Negating a partition relation by a family of simplified morasses

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Abstract

We show the consistency of GCH + $\omega_3 \not\longrightarrow (\omega_2 + \omega_1)^2_\omega$.

Introduction

In [HL] (4.5 Question on p. 142) Hajnal and Larson ask if $\omega_3 \longrightarrow (\omega_2 + 2)^2_\omega$ is provable in ZFC + GCH. We show $\omega_3 \not\longrightarrow (\omega_2 + \omega_1)^2_\omega$ assuming a family of simplified morasses. The members include a simplified $(\omega_2, 1)$-morass $\mathcal{F} \subseteq [\omega_3]^{<\omega_2}$, a kind of $(\omega_1, 1)$-morass $\mathcal{N}$, and an $(\omega, 1)$-morass $\mathcal{A} \subseteq [\omega_1]^{<\omega}$. To prepare a universe where such a family exists, we start with a ground model V where, by [V1], GCH holds and $\mathcal{F}$ exists. By a mild modification to [M], construct a poset $P \subseteq H_{\omega_2}$ that is σ-closed, has the ω_2-cc, and forces the kind $\mathcal{N} \subset ([H_{\omega_2}]^\omega)^V$ preserving GCH. By [V2], $\mathcal{A}$ is provable in ZFC. More precisely, we are in the generic extension $V[G]$ of V s.t. there exists $(\mathcal{F}, \mathcal{N}, \mathcal{A})$ that satisfies the following.

(0) The cofinalities and so the cardinalities are preserved in $V[G]$. Also GCH is preserved in $V[G]$.

(1) The simplified $(\omega_2, 1)$-morass $\mathcal{F} \subseteq ([\omega_3]^{<\omega_2})^V$ belongs to V with the representation in V

$$\big(\langle \varphi_\xi \mid \xi \leq \omega_2 \rangle,\ \langle F_{\eta\xi} \mid \eta < \xi \leq \omega_2 \rangle\big).$$

However, $\mathcal{F}$ is upwardly absolute. In particular, $\mathcal{F} = \{g[\varphi_\xi] \mid \xi < \omega_2,\ g \in F_{\xi\omega_2}\}$, $\mathrm{rank}(g[\varphi_\xi]) = \xi$, and $\mathcal{F}$ satisfies the following. For each $X, Y \in \mathcal{F}$ and $x < \omega_3$, if $\mathrm{rank}(X) = \mathrm{rank}(Y)$ w.r.t. $\mathcal{F}$ and $x \in X \cap Y$, then

$$X \cap (x+1) = Y \cap (x+1).$$

(2) In V, let $\mathcal{C}_0 := \{N \in [H_{\omega_2}]^\omega \mid N$ is an elementary substructure of $H_{\omega_2}\}$, where H_{ω_2} represents any of your favorite structure $(H_{\omega_2}, \in \cap (H_{\omega_2} \times H_{\omega_2}), \cdots)$ in V. Now, the kind $\mathcal{N} \subseteq (\mathcal{C}_0)^V$ is forced over V. It relies on the two binary relations on $\mathcal{C}_0$ defined in V by

$$N_1 =_{\omega_1} N_2, \text{ if } \omega_1 \cap N_1 = \omega_1 \cap N_2,$$

$$N_1 <_{\omega_1} N_2, \text{ if } \omega_1 \cap N_1 < \omega_1 \cap N_2.$$

The following holds in $V[G]$.

∗ (elementary) For each $N \in \mathcal{N}$, we have $N \prec (H_{\omega_2}, \in \cap (H_{\omega_2} \times H_{\omega_2}), \cdots)^V$.

∗ (isomorphic) For each $N_1, N_2 \in \mathcal{N}$ with $N_1 =_{\omega_1} N_2$, there exists (in V) the unique isomorphism $\phi_{N_1N_2} : (N_1, \in \cap (N_1 \times N_1), \cdots) \longrightarrow (N_2, \in \cap (N_2 \times N_2), \cdots)$ s.t. $\phi_{N_1N_2}(x) = x$ for all $x \in N_1 \cap N_2$.

∗ (up) For each $N_3, N_2 \in \mathcal{N}$ with $N_3 <_{\omega_1} N_2$, there exists $N_1 \in \mathcal{N}$ s.t. $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.

∗ (down) For each $N_1, N_2, N_3 \in \mathcal{N}$, if $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$, then $\phi_{N_1N_2}(N_3) \in \mathcal{N}$.

∗ (partition) $\mathcal{N} = \mathrm{zero}(\mathcal{N}) \cup \mathrm{suc}(\mathcal{N}) \cup \lim(\mathcal{N})$, where

$$\mathrm{zero}(\mathcal{N}) = \{N \in \mathcal{N} \mid \mathcal{N} \cap N = \emptyset\},$$

$$\mathrm{suc}(\mathcal{N}) = \mathrm{suc}_1(\mathcal{N}) \cup \mathrm{suc}_2(\mathcal{N}),$$

$$\mathrm{suc}_1(\mathcal{N}) = \{N \in \mathcal{N} \mid \text{there exists } N_0 \text{ s.t. } \mathcal{N} \cap N = (\mathcal{N} \cap N_0) \cup \{N_0\}\},$$

$$\mathrm{suc}_2(\mathcal{N}) = \{N \in \mathcal{N} \mid \text{there exsist } N_1, N_2 \text{ s.t. the following 3 items hold}\},$$

$$N_1 =_{\omega_1} N_2,$$

$$\mathcal{N} \cap N = (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2) \cup \{N_1, N_2\},$$
$$\Delta := (\omega_2 \cap N_1) \cap (\omega_2 \cap N_2) < (\omega_2 \cap N_1) \setminus \Delta < (\omega_2 \cap N_2) \setminus \Delta \neq \emptyset,$$
$$\lim(\mathcal{N}) = \{N \in \mathcal{N} \mid N = \bigcup(\mathcal{N} \cap N)\}.$$

$*$ (cofinal) $\bigcup \mathcal{N} = (H_{\omega_2})^V$.

