcal{P}$ whose elements are all ccc, if κ is $\mathcal{P}$-Laver-generically supercompact, then* $\mathsf{MA}^{++<\kappa}(\mathcal{P},<\kappa)$ *holds.*

(2) *If $\aleph_2$ is Laver-generically supercompact for an iterable class $\mathcal{P}$ of posets, then* $\mathsf{MA}^{+\omega_1}(\mathcal{P})$ *holds.* ❑

Proposition 3.6 *If* ZFC + *"there are two supercompact cardinals" is consistent, then* ZFC + FRP + *"there is a tightly ccc-Laver-gen. supercompact" is consistent as well.*

Proof. Let κ_0 and κ_1 with $\kappa_0 < \kappa_1$ be two supercompact cardinals. We can use κ_0 to force $\mathsf{MA}^+(\sigma\text{-closed})$ by a poset of size κ_0. In the generic extension we have FRP and κ_2 is still supercompact. Now we use κ_1 to force that κ_1 is tightly ccc-Laver-gen. supercompact in the generic extension as described in Theorem 3.3, (3). FRP still holds in the second generic extension since FRP is preserved by ccc forcing (Theorem 3.4 in [14]). ❑ (Proposition 3.6)

These results together with some other implications proved [16] as well as some results that are going to be discussed in [13] are integrated in the diagram of Figure 1 to obtain the following:

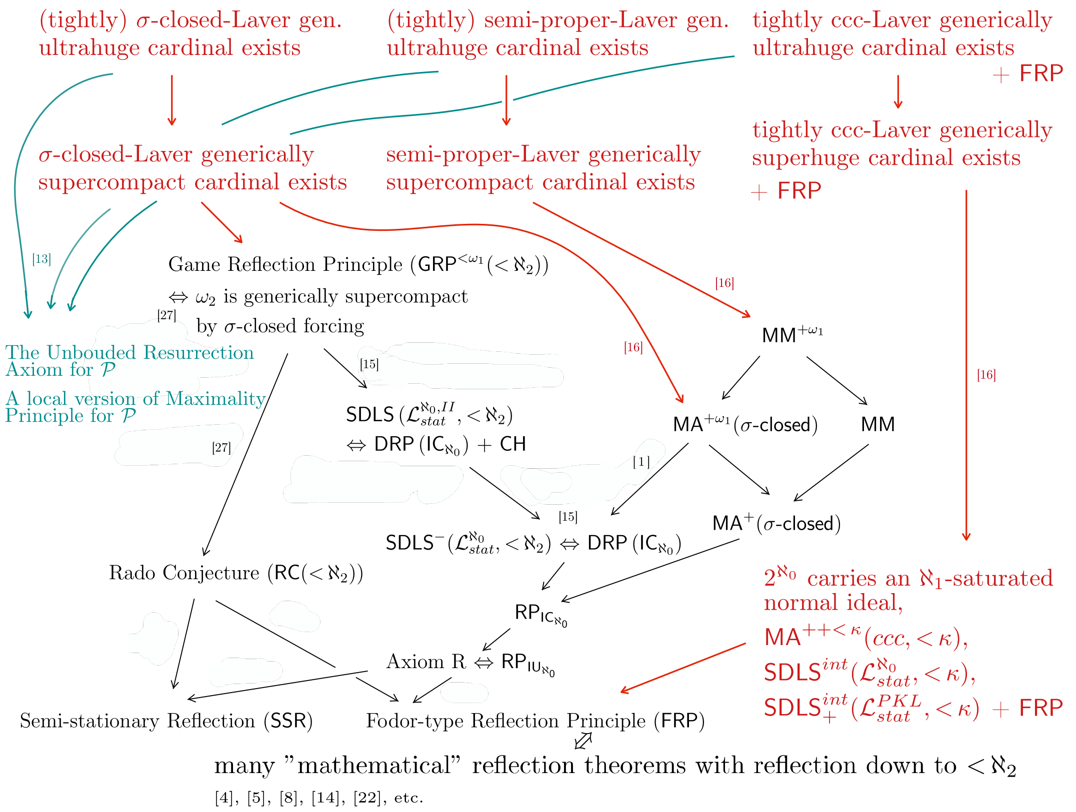

Figure 3.

