THE WEAK DIAMOND

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1. Introduction

D. Jensen proposed the diamond principle $\diamondsuit$ in [2]. It asserts the existence of a sequence that guesses every subset of ω_1, which is called a *$\diamondsuit$-sequence*. He showed that $V = L$ implies $\diamondsuit$ and also $\diamondsuit$ implies the existence of a Suslin tree. Since then, numerous variations have been proposed, studied, and applied.

The weak diamond principle is one of such variations, proposed by K. Devlin and S. Shelah in [1]. They showed that this principle is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$. In particular, CH implies the weak diamond principle, which is very rare among variations of the diamond principle. Moreover, the argument used to prove this fact is unique and interesting.

The purpose of this paper is to present the proof that $2^{\aleph_0} < 2^{\aleph_1}$ implies the weak diamond principle in a more intuitive way to help understand the idea behind it. Please keep in mind that the proof is essentially the same, although the presentation was modified, and more explanations are provided.

2. Definition and interpretation

The weak diamond is defined as follows by K. Devlin and S. Shelah in [1]: for every function $F : 2^{<\omega_1} \to 2$, there exists a function $g : \omega_1 \to 2$ such that for every function $f : \omega_1 \to 2$, there are stationarily many $\alpha < \omega_1$ such that $F(f \restriction \alpha) = g(\alpha)$.

The following equivalent formulation may be more intuitive. for every sequence $\langle F_\alpha : \alpha < \omega_1 \rangle$ of functions with $F_\alpha : \mathcal{P}(\alpha) \to 2$, there exists a function $g : \omega_1 \to 2$ such that for every subset X of ω_1, there are stationarily many $\alpha < \omega_1$ such that $F_\alpha(X \cap \alpha) = g(\alpha)$.

Devlin and Shelah proved the following theorem in [1].

Theorem 2.1. *The weak diamond is equivalent to* $2^{\aleph_0} < 2^{\aleph_1}$.

In particular, CH implies the weak diamond. The rest of this paper is devoted to the proof of one direction of this theorem, namely $2^{\aleph_0} < 2^{\aleph_1}$ implies the weak diamond.

Suppose not, i.e. $2^{\aleph_0} < 2^{\aleph_1}$ but the weak diamond does not hold. It means that there exists a function $F : 2^{<\omega_1} \to 2$ such that for every function $g : \omega_1 \to 2$, there exists a function $f : \omega_1 \to 2$ such that for club many $\alpha < \omega_1$, $F(f \restriction \alpha) \neq g(\alpha)$. By considering $g' : \omega_1 \to 2$ defined by $g'(\alpha) = 1 - g(\alpha)$, we can see that for every function $g : \omega_1 \to 2$, there exists a function $f : \omega_1 \to 2$

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such that for club many $\alpha < \omega_1$, $F(f \restriction \alpha) = g(\alpha)$. It is why this 2-color weak diamond is so distinct from the weak diamond of 3-color or more.

Before going into the details, we will explain our strategy. Let X be the set of all sequences $\langle s_\alpha : \alpha < \omega^2 \rangle$ such that there exists a $\delta < \omega_1$ such that for every $\alpha < \omega^2$, s_α is a function from δ into 2. Note $|X| = 2^{\aleph_0}$. We shall define an injection function $\varphi : 2^{\omega_1} \to X$. Of course, this is a contradiction. To show that φ is injective, we shall define a function $\sigma : X \to 2^{\omega_1}$ such that for every $f \in 2^{\omega_1}$, $\sigma \circ \varphi(f) = f$.

The definition of φ goes as follows. Let $f : \omega_1 \to 2$. Inductively, we shall define a sequence $\langle f_\alpha : \alpha < \omega^2 \rangle$ in 2^{ω_1} with $f_n = f$ for all $n < \omega$. This sequence is designed so that the lower part $\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle$ reflect the information about the higher part. φ is defined to be $\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle$ for some nice $\delta < \omega_1$. It will be shown that we can reconstruct $\langle f_\alpha : \alpha < \omega^2 \rangle$ from this sequence of short functions. In a sense, we "slide down" the information about one tall function f into a wide sequence of shorter functions.