In particular, $\mathcal{N}$ satisfies the following. For each $N_1, N_2 \in \mathcal{N}$ and $x < \omega_2$, if $N_1 =_{\omega_1} N_2$ and $x \in N_1 \cap N_2$, then

$$N_1 \cap (x+1) = N_2 \cap (x+1).$$

To see this, assume $0 < x$ and let $e \in \underline{N_1 \cap N_2}$ be a common onto map $e : \omega_1 \longrightarrow x$. Then

$$N_1 \cap x = e[\,\omega_1 \cap N_1\,] = e[\,\omega_1 \cap N_2\,] = N_2 \cap x.$$

(3) The simplified $(\omega, 1)$-morass $\mathcal{A} \subseteq [\omega_1]^{<\omega}$ provable to exist in ZFC. Hence, $\mathcal{A}$ satisfies the following. For each $a, b \in \mathcal{A}$ and $x < \omega_1$, if $\mathrm{rank}(a) = \mathrm{rank}(b)$ w.r.t. $\mathcal{A}$ and $x \in a \cap b$, then

$$a \cap (x+1) = b \cap (x+1).$$

Question. (1) Does a gap two simplified $(\omega_1, 2)$-morass together with GCH, or, $V = L$ suffice for $\omega_3 \not\longrightarrow (\omega_2 + \omega_1)^2_\omega$ + GCH ?

(2) Is $\omega_3 \longrightarrow (\omega_2 + \omega_1)^2_\omega$ ever consistent by a large cardinal ? ([HL])

The $(\omega_2, 1)$-Simplified Morass

There are several formulations in [V1], [I], and others. Here is the $(\omega_2, 1)$-simplified morass in this note.

Definition. $\big(\langle \varphi_\xi \mid \xi \le \omega_2 \rangle,\ \langle F_{\eta\xi} \mid \eta < \xi \le \omega_2 \rangle\big)$ is the $(\omega_2, 1)$-simplified morass, if

- $\varphi_0 = 1$, $\varphi_{\omega_2} = \omega_3$, and for any $\xi < \omega_2$, we have $\varphi_\xi < \omega_2$.
- For any $\eta < \xi \le \omega_2$, $F_{\eta\xi}$ is a set of order preserving maps from $(\varphi_\eta, <)$ into $(\varphi_\xi, <)$ s.t. if $\xi < \omega_2$, then $|\,F_{\eta\xi}\,| < \omega_2$.
- For any $\eta < \xi < \zeta \le \omega_2$, we have $F_{\eta\zeta} = F_{\xi\zeta} \circ F_{\eta\xi}$.
- For any $\xi < \omega_2$, there exists σ_ξ s.t. $0 \le \sigma_\xi < \varphi_\xi$, $\varphi_{\xi+1} = \varphi_\xi + (\varphi_\xi - \sigma_\xi)$, $F_{\xi\xi+1} = \{b_\xi, \mathrm{id}_\xi\}$, where $b_\xi(x) = x$ (for any $x < \sigma_\xi$), $b_\xi(x) = \varphi_\xi + (x - \sigma_\xi)$ (for any $\sigma_\xi \le x < \varphi_\xi$), and $\mathrm{id}_\xi(x) = x$ for all $x < \varphi_\xi$.
- For any limit ordinal $\zeta \le \omega_2$, if $\eta_1, \eta_2 < \zeta$, $f_1 \in F_{\eta_1\zeta}$, and $f_2 \in F_{\eta_2\zeta}$, then there exists (ξ, h, g_1, g_2) s.t. $\eta_1, \eta_2 < \xi < \zeta$, $h \in F_{\xi\zeta}$, $g_1 \in F_{\eta_1\xi}$, and $g_2 \in F_{\eta_2\xi}$ s.t. $f_1 = h \circ g_1$ and $f_2 = h \circ g_2$.
- For any limit ordinal $\zeta \le \omega_2$, we have $\varphi_\zeta = \bigcup\{f[\,\varphi_\eta\,] \mid \eta < \zeta, f \in F_{\eta\zeta}\}$.

Now collect all of the images $\mathcal{F} = \{f[\,\varphi_\xi\,] \mid \xi < \omega_2, f \in F_{\xi\omega_2}\}$. Then $\mathcal{F} \subseteq [\omega_3]^{<\omega_2}$ satisfies, among others,

- (crutial for simplified morasses) Let $\eta < \xi \le \omega_2$, $f, g \in F_{\eta\xi}$, and $x_1, x_2 \in \varphi_\eta$. If $f(x_1) = g(x_2)$, then $x_1 = x_2$ and $f(y) = g(y)$ for any $y < x_1 = x_2$. In particular, let $f, g \in F_{\eta\omega_2}$ and denote $X = f[\,\varphi_\eta\,]$ and $Y = g[\,\varphi_\eta\,]$. Then

  $*$ If $x \in X \cap Y$, then $X \cap (x+1) = Y \cap (x+1)$.

  $*$ If $X \subseteq Y$, then $X = Y$.
- (well founded) There exists no $\langle X_n \mid n < \omega \rangle$ s.t. for any $n < \omega$, $X_n \in \mathcal{F}$, $X_{n+1} \subseteq X_n$ and $X_{n+1} \neq X_n$. Recursively define the $\mathrm{rank}(X) = \sup\{\mathrm{rank}(Y) + 1 \mid Y \in \mathcal{F}, Y \subseteq X, Y \neq X\} < \omega_2$ for any $X \in \mathcal{F}$. Then for any $X \in \mathcal{F}$ and $\xi < \omega_2$, we have $\mathrm{rank}(X) = \xi$ iff there exists $f \in F_{\xi\omega_2}$ with $X = f[\,\varphi_\xi\,]$.

- (up and commom head) Let $X_3 \in \mathcal{F}$ and $\mathrm{rank}(X_3) < \xi < \omega_2$. Then there exists $X_1 \in \mathcal{F}$ s.t. $X_3 \subseteq X_1$ and $\mathrm{rank}(X_1) = \xi$. Let (Y, Z, x) s.t. $Y, Z \in \mathcal{F}$ and $x \in Y \cap Z$. Then

  $*$ If $\mathrm{rank}(Y) = \mathrm{rank}(Z)$, then $Y \cap (x+1) = Z \cap (x+1)$.

  $*$ If $\mathrm{rank}(Y) \leq \mathrm{rank}(Z)$, then $Y \cap (x+1) \subseteq Z \cap (x+1)$.

The Coloring

Definition. Let $x < y < \omega_3$. Then $f : [\omega_3]^2 \longrightarrow \omega \times \omega$ assigns $f(\{x, y\})$ as follows.

Step 1. Let $\xi < \omega_2$ be the least s.t. there exists $X \in \mathcal{F}$ that satisfies $\mathrm{rank}(X) = \xi$ and $\{x, y\} \subseteq X$. If $Y \in \mathcal{F}$, $\mathrm{rank}(Y) = \xi$ and $\{x, y\} \subseteq Y$, then $X \cap (y+1) = Y \cap (y+1)$. Hence, there exists the unique $x_\xi < y_\xi < \varphi_\xi$ s.t. there exists $g \in F_{\xi\omega_2}$ that satisfies $g(x_\xi) = x$ and $g(y_\xi) = y$. Focus on the two elements set $\{y_\xi, \varphi_\xi\} \subseteq \omega_2$.