4 Maximality Principle

Maximality Principle (MP) in its non parameterized form as introduced by Joel Hamkins in [24] is formulated in an infinite set of formulas asserting that all buttons are already pushed. I.e., for any $\mathcal{L}_\in$-sentence φ, if, for a poset $\mathbb{P}$,

$$\Vdash_{\mathbb{Q}} \text{“}\varphi\text{”} \text{ for all } \mathbb{Q} \text{ with } \mathbb{P} \leqslant \mathbb{Q}, \tag{4.1}$$

then φ holds.

If (4.1) holds, then we shall say that φ is a *button with the push* $\mathbb{P}$. One of the easy consequence of MP is the following:

Proposition 4.1 MP *implies* $\mathsf{V} \neq \mathsf{L}$. ❑

For an $\mathcal{L}_\in$-sentence φ let mp_φ be the $\mathcal{L}_\in$-sentence:

$$\exists P\,(P \text{ is a poset} \land \forall Q(P \leqslant Q \rightarrow \Vdash_Q \text{“}\varphi\text{”})) \rightarrow \varphi. \tag{4.2}$$

Formally we define MP to be the collection of all $\mathcal{L}_\in$-sentence of the form mp_φ for $\mathcal{L}_\in$-sentence φ.

Lemma 4.2 *Suppose that φ is an $\mathcal{L}_\in$-sentence. If* ZFC *is consistent, then so is* ZFC + mp_φ.

Proof. Suppose otherwise. Then we have

$$\mathsf{ZFC} \vdash \neg mp_\varphi. \tag{4.3}$$

Note that

$$\neg mp_\varphi \leftrightarrow \exists P\,(P \text{ is a poset} \land \forall Q(P \leqslant Q \rightarrow \Vdash_Q \text{“}\varphi\text{”})) \land \neg\varphi. \tag{4.4}$$

In ZFC, let $\mathbb{P}$ be a poset as above. Then $\Vdash_{\mathbb{P}}$ “φ”. On the other hand, since $\Vdash_{\mathbb{P}}$ “ψ” for all $\psi \in \mathsf{ZFC}$ and by (4.3) and (4.4), we have $\Vdash_{\mathbb{P}}$ “$\neg\varphi$” which is equivalent to $\neg \Vdash_{\mathbb{P}}$ “φ”.

Thus we obtained a proof of contradiction from ZFC. This is a contradiction to our assumption. ❑ (Lemma 4.2)

Lemma 4.3 *For any $\mathcal{L}_\in$-sentences $\varphi_0,\ldots,\varphi_{n-1}$, we have*
$\mathsf{ZFC} \vdash (mp_{\varphi_0} \land \cdots \land mp_{\varphi_{n-1}}) \leftrightarrow mp_{\varphi_0\land\cdots\land\varphi_{n-1}}$.

Proof. If $\mathbb{P}_0,\ldots,\mathbb{P}_{n-1}$ are pushes of the buttons $\varphi_0,\ldots,\varphi_{n-1}$ resp., then $\mathbb{P}_0 \times \cdots \times \mathbb{P}_{n-1}$ is a push for $\varphi_0 \land \cdots \land \varphi_{n-1}$. ❑ (Lemma 4.3)

Theorem 4.4 (Hamkins, [24]) *If* ZFC *is consistent then so is* ZFC + MP.