Let τ be a bijection from 2^ω onto the set of all countable sequences $\langle s_\alpha : \alpha < \eta \rangle$ such that there exists a $\delta < \omega_1$ such that for every $\alpha < \eta$, s_α is a function from δ into 2.

We begin with the definition of φ. Let $f : \omega_1 \to 2$. We shall define functions f_α and g_α for For every $n < \omega$, let $f_n = f$. For every $\alpha < \omega_1$ and $n < \omega$, let $g_n(\alpha) = F(f \restriction \alpha)$. Define $D_0 = D_1 = \omega_1$. Now suppose that for some $n \in (0, \omega)$, we have defined D_n and f_α and g_α for all $\alpha < \omega n$. For every $\delta < \omega_1$, we shall define $g_\alpha(\delta)$ as follows. If $\delta \notin D_n$, then let $g_\alpha(\delta) = 0$ (this is an ignorable case). Suppose $\delta \in D_n$. Set $\gamma_n = \min(D_n \setminus (\delta + 1))$. Let $x_{n,\delta} \in 2^\omega$ be so that $\tau(x_{n,\delta}) = \langle f_\alpha \restriction \gamma_n : \alpha < \omega n \rangle$. Define $g_{\omega n+m}(\delta) = x_{n,\delta}(m)$.

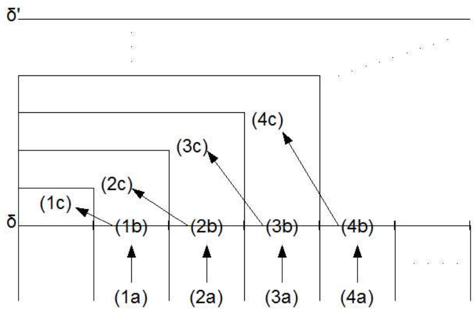
By assumption, for every $m < \omega$, there exist a $f_{\omega n+m} : \omega_1 \to 2$ such that for club many $\xi < \omega_1$, $F(f_{\omega n+m} \restriction \xi) = g_{\omega n+m}(\xi)$. Let D_{n+1} be a club subset of ω_1 such that for every $\xi \in D_{n+1}$ and $m < \omega$, $F(f_{\omega n+m} \restriction \xi) = g_{\omega n+m}(\xi)$. It completes the definition of f_α and g_α for $\alpha < \omega^2$ and D_n for $n < \omega$. Let $\delta = \min \bigcap_{n<\omega} D_n$ and $\varphi(f) = \langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle$.

The point of this construction is:

- (i) Let $n \in (0, \omega)$. For every $m < \omega$ and $\delta \in D_{n+1}$, we have $F(f_{\omega n+m} \restriction \delta) = g_{\omega n+m}(\delta)$. So, if we know $f_{\omega n+m} \restriction \delta$, then we can compute $g_{\omega n+m}(\delta)$.
- (ii) Recall that for every $m < \omega$, $x_{n,\delta}(m) = g_{\omega n+m}(\delta)$. If we know $g_{\omega n+m}(\delta)$, we can compute $x_{n,\delta}$.
- (iii) Let $\gamma_n = \min(D_n \setminus (\delta + 1))$ for each $n < \omega$. Recall $\tau(x_{n,\delta}) = \langle f_\alpha \restriction \gamma_n : \alpha < \omega n \rangle$. So, from $x_{n,\delta}$, we can compute $\langle f_\alpha \restriction \gamma_n : \alpha < \omega n \rangle$.
- (iv) By doing this for every $n \in (0, \omega)$, we can compute $\langle f_\alpha \restriction \delta' : \alpha < \omega n \rangle$ where $\delta' = \min(\bigcap_{n<\omega} D_n \setminus (\delta + 1))$.