Step 2. Let $\alpha < \omega_1$ be the least s.t. there exists $N \in \mathcal{N}$ that satisfies $\omega_1 \cap N = \alpha$ and $\{y_\xi, \varphi_\xi\} \subseteq N$. If $N' \in \mathcal{N}$, $\omega_1 \cap N' = \alpha$, and $\{y_\xi, \varphi_\xi\} \subseteq N'$, then $N' \cap (\varphi_\xi + 1) = N \cap (\varphi_\xi + 1)$. Hence, the set of collapses $\{c_N(y_\xi), c_N(\varphi_\xi), c_N(\omega_2)\}$ is independent of the actual choices of N, where c_N is the transitive collapse of N and $c_N(\omega_2) := \{c_N(i) \mid i \in \omega_2 \cap N\}$. Focus on the two elements set $\{c_N(\varphi_\xi), c_N(\omega_2)\} \subseteq \omega_1$.

Step 3. Let $n < \omega$ be the least s.t. there exists $a \in \mathcal{A}$ that satisfies $\mathrm{rank}(a) = n$ and $\{c_N(\varphi_\xi), c_N(\omega_2)\} \subseteq a$. If $b \in \mathcal{A}$, $\mathrm{rank}(b) = n$, and $\{c_N(\varphi_\xi), c_N(\omega_2)\} \subseteq b$, then $a \cap (c_N(\omega_2) + 1) = b \cap (c_N(\omega_2) + 1)$. Hence, the set of collases $\{c_a(c_N(\varphi_\xi)), c_a(c_N(\omega_2))\}$ is independent of the actual choices of a, where c_a is the transitive collapse of a.

Let us focus on n and $c_a\big(c_N(\varphi_\xi)\big)$. Assign $f(\{x, y\}) = \big(n, c_a\big(c_N(\varphi_\xi)\big)\big) \in \omega \times \omega \equiv \omega$.

Lemma. Let A and B be subsets of ω_3 s.t. $A < B$, $A \sim \omega_2$, and $B \sim \omega_1$. Then there exist $u < u'$ in A and $s \in B$ s.t. $f(\{u, s\}) \neq f(\{u', s\})$. Hence, there exists no f-homogeneous subset of ω_3 with the order-type $\omega_2 + \omega_1$.

Proof. Let $t = \sup(B)$. We have 3 stages.

Stage 1. Let us fix a sequence $\langle X_i \mid i < \omega_2 \rangle$ s.t. $X_i \in \mathcal{F}$, $\mathrm{rank}(X_i) = i$, and $t \in X_i$. Then $\langle t \cap X_i \mid i < \omega_2 \rangle$ is continuously $\subseteq$-increasing and $\bigcup\{t \cap X_i \mid i < \omega_2\} = t < \omega_3$. Let $\eta < \omega_2$ be sufficiently large s.t. $\omega_1 \sim B \subseteq \bigcup\{X_i \cap t \mid i < \eta\}$. Hence, $B \cup \{t\} \subseteq X_\eta$. Let us pick

- $u \in A \sim \omega_2$ s.t. $u \notin X_\eta$.

  Let ξ_1 be the least $\xi < \omega_2$ s.t. $u \in X_\xi$. Then $\eta < \xi_1$ and it is routine to deduce

- For any $s \in B$, ξ_1 is the least $\xi < \omega_2$ s.t. there exists $Y \in \mathcal{F}$ that satisfies $\mathrm{rank}(Y) = \xi$ and $\{u, s\} \subseteq Y$.

  Since $\varphi_{\xi_1} < \omega_2 \leq \sup(A)$, we can fix $i < \omega_2$ s.t. $\xi_1 < i$ and $\varphi_{\xi_1} \cup \{\varphi_{\xi_1}, u\} \cup B \cup \{t\} \subseteq X_i$. Let

- $u' \in A \sim \omega_2$ s.t. $\max\{\varphi_{\xi_1}, u\} < u'$ and $u' \notin X_i$.

  Let $\xi_2 < \omega_2$ be the least $\xi < \omega_2$ that satisfies $u' \in X_\xi$. Then $i < \xi_2$. Hence,

- $\xi_1 < \xi_2 < \omega_2$.

  It is routine to deduce

- For any $s \in B$, ξ_2 is the least $\xi < \omega_2$ s.t. there exists $Y \in \mathcal{F}$ that satisfies $\mathrm{rank}(Y) = \xi$ and $\{u', s\} \subseteq Y$.

Let us transitve collase $c_{X_{\xi_1}} : (X_{\xi_1}, <, t, u) \sim (\varphi_{\xi_1}, <, t_{\xi_1}, u_{\xi_1})$. Take $X'_{\xi_2} \in \mathcal{F}$ s.t. $\mathrm{rank}(X'_{\xi_2}) = \mathrm{rank}(X_{\xi_2})$ and $X_{\xi_1} \subseteq X'_{\xi_2}$. Since $t \in X_{\xi_2} \cap X'_{\xi_2}$, we have $X_{\xi_2} \cap (t+1) = X'_{\xi_2} \cap (t+1)$. Hence $\varphi_{\xi_1} \cup \{\varphi_{\xi_1}, u, u'\} \cup B \cup \{t\} \subseteq X'_{\xi_2}$. Let us transitive collase $c_{X'_{\xi_2}} : (X'_{\xi_2}, <, t, u') \sim (\varphi_{\xi_2}, <, t_{\xi_2}, u'_{\xi_2})$. Let $f \in F_{\xi_1\xi_2}$ be the composition $f = c_{X'_{\xi_2}} \circ c^{-1}_{X_{\xi_1}}$. Then

- $u_{\xi_1} < t_{\xi_1} < \varphi_{\xi_1} < u'_{\xi_2} < t_{\xi_2} < \varphi_{\xi_2} < \omega_2$ and $f(t_{\xi_1}) = t_{\xi_2}$.

Stage 2. Fix $N_0 \in \mathcal{N}$ s.t. $\{u_{\xi_1}, t_{\xi_1}, \varphi_{\xi_1}, u'_{\xi_2}, t_{\xi_2}, \varphi_{\xi_2}, f\} \subseteq N_0$.