Proof. Assume toward a contradiction that $\mathsf{ZFC} + \mathsf{MP}$ is inconsistent. Then there are $\mathcal{L}_\in$-sentences $\varphi_0, \ldots, \varphi_{n-1}$ such that $\mathsf{ZFC} + mp_{\varphi_0} + \cdots + mp_{\varphi_{n-1}}$ is inconsistent. By Lemma 4.3, it follows that $\mathsf{ZFC} + mp_{\varphi_0 \wedge \cdots \wedge \varphi_{n-1}}$ is inconsistent. But this is a contradiction to Lemma 4.2. ❑ (Theorem 4.4)

Practically the same proof as above, we can prove also the following:

Theorem 4.5 *Suppose that "x-large cardinal" is a notion of a large cardinal such that, if κ is an x-large cardinal then this is preserved by any set-forcing of size $< \kappa$. If* $\mathsf{ZFC}+$ *"there are class many x-large cardinals" is consistent, then so is* $\mathsf{ZFC} + \mathsf{MP} +$ *"there are class many x-large cardinals".*

Proof. Working in the theory $\mathsf{ZFC} +$ "there are class many x-large cardinals" we have that $\Vdash_{\mathbb{P}}$ "there are class many x-large cardinals" holds for any poset $\mathbb{P}$. Thus Lemma 4.2 with ZFC replaced by $\mathsf{ZFC} +$ "there are class many x-large cardinals" holds. ❑ (Theorem 4.5)

Theorem 4.6 (Hamkins, [24]) MP *is preserved by any set-generic extension.*

Proof. The theorem follows immediately from the following Lemma. ❑ (Theorem 4.6)

For an $\mathcal{L}_\in$-sentence φ let mp^+_φ be the $\mathcal{L}_\in$-sentence:

$$\begin{aligned}(4.5)\qquad &\exists P\,(P \text{ is a poset} \wedge \forall Q(P \leqslant Q \rightarrow \Vdash_Q \text{“}\varphi\text{”}))\\ &\quad\rightarrow \forall R(R \text{ is a poset } \rightarrow \Vdash_R \text{“}\varphi\text{”}).\end{aligned}$$

Let MP^+ be the collection of $\mathcal{L}_\in$-sentences of the form mp^+_φ for all $\mathcal{L}_\in$-sentences φ.

Lemma 4.7 MP *and* MP^+ *are equivalent over* ZFC.

Proof. It is clear that MP^+ implies MP.

To see that MP implies MP^+, let φ be an arbitrary $\mathcal{L}_\in$-sentence. We write $\Box\varphi$ for $\forall R(R$ is a poset $\rightarrow \Vdash_R$ "φ"$)$.

It is easy to see that we have $\Box\varphi \leftrightarrow \Box\Box\varphi$. Thus mp^+_φ is equivalent to $mp_{\Box\varphi}$. The latter sentence is a member of MP. ❑ (Lemma 4.7)

A sort of inverse of Theorem 4.5 also holds:

Theorem 4.8 *Suppose that* MP *holds. If "x-large cardinal" is a notion of large cardinal such that* ① *"κ is an x-large cardinal" implies that κ is inaccessible;*
② *"κ is an x-large cardinal" can not be destroyed by forcing of size $< \kappa$;*
③ *no new x-large cardinal is created by set-forcing.*
If there is an x-large cardinal, then there are cofinally many x-large cardinals in V.

Proof. Suppose otherwise. Let κ_0 be a x-large cardinal, and $\kappa_1 > \kappa_0$ be a cardinal above which there are no x-large cardinals.

Let $\mathbb{P}$ be a poset which collapses κ_1 to, say cardinality ω_1, and let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter. Then by ① and ②, there is no x-large cardinal in $\mathsf{V}[\mathbb{G}]$. Also there is no x-large cardinal in any further generic extension by ③.

By MP it follows that there is no x-large cardinal in V but this is a contradiction to the assumption of the theorem. ❑ (Theorem 4.8)

5 Independence

Following Theorem 4.5 and Theorem 4.8, we introduce the meta-definition of normality and suspicious normality of a notion of large cardinal.

Let us say a notion of large cardinal (call this notion "x-large cardinal") *normal* if

① "κ is an x-large cardinal" implies that κ is inaccessible;

② "κ is an x-large cardinal" cannot be destroyed by a forcing of size $< \kappa$;

③ No new x-large cardinal can be created by small set-forcing; and

④ ZFC + "there are unboundedly many x-large cardinals" is consistent.

Note that most of the known notions of large cardinal are normal in the sense above under the assumption of the consistency of the existence of a sufficiently large cardinal.