The following figure visualizes how this argument works.

From $\langle f_{\omega n+m} \restriction \delta : m < \omega \rangle$ (shown as (1a), (2a), ...), we can find $\langle g_{\omega n+m}(\delta) : m < \omega \rangle$ and hence $x_{n,\delta}$ (shown as (1b), (2b), ...). Each $x_{n,\delta}$ codes the box $\langle f_\alpha \restriction \gamma_n : \alpha < \omega n \rangle$ where $\gamma_n = \min(D_n \setminus (\delta+1))$ (shown as (1c), (2c), ...). By doing this for all $n \in (0, \omega)$, we can find all $\langle f_\alpha \restriction \delta' : \alpha < \omega^2 \rangle$



where $\delta' = \min(D \setminus (\delta+1))$. It is a little surprising that since we only need the values of f_α below δ to find $f_\alpha \restriction \delta'$, we can pass the limit stages.

Let us do it more formally. We shall define $\bar{\sigma} : X \to X$ as follows. Let $\langle s_\alpha : \alpha < \omega^2 \rangle \in X$ and $\mathrm{dom}(s_0) = \delta$ (note that by the definition of X, $\mathrm{dom}(s_\alpha) = \delta$ for every $\alpha < \omega^2$). For every $n \in (0, \omega)$, define $y_{n,\delta} : \omega \to 2$ by for every $m < \omega$, $y_{n,\delta}(m) = F(s_{\omega n + m})$. Let $\langle t_{n,\alpha} : \alpha < \eta_n \rangle = \tau(y_{n,\delta})$. If for every $n \in (0, \omega)$, $\eta_n = \omega n$ and for every $\alpha < \omega n$, $t_{n+1,\alpha}$ is an extension of $t_{n,\alpha}$, then for every $\bar{n} < \omega$ and $\alpha \in [\omega \bar{n}, \omega(\bar{n}+1))$, let $t_\alpha = \bigcup_{\bar{n} < n < \omega} t_{n,\alpha}$. It is easy to see that $\langle t_\alpha : \alpha < \omega^2 \rangle \in X$. Let $\bar{\sigma}(\langle s_\alpha : \alpha < \omega^2 \rangle) = \langle t_\alpha : \alpha < \omega^2 \rangle$. Otherwise, let $\bar{\sigma}(\langle s_\alpha : \alpha < \omega^2 \rangle) = \varnothing$ (this is ignorable).

We shall define σ as follows. Let $\langle s_\alpha : \alpha < \omega^2 \rangle \in X$. For each $\alpha < \omega^2$, set $s^0_\alpha = s_\alpha$. We shall inductively define $\langle s^\xi_\alpha : \alpha < \omega^2 \rangle \in X$ for all $\xi < \omega_1$. Suppose that $\langle s^\xi_\alpha : \alpha < \omega^2 \rangle$ has been defined. Let $\langle t_\alpha : \alpha < \omega^2 \rangle = \bar{\sigma}(\langle s^\xi_\alpha : \alpha < \omega^2 \rangle)$. If for every $\alpha < \omega^2$, t_α extends s^ξ_α, then we let $s^{\xi+1}_\alpha = t_\alpha$ for every $\alpha < \omega^2$. Otherwise, stop the induction and let $\sigma(\langle s_\alpha : \alpha < \omega^2 \rangle)$ be just any function from ω_1 into 2. If ξ is limit, for every $\alpha < \omega^2$, let $s^\xi_\alpha = \bigcup_{\zeta < \xi} s^\zeta_\alpha$. Let $\sigma(\langle s_\alpha : \alpha < \omega^2 \rangle) = \bigcup_{\xi < \omega_1} s^\xi_0$.