Remember $B \subseteq X_{\xi_1} \sim \varphi_{\xi_1}$. Let $s \in B \sim \omega_1$ s.t.

- $s_{\xi_1} := c_{X_{\xi_1}}(s) \notin N_0$.

Let $s_{\xi_2} := c_{X'_{\xi_2}}(s)$. Then $f(s_{\xi_1}) = c_{X'_{\xi_2}}\big(c^{-1}_{X_{\xi_1}}(s_{\xi_1})\big) = c_{X'_{\xi_2}}(s) = s_{\xi_2}$. Since $u' \in A$, $A < B$, and $s \in B \subseteq t$, we have

- $u'_{\xi_2} < f(s_{\xi_1}) = s_{\xi_2} < t_{\xi_2} < \varphi_{\xi_2}$.

Claim. Let $\alpha < \omega_1$ be the least s.t. there exists $N \in \mathcal{N}$ that satisfies $\omega_1 \cap N = \alpha$ and $\{s_{\xi_1}, \varphi_{\xi_1}\} \subseteq N$. Then there is one with $N_0 \in N$.

Proof. We argue in 3 cases.

Case. $N =_{\omega_1} N_0$: Then $\varphi_{\xi_1} \in N \cap N_0 \cap \omega_2$ and so $s_{\xi_1} \in N \cap \varphi_{\xi_1} = N_0 \cap \varphi_{\xi_1}$. Hence, $s_{\xi_1} \in N_0$. This is absurd.

Case. $N <_{\omega_1} N_0$: Then fix $N'_0 \in \mathcal{N}$ s.t. $N \in N'_0$ and $N'_0 =_{\omega_1} N_0$. Then $\varphi_{\xi_1} \in N'_0 \cap N_0$ and so $s_{\xi_1} \in N \cap \varphi_{\xi_1} \subseteq N'_0 \cap \varphi_{\xi_1} = N_0 \cap \varphi_{\xi_1}$. Hence, $s_{\xi_1} \in N_0$. This is absurd.

Case. $N_0 <_{\omega_1} N$: Let $N' \in \mathcal{N}$ s.t. $N_0 \in N'$ and $N =_{\omega_1} N'$. Then $\varphi_{\xi_1} \in N \cap N'$ and so $s_{\xi_1} \in N \cap \varphi_{\xi_1} = N' \cap \varphi_{\xi_1}$. Hence, $N_0 \in N' =_{\omega_1} N$ and $\{s_{\xi_1}, \varphi_{\xi_1}\} \subseteq N'$. We may work with this N'.

□

Fix any $N \in \mathcal{N}$ s.t. $\alpha = \omega_1 \cap N$, $\{s_{\xi_1}, \varphi_{\xi_1}\} \subseteq N$, and $N_0 \in N$.

Claim. The same $\alpha = \omega_1 \cap N < \omega_1$ is the least s.t. there exists $N' \in \mathcal{N}$ that satisfies $\omega_1 \cap N' = \alpha$ and $\underline{\{s_{\xi_2}, \varphi_{\xi_2}\}} \subseteq N'$.

Proof. Since $\{f, \varphi_{\xi_2}\} \subseteq N_0 \in N$ and $s_{\xi_1} \in N$, we have $s_{\xi_2} = f(s_{\xi_1}) \in N$. Hence,

- $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N$.

Since $\omega_1 \cap N$ is the least with $\{s_{\xi_1}, \varphi_{\xi_1}\} \subseteq N$ and $N_0 \in N$, we must have $N \in \mathrm{suc}(\mathcal{N})$.

Case. $N \in \mathrm{suc}_2(\mathcal{N})$: Let $N_1, N_2 \in \mathcal{N}$ as in the definition of $\mathrm{suc}_2(\mathcal{N})$. Since $N_0 \in \mathcal{N} \cap N = (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2) \cup \{N_1, N_2\}$, we have $k \in \{1, 2\}$ s.t. $N_0 \in (\mathcal{N} \cap N_k) \cup \{N_k\}$. Hence,

- $N_0 \subseteq N_k$.

Since $\omega_1 \cap N$ is the least with $\{s_{\xi_1}, \varphi_{\xi_1}\} \subseteq N$ and $\varphi_{\xi_1} \in N_0 \subseteq N_k \in N$. We have

- $s_{\xi_1} \notin N_k$.

Subclaim. $\{s_{\xi_2}, \varphi_{\xi_2}\} \nsubseteq N_1$ and $\{s_{\xi_2}, \varphi_{\xi_2}\} \nsubseteq N_2$.

Proof. If $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N_k$, then $s_{\xi_1} = f^{-1}(s_{\xi_2}) \in N_k$, as $f \in N_0 \subseteq N_k$. This is absurd. If $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N_{\overline{k}}$, where $\overline{k} \in \{1, 2\}$ and $\overline{k} \neq k$, then $\varphi_{\xi_2} \in N_0 \cap N_{\overline{k}} \subseteq N_k \cap N_{\overline{k}}$ and so $s_{\xi_2} \in N_{\overline{k}} \cap \varphi_{\xi_2} = N_k \cap \varphi_{\xi_2}$. Hence, $s_{\xi_1} = f^{-1}(s_{\xi_2}) \in N_k$, as $f \in N_0 \subseteq N_k$. This is absurd.

□

Subclaim. The same $\alpha = \omega_1 \cap N$ is the least s.t. there exists $\underline{N} \in \mathcal{N}$ that satisfies $\omega_1 \cap \underline{N} = \alpha$ and $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq \underline{N}$,

Proof. Let $\underline{N} \in \mathcal{N}$ s.t, $\omega_1 \cap \underline{N} < \omega_1 \cap N$ and $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq \underline{N}$. Then take $N'' \in \mathcal{N}$ s.t. $\underline{N} \in N''$ and $N'' =_{\omega_1} N$. Then $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N'' \cap N$. Hence, $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq \phi_{N''N}(\underline{N}) \in N$. Since $\phi_{N''N}(\underline{N}) \in (\mathcal{N} \cap N_1) \cup (\mathcal{N} \cap N_2) \cup \{N_1, N_2\}$, we have $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N_1$ or $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N_2$. This is absurd.

□

Case. $N \in \mathrm{suc}_1(\mathcal{N})$: Let $N_3 \in \mathcal{N}$ s.t. $\mathcal{N} \cap N = (\mathcal{N} \cap N_3) \cup \{N_3\}$. Then $N_0 \in (\mathcal{N} \cap N_3) \cup \{N_3\}$. Since $N_0 \subseteq N_3 \in N$ and $\varphi_{\xi_1} \in N_0$, we have $s_{\xi_1} \notin N_3$.

Subclaim. $s_{\xi_2} \notin N_3$ and so $\{s_{\xi_2}, \varphi_{\xi_2}\} \not\subseteq N_3$.

Proof. If $s_{\xi_2} \in N_3$, then $s_{\xi_1} = f^{-1}(s_{\xi_2}) \in N_3$, as $f \in N_3$. This is absurd.

□

Subclaim. The same $\alpha = \omega_1 \cap N < \omega_1$ is the least s.t. there exists $\underline{N} \in \mathcal{N}$ that satisfies $\omega_1 \cap \underline{N} = \alpha$ and $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq \underline{N}$.

Proof. Let $\underline{N} \in \mathcal{N}$ s.t, $\omega_1 \cap \underline{N} < \omega_1 \cap N$ and $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq \underline{N}$. Then take $N'' \in \mathcal{N}$ s.t. $\underline{N} \in N''$ and $N'' =_{\omega_1} N$. Then $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N'' \cap N$. Hence, $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq \phi_{N''N}(\underline{N}) \in N$. Since $\phi_{N''N}(\underline{N}) \in (\mathcal{N} \cap N_3) \cup \{N_3\}$, we have $\{s_{\xi_2}, \varphi_{\xi_2}\} \subseteq N_3$. This is absurd.

□

□

Stage 3. Since $s_{\xi_1} < \varphi_{\xi_1} < s_{\xi_2} < \varphi_{\xi_2}$ in N, we have $c_N(s_{\xi_1}) < c_N(\varphi_{\xi_1}) < c_N(s_{\xi_2}) < c_N(\varphi_{\xi_2}) < c_N(\omega_2) < \omega_1$. Let $n_1 < \omega$ be the least s.t. there exists $a_1 \in \mathcal{A}$ that satisfies $\mathrm{rank}(a_1) = n_1$ and $\{c_N(\varphi_{\xi_1}), c_N(\omega_2)\} \subseteq a_1$. Let $n_2 < \omega$ be the least s.t. there exists $a_2 \in \mathcal{A}$ that satisfies $\mathrm{rank}(a_2) = n_2$ and $\{c_N(\varphi_{\xi_2}), c_N(\omega_2)\} \subseteq a_2$.

Claim. Either $n_1 \neq n_2$ or $c_{a_1}\big(c_N(\varphi_{\xi_1})\big) \neq c_{a_2}\big(c_N(\varphi_{\xi_2})\big)$. Hence, $f(\{u, s\}) \neq f(\{u', s\})$.

Proof. Suppose $n_1 = n_2$. Then $c_N(\omega_2) \in a_1 \cap a_2$ and so $a_1 \cap (c_N(\omega_2) + 1) = a_2 \cap (c_N(\omega_2) + 1)$. Hence, $c_N(\varphi_{\xi_1}) < c_N(\varphi_{\xi_2})$ in a_1 and so $c_{a_1}\big(c_N(\varphi_{\xi_1})\big) < c_{a_1}\big(c_N(\varphi_{\xi_2})\big)$. But $a_1 \cap c_N(\varphi_{\xi_2}) = a_2 \cap c_N(\varphi_{\xi_2})$ and so $c_{a_1}\big(c_N(\varphi_{\xi_2})\big) = c_{a_2}\big(c_N(\varphi_{\xi_2})\big)$. Hence, $c_{a_1}\big(c_N(\varphi_{\xi_1})\big) < c_{a_2}\big(c_N(\varphi_{\xi_2})\big)$.

□

□

Forcing $\mathcal{N}$

For the sake of convenience, prepared is a part of [M] adjusting to this note.

Notation. Let $\kappa = \omega_2$. Let N be a countable subset of H_κ. Denote $N \prec H_\kappa$ to express the substructure $(N, \in \cap (N \times N), \cdots)$ is an elementary substructure of your favorite structure $(H_\kappa, \in \cap (H_\kappa \times H_\kappa), \cdots)$. It was written $N \in \mathcal{C}_0$ in the previous section. For any two countable $N, N' \prec H_\kappa$, considered was the binary relations

$$N =_{\omega_1} N' \text{ iff } \omega_1 \cap N = \omega_1 \cap N',$$

$$N <_{\omega_1} N' \text{ iff } \omega_1 \cap N < \omega_1 \cap N'.$$

If two substructures $(N, \in \cap (N \times N), \cdots)$ and $(N', \in \cap (N' \times N'), \cdots)$ are isomorphic, then there is a unique isomorphism. Denote $\phi_{NN'} : N \longrightarrow N'$ to express $(N, \in \cap (N \times N), \cdots)$ and $(N', \in \cap (N' \times N'), \cdots)$ are isomorphic with the isomorphism $\phi_{NN'}$. Note that if $X, Y \in [H_\kappa]^\omega$, $Y \prec H_\kappa$, and $X \in Y$, then $X = \{e_X(n) \mid n < \omega\} \subseteq Y$ and $X \neq Y$, where $e_X : \omega \longrightarrow X$ onto with $e_X \in Y$.

We force $\mathcal{N}$ in such a way that $T^p \in \mathcal{N}$ and $(\mathcal{N} \cap T^p) \cup \{T^p\} = \mathcal{N}^p$ for every $p \in G$.

Definition. Let $p = \mathcal{N}^p \in P$, if

- (countable with the top) $\mathcal{N}^p \subseteq [H_\kappa]^\omega$, $|\mathcal{N}^p| < \omega_1$, and there exists $T^p \in \mathcal{N}^p$ s.t. $\mathcal{N}^p = (\mathcal{N}^p \cap T^p) \cup \{T^p\}$.
- (elementary) For any $N \in \mathcal{N}^p$, we have $N \prec H_\kappa$.
- (isomorphic) For any $N_1, N_2 \in \mathcal{N}^p$ with $N_1 =_{\omega_1} N_2$, there exists the isomorphism $\phi_{N_1 N_2} : N_1 \longrightarrow N_2$ that also satisfies $\phi_{N_1 N_2}(x) = x$ for all $x \in N_1 \cap N_2$.

- (up) For any $N_3, N_2 \in \mathcal{N}^p$ with $N_3 <_{\omega_1} N_2$, there exists $N_1 \in \mathcal{N}^p$ s.t. $N_3 \in N_1$ and $N_1 =_{\omega_1} N_2$.
- (down) For any $N_1, N_2, N_3 \in \mathcal{N}^p$ with $N_3 \in N_1 =_{\omega_1} N_2$, we have $\phi_{N_1 N_2}(N_3) \in \mathcal{N}^p$.
- (partition) $\mathcal{N}^p = \mathrm{zero}(\mathcal{N}^p) \cup \mathrm{suc}(\mathcal{N}^p) \cup \lim(\mathcal{N}^p)$, where

$$\mathrm{zero}(\mathcal{N}^p) = \{N \in \mathcal{N}^p \mid \mathcal{N}^p \cap N = \emptyset\},$$

$$\mathrm{suc}(\mathcal{N}^p) = \mathrm{suc}_1(\mathcal{N}^p) \cup \mathrm{suc}_2(\mathcal{N}^p),$$

$$\mathrm{suc}_1(\mathcal{N}^p) = \big\{N \in \mathcal{N}^p \mid \text{there exists } N_0 \text{ s.t. } \mathcal{N}^p \cap N = (\mathcal{N}^p \cap N_0) \cup \{N_0\}\big\},$$

$$\mathrm{suc}_2(\mathcal{N}^p) = \big\{N \in \mathcal{N}^p \mid \text{there exists } N_1, N_2 \text{ s.t. the following 3 items hold}\big\},$$

$$N_1 =_{\omega_1} N_2,$$

$$\mathcal{N}^p \cap N = (\mathcal{N}^p \cap N_1) \cup (\mathcal{N}^p \cap N_2) \cup \{N_1, N_2\},$$

$$\Delta := (\omega_2 \cap N_1) \cap (\omega_2 \cap N_2) < (\omega_2 \cap N_1) \setminus \Delta < (\omega_2 \cap N_2) \setminus \Delta \neq \emptyset,$$

$$\lim(\mathcal{N}^p) = \{N \in \mathcal{N}^p \mid \bigcup(\mathcal{N}^p \cap N) = N\}.$$

For $p, q \in P$, let $q \leq p$ in P, if $\mathcal{N}^q \supseteq \mathcal{N}^p$ and $\mathcal{N}^q \cap T^p = \mathcal{N}^p \cap T^p$.

Lemma. (1) (abusive) T^p is unique to p.

(2) $q \leq p$ in P iff $T^p \in \mathcal{N}^q$ and $(\mathcal{N}^q \cap T^p) \cup \{T^p\} = \mathcal{N}^p$.

(3) P is a poset with $P \subseteq H_\kappa$.

Proof. (1): Suppose $\mathcal{N}^p = (\mathcal{N}^p \cap T) \cup \{T\} = (\mathcal{N}^p \cap T') \cup \{T'\}$. Suppose $T \neq T'$. Then $T \in \mathcal{N}^p \cap T'$ and $T' \in \mathcal{N}^p \cap T$. Hence, $T \in T'$ and $T' \in T$. This is absurd.

(2): Let $q \leq p$ in P. Then $T^p \in \mathcal{N}^q$ and $\mathcal{N}^q \cap T^p = \mathcal{N}^p \cap T^p$. Hence, $(\mathcal{N}^q \cap T^p) \cup \{T^p\} = (\mathcal{N}^p \cap T^p) \cup \{T^p\} = \mathcal{N}^p$. Conversely, let $T^p \in \mathcal{N}^q$ and $(\mathcal{N}^q \cap T^p) \cup \{T^p\} = \mathcal{N}^p$. Then $\mathcal{N}^q \supseteq \mathcal{N}^p$ and $((\mathcal{N}^q \cap T^p) \cup \{T^p\}) \cap T^p = \mathcal{N}^p \cap T^p$. Hence, $\mathcal{N}^q \cap T^p = \mathcal{N}^p \cap T^p$.

(3): (reflexive): Let $p \in P$. Then $T^p \in \mathcal{N}^p$ and $(\mathcal{N}^p \cap T^p) \cup \{T^p\} = \mathcal{N}^p$. Hence, $p \leq p$ in P.

(transitive): Let $r \leq q \leq p$ in P. Then $\mathcal{N}^r \supseteq \mathcal{N}^q \supseteq \mathcal{N}^p$, $\mathcal{N}^r \cap T^q = \mathcal{N}^q \cap T^q$, and $\mathcal{N}^q \cap T^p = \mathcal{N}^p \cap T^p$. Hence, $\mathcal{N}^r \supseteq \mathcal{N}^p$ and $T^q \supseteq T^p$. Hence, $\mathcal{N}^r \cap T^p = (\mathcal{N}^r \cap T^q) \cap T^p = (\mathcal{N}^q \cap T^q) \cap T^p = \mathcal{N}^q \cap T^p = \mathcal{N}^p \cap T^p$. □

Lemma. (Dense) For any $p \in P$ and $x \in H_\kappa$, there exists $q \leq p$ in P s.t. $x \in T^q$.

Proof. Let $N \in [H_\kappa]^\omega$ with $\{x, p\} \subseteq N \prec H_\kappa$. Let $\mathcal{N}^q = \mathcal{N}^p \cup \{N\}$. Then $\mathcal{N}^p \subseteq N$ and $T^p \subseteq N$. Hence, $\mathcal{N}^q \cap N = \mathcal{N}^p = (\mathcal{N}^p \cap T^p) \cup \{T^p\} = (\mathcal{N}^q \cap T^p) \cup \{T^p\}$. Hence,

- $\mathcal{N}^q \cap N = (\mathcal{N}^q \cap T^p) \cup \{T^p\}$.
- (localized) For any $X \in \mathcal{N}^p$, we have $\mathcal{N}^q \cap X = (\mathcal{N}^q \cap N) \cap X = \mathcal{N}^p \cap X$.
- (countable with the top) $\mathcal{N}^q = (\mathcal{N}^q \cap N) \cup \{N\}$.
- (elementary) For any $X \in \mathcal{N}^q$, we have $X \prec H_\kappa$.
- (isomorphism) Let $X_1, X_2 \in \mathcal{N}^q$ with $X_1 =_{\omega_1} X_2$. We want the isomorphism $\phi_{X_1 X_2} : X_1 \longrightarrow X_2$ that satisfies $\phi_{X_1 X_2}(x) = x$ for all $x \in X_1 \cap X_2$. But we may assume that $X_1 \neq X_2$. Then $\{X_1, X_2\} \subseteq \mathcal{N}^p$. Hence, $\phi_{X_1 X_2}$ exists.
- (up) Let $X_3, X_2 \in \mathcal{N}^q$ with $X_3 <_{\omega_1} X_2$. If $X_2 = N$, then $X_3 \in N =_{\omega_1} X_2$. If $X_2 \neq N$, then $\{X_3, X_2\} \subseteq \mathcal{N}^p$. Hence, there exists $X_1 \in \mathcal{N}^p \subseteq \mathcal{N}^q$ with $X_3 \in X_1 =_{\omega_1} X_2$.

- (down) Let $X_1, X_2, X_3 \in \mathcal{N}^q$ and $X_3 \in X_1 =_{\omega_1} X_2$. We may assume $X_1 \neq X_2$. Then $\{X_1, X_2\} \subseteq \mathcal{N}^p$. Hence, $X_3 \in \mathcal{N}^q \cap X_1 = \mathcal{N}^p \cap X_1$. Hence, $X_3 \in \mathcal{N}^p$ and so $\phi_{X_1 X_2}(X_3) \in \mathcal{N}^p \subseteq \mathcal{N}^q$.
- (partition) We observe $\mathcal{N}^q = \text{zero}(\mathcal{N}^q) \cup \text{suc}(\mathcal{N}^q) \cup \lim(\mathcal{N}^q)$ as follows.

  * $\text{zero}(\mathcal{N}^p) \subseteq \text{zero}(\mathcal{N}^q)$: Let $X \in \text{zero}(\mathcal{N}^p)$. Then $\mathcal{N}^q \cap X = \mathcal{N}^p \cap X = \emptyset$.

  * $\text{suc}_1(\mathcal{N}^p) \subseteq \text{suc}_1(\mathcal{N}^q)$: Let $X \in \text{suc}_1(\mathcal{N}^p)$. Then there exists X_0 s.t. $\mathcal{N}^p \cap X = (\mathcal{N}^p \cap X_0) \cup \{X_0\}$. Since $X, X_0 \in \mathcal{N}^p$, we have $\mathcal{N}^q \cap X = \mathcal{N}^p \cap X$ and $\mathcal{N}^q \cap X_0 = \mathcal{N}^p \cap X_0$. We conclude $\mathcal{N}^q \cap X = (\mathcal{N}^q \cap X_0) \cup \{X_0\}$.

  * $\text{suc}_2(\mathcal{N}^p) \subseteq \text{suc}_2(\mathcal{N}^q)$: Similar. Let $X \in \text{suc}_2(\mathcal{N}^p)$. Then there exist X_1, X_2 s.t. $\mathcal{N}^p \cap X = (\mathcal{N}^p \cap X_1) \cup (\mathcal{N}^p \cap X_2) \cup \{X_1, X_2\}$ among others. Since $X, X_1, X_2 \in \mathcal{N}^p$, we have $\mathcal{N}^q \cap X = \mathcal{N}^p \cap X$, $\mathcal{N}^q \cap X_1 = \mathcal{N}^p \cap X_1$, and $\mathcal{N}^q \cap X_2 = \mathcal{N}^p \cap X_2$. Hence, $\mathcal{N}^q \cap X = (\mathcal{N}^q \cap X_1) \cup (\mathcal{N}^q \cap X_2) \cup \{X_1, X_2\}$ among others.

  * $\lim(\mathcal{N}^p) \subseteq \lim(\mathcal{N}^q)$: Let $X \in \lim(\mathcal{N}^p)$. Then $X = \bigcup(\mathcal{N}^p \cap X) = \bigcup(\mathcal{N}^q \cap X)$.

  * $N \in \text{suc}_1(\mathcal{N}^q)$: We have $\mathcal{N}^q \cap N = (\mathcal{N}^q \cap T^p) \cup \{T^p\}$.

  Hence, $q \in P$ with $x \in T^q = N$. Since $\mathcal{N}^p \subseteq \mathcal{N}^q$ and $\mathcal{N}^q \cap T^p = \mathcal{N}^p \cap T^p$, we have $q \leq p$ in P.

□

Lemma. P is σ-closed.

Proof. Let $p_{n+1} \leq p_n$ and $p_n \neq p_{n+1}$ for all $n < \omega$. First observe $T^{p_n} \neq T^{p_{n+1}}$. Suppose not. Since $\mathcal{N}^{p_{n+1}} \cap T^{p_n} = \mathcal{N}^{p_n} \cap T^{p_n}$, we have $\mathcal{N}^{p_{n+1}} = (\mathcal{N}^{p_{n+1}} \cap T^{p_{n+1}}) \cup \{T^{p_{n+1}}\} = (\mathcal{N}^{p_n} \cap T^{p_n}) \cup \{T^{p_n}\} = \mathcal{N}^{p_n}$. This is absurd. Let $T = \bigcup\{T^{p_n} \mid n < \omega\}$. Since $T^{p_{n+1}} \neq T^{p_n}$ and $T^{p_n} \in \mathcal{N}^{p_{n+1}} = (\mathcal{N}^{p_{n+1}} \cap T^{p_{n+1}}) \cup \{T^{p_{n+1}}\}$, we have $T^{p_n} \in T^{p_{n+1}}$ and so $T^{p_n} \in T$. Since T is the union of the $\in$-increasing countable elementary substructures of H_κ, T itself is also a countable elementary substructure of H_κ. Let $p = \mathcal{N}^p = \bigcup\{\mathcal{N}^{p_n} \mid n < \omega\} \cup \{T\}$. We claim $p \in P$ and for all $n < \omega$, $p \leq p_n$ in P. We check these item by item. Let $n < \omega$ and $X \in \mathcal{N}^{p_n}$. Then for all $k < \omega$ with $k \geq n$, since $p_k \leq p_n$ in P, we have $\mathcal{N}^{p_k} \cap T^{p_n} = \mathcal{N}^{p_n} \cap T^{p_n}$ and so $\mathcal{N}^{p_k} \cap X = \mathcal{N}^{p_n} \cap X$. In turn, we have $\mathcal{N}^p \cap X = \mathcal{N}^{p_n} \cap X$. Hence,

- (localized) For any $X \in \mathcal{N}^{p_n}$, we have $\mathcal{N}^p \cap X = \mathcal{N}^{p_n} \cap X$.
- (countable with the top) $p \subseteq [H_\kappa]^\omega$ and $|\,p\,| < \omega_1$. We observe $\mathcal{N}^p = (\mathcal{N}^p \cap T) \cup \{T\}$. But we have $\mathcal{N}^p \cap T = \bigcup\{\mathcal{N}^{p_n} \mid n < \omega\}$.
- (elementary) $\mathcal{N}^p = \bigcup\{\mathcal{N}^{p_n} \mid n < \omega\} \cup \{T\}$. Hence, for any $X \in \mathcal{N}^p$, we have $X \prec H_\kappa$.
- (isomorphism) Let $N_1, N_2 \in \mathcal{N}^p$ with $N_1 =_{\omega_1} N_2$. We may assume $N_1 \neq N_2$. Then $\{N_1, N_2\} \subseteq \mathcal{N}^{p_n}$ for some $n < \omega$. Hence there is the isomorphism $\phi_{N_1 N_2} : N_1 \longrightarrow N_2$ s.t. $\phi_{N_1 N_2}(x) = x$ for all $x \in N_1 \cap N_2$.
- (up) Let $N_3, N_2 \in \mathcal{N}^p$ with $N_3 <_{\omega_3} N_2$. We may assume $N_2 = T$. Then $N_3 \in T =_{\omega_1} N_2$.
- (down) Let $N_1, N_2, N_3 \in \mathcal{N}^p$ s.t. $N_3 \in N_1 =_{\omega_1} N_2$. We may assume $N_1 \neq N_2$. Then $N_1, N_2, N_3 \in \mathcal{N}^{p_n}$ for some $n < \omega$. Hence, $\phi_{N_1 N_2}(N_3) \in \mathcal{N}^{p_n} \subseteq \mathcal{N}^p$.
- (partition) Need to observe $\mathcal{N}^p = \text{zero}(\mathcal{N}^p) \cup \text{suc}(\mathcal{N}^p) \cup \lim(\mathcal{N}^p)$.

  * $\text{zero}(\mathcal{N}^{p_n}) \subseteq \text{zero}(\mathcal{N}^p)$: Let $X \in \text{zero}(\mathcal{N}^{p_n})$. Since $X \in \mathcal{N}^{p_n}$, we have $\mathcal{N}^p \cap X = \mathcal{N}^{p_n} \cap X$. Hence, $\mathcal{N}^p \cap X = \mathcal{N}^{p_n} \cap X = \emptyset$.

  * $\text{suc}_1(\mathcal{N}^{p_n}) \subseteq \text{suc}_1(\mathcal{N}^p)$: Let $X \in \text{suc}_1(\mathcal{N}^{p_n})$. Then there exists X_0 s.t. $\mathcal{N}^{p_n} \cap X = (\mathcal{N}^{p_n} \cap X_0) \cap \{X_0\}$. Since $X, X_0 \in \mathcal{N}^{p_n}$, we have $\mathcal{N}^p \cap X = \mathcal{N}^{p_n} \cap X$ and $\mathcal{N}^p \cap X_0 = \mathcal{N}^{p_n} \cap X_0$. Hence, $\mathcal{N}^p \cap X = (\mathcal{N}^p \cap X_0) \cap \{X_0\}$.

  * $\text{suc}_2(\mathcal{N}^{p_n}) \subseteq \text{suc}_2(\mathcal{N}^p)$: Let $X \in \text{suc}_2(\mathcal{N}^{p_n})$. Then there exists X_1, X_2 s.t. $X_1 =_{\omega_1} X_2$, $\mathcal{N}^{p_n} \cap X = (\mathcal{N}^{p_n} \cap X_1) \cup (\mathcal{N}^{p_n} \cap X_2) \cup \{X_1, X_2\}$, and $\Delta := (\omega_2 \cap X_1) \cap (\omega_2 \cap X_2) < (\omega_2 \cap X_1) \setminus \Delta < (\omega_2 \cap X_2) \setminus \Delta \neq \emptyset$. Since $\mathcal{N}^p \cap X = \mathcal{N}^{p_n} \cap X$, $\mathcal{N}^p \cap X_1 = \mathcal{N}^{p_n} \cap X_1$, and $\mathcal{N}^p \cap X_2 = \mathcal{N}^{p_n} \cap X_2$, conclude $\mathcal{N}^p \cap X = (\mathcal{N}^p \cap X_1) \cup (\mathcal{N}^p \cap X_2) \cup \{X_1, X_2\}$.

* $\lim(\mathcal{N}^{p_n}) \subseteq \lim(\mathcal{N}^p)$: Let $X \in \lim(\mathcal{N}^{p_n})$. Then $X = \bigcup(\mathcal{N}^{p_n} \cap X) = \bigcup(\mathcal{N}^p \cap X)$.
* $T \in \lim(\mathcal{N}^p)$: Since $T^{p_n} \in \mathcal{N}^{p_n} \cap T \subseteq \mathcal{N}^p \cap T$ and $T = \bigcup\{T^{p_n} \mid n < \omega\}$, conclude $T = \bigcup(\mathcal{N}^p \cap T)$.

Hence, $p \in P$. Since $\mathcal{N}^p \supseteq \mathcal{N}^{p_n}$ and $\mathcal{N}^p \cap T^{p_n} = \mathcal{N}^{p_n} \cap T^{p_n}$, conclude $p \leq p_n$ in P for all $n < \omega$.

□

Lemma. (CH) P has the ω_2-cc.

Proof. Let us denote $S^2_1 = \{i < \kappa \mid \mathrm{cf}(i) = \omega_1\}$. Let $\langle p_i \mid i \in S^2_1\rangle$ be an indexed family of conditions of P. Let N_i be a countable elementary substructure of H_κ with $\{i, p_i\} \subseteq N_i$. By CH, we may assume that the N_i form a Δ-system, furthermore for $i < j$, $\Delta := (\omega_2 \cap N_i) \cap (\omega_2 \cap N_j) < (\omega_2 \cap N_i) \setminus \Delta < (\omega_2 \cap N_j) \setminus \Delta \neq \emptyset$, the N_i are isomorphic s.t. for $i < j$, the isomorphism $\phi_{N_iN_j} : N_i \longrightarrow N_j$ satisfies $\phi_{N_iN_j}(i) = j$, $\phi_{N_iN_j}(p_i) = p_j$, and $\phi_{N_iN_j}(x) = x$ for all $x \in N_i \cap N_j$. Fix any $i < j$ and let T be any countable elementary substructure of H_κ s.t. $\{N_i, N_j\} \subseteq T$. Let $p = \mathcal{N}^p = \mathcal{N}^{p_i} \cup \mathcal{N}^{p_j} \cup \{N_i, N_j, T\}$. We claim $p \in P$ and $p \leq p_i, p_j$ in P. We check these item by item. Since $\mathcal{N}^{p_i}$ is countable and $\mathcal{N}^{p_i} \in N_i$, we have $\mathcal{N}^{p_i} \subseteq N_i$. Similarly, we have $\mathcal{N}^{p_j} \subseteq N