Example 5.1 *The notion of super almost-huge cardinal is normal under the consistency of* ZFC + *"there is a huge cardinal".*

The example above follows from the next theorem which should be a folklore:

Theorem 5.2 *Suppose that* κ *is huge. Then,* $\{\alpha < \kappa : V_\kappa \models \text{"}\alpha \text{ is super almost-huge"}\}$ *is a normal measure 1 subset of* κ. ❑

A normal notion of large cardinal "x-large cardinal" is *suspiciously normal* if "small" in ③ of the definition of normality is dropped and the whole condition ④ is dropped. The notion of "x-large cardinal" in Theorem 4.8 is rather suspiciously normal.

Example 5.3 *The notion of inaccessible cardinal is suspiciously normal.* ❑

Theorem 5.4 *Suppose that* $\mathcal{P}$ *is an iterable class of posets, and "x-large cardinal" is a normal notion of large cardinal such that its (tightly) Laver-generic version is well-defined and an x-large cardinal* κ *can be forced to be a (tightly)* $\mathcal{P}$*-Laver generic x-large cardinal by a set forcing of size* κ*, then* MP *is consistent with* ZFC + *"there exists a (tightly)* $\mathcal{P}$*-gen. Laver-gen. x-large cardinal".*

If, in addition, "there exist y-large cardinals above an x-large cardinal but only boundedly many" is consistent for a suspiciously normal notion of large cardinal "y-large cardinal", then MP *is independent over* ZFC + *"there exists a (tightly)* $\mathcal{P}$*-gen. Laver-gen. x-large cardinal".*

Proof. The theory ZFC + MP + "there are class many x-large cardinals" is consistent by Theorem 4.5.

Starting from a model of this theory, we can force the existence of (tightly) $\mathcal{P}$-Laver-gen. x-large cardinal by a set-forcing (of size κ) then MP survives in the generic extension by Theorem 4.6. This shows the consistence of ZFC + MP + "there is a (tightly) $\mathcal{P}$-Laver gen. x-large cardinal".

For the second assertion of the theorem, we start from a model with an x-large cardinal κ_0 and with at least one but only set many y-large cardinals above κ_0.

Working in such a model V, force the existence of (tightly) $\mathcal{P}$-Laver gen. x-large cardinal using κ_0.

Let $\mathsf{V}[\mathbb{G}]$ be the generic extension. By the properties ② and ③ of suspicious normality, there are y-large cardinals above κ_0 in $\mathsf{V}[\mathbb{G}]$ but they are are only set many.

By Theorem 4.8, it follows that $\mathsf{V}[\mathbb{G}] \models \neg\mathsf{MP}$. ❑ (Theorem 5.4)

Corollary 5.5 *Suppose that "x-large cardinal" is one of supercompact, super-almost-huge, superhuge, or ultrahuge. Then the theory* ZFC + MP + *there is a/the* $\mathcal{P}$*-Laver-ge. x-large cardinal" is consistent for an iterable class* $\mathcal{P}$ *of posets in Theorem 3.3 assuming the consistency of the existence of a sufficiently large cardinal.* ❑

Corollary 5.6 *Suppose that* $\mathcal{P}$ *is an iterable class of posets for which a forcing construction of (tightly)* $\mathcal{P}$*-Laver gen. supercompact cardinal starting from a supercompact* κ *and forcing with a poset of size* κ *is available (Note that this is the case for* $\mathbb{P}$ *being either the class of all* σ*-closed posets or any reasonable subclass of ccc posets).*
Then MP *is independent over* ZFC + *"there is a (tightly)* $\mathcal{P}$*-Laver gen. supercompact cardinal"*

Proof. Use "inaccessible" as the notion of y-large cardinal in Theorem 5.4 (cf. Exercise 22.8, (a) in [26]). ❑ (Corollary 5.6)

Problem 5.7 *Is* MP *independent over* ZFC + *"there is a (tightly)* $\mathcal{P}$*-Laver gen. huge cardinal"? How about with* $\mathcal{P}$*-Laver gen. superhuge?*

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