Now, it suffices to show that for every $f : \omega_1 \to 2$, $\sigma \circ \varphi(f) = f$. Let f_α, g_α, $x_{n,\delta}$, D_n be as in the definition of $\varphi(f)$. Define $D = \bigcap_{n<\omega} D_n$ and let $\langle \delta_\xi : \xi < \omega_1 \rangle$ be the increasing enumeration of D. Then, $\varphi(f) = \langle f_\alpha \restriction \delta_0 : \alpha < \omega^2 \rangle$.

Claim 1. Let $\delta \in D$. Then, $\bar{\sigma}(\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle) = \langle f_\alpha \restriction \delta' : \alpha < \omega^2 \rangle$ where $\delta' = \min(D \setminus (\delta+1))$.

$\vdash$ For every $n \in (0, \omega)$, define $y_{n,\delta} \in 2^\omega$ by $y_{n,\delta}(m) = F(f_{\omega n + m} \restriction \delta)$. Since $\delta \in D \subseteq D_{n+1}$, we have $g_{\omega n + m}(\delta) = F(f_{\omega n + m} \restriction \delta)$ for every $m < \omega$. Since $\delta \in D \subseteq D_n$, we have $x_{n,\delta}(m) = g_{\omega n + m}(\delta)$. Therefore, we have $x_{n,\delta} = y_{n,\delta}$. So, $\tau(y_{n,\delta}) = \tau(x_{n,\delta}) = \langle f_\alpha \restriction \gamma_n : \alpha < \omega n \rangle$ where $\gamma_n = \min(D_n \setminus (\delta+1))$. Note that $\sup_{n<\omega} \gamma_n = \delta'$. By the definition of $\bar{\sigma}$, we have $\bar{\sigma}(\langle f_\alpha \restriction \delta : \alpha < \omega^2 \rangle) = \langle f_\alpha \restriction \delta' : \alpha < \omega^2 \rangle$. $\dashv$ (Claim 1)

Let $s_\alpha = f_\alpha \restriction \delta_0$ for every $\alpha < \omega^2$ and define $\langle s^\xi_\alpha : \xi < \omega_1 \text{ and } \alpha < \omega^2 \rangle$ as in the definition of $\sigma(\langle s_\alpha : \alpha < \omega^2 \rangle)$.

Claim 2. For every $\xi < \omega_1$ and $\alpha < \omega^2$, $s^\xi_\alpha = f_\alpha \restriction \delta_\xi$.

$\vdash$ Go by induction on $\xi < \omega_1$. The case $\xi = 0$ is just by definition. Suppose that $s^\xi_\alpha = f_\alpha \restriction \delta_\xi$ for all $\alpha < \omega^2$. Then, by Claim 1, $s^{\xi+1}_\alpha = f_\alpha \restriction$

$\delta_{\xi+1}$. Suppose that ξ is a limit ordinal and $s^\zeta_\alpha = f_\alpha \restriction \delta_\zeta$ for every $\zeta < \xi$ and $\alpha < \omega^2$. Then,

$$s^\xi_\alpha = \bigcup_{\zeta<\xi} s^\zeta_\alpha = \bigcup_{\zeta<\xi} f_\alpha \restriction \delta_\zeta = f_\alpha \restriction \delta_\xi$$

⊣ (Claim 2)

Therefore, we have

$$\begin{aligned} \sigma \circ \varphi(f) &= \sigma(\langle f_\alpha \restriction \delta_0 : \alpha < \omega^2 \rangle) \\ &= \bigcup_{\xi<\omega_1} s^\xi_0 \\ &= \bigcup_{\xi<\omega_1} f_0 \restriction \delta_\xi \\ &= f_0 = f \end{aligned}$$

References

1. K. J. Devlin and S. Shelah, *A weak version of ◇ which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math. **29** (1978), no. 2-3, 239–247.
2. Ronald B. Jensen, *Souslin's Hypothesis is incompatible with V=L*, Notices of the American Mathematical Society **15** (1968), 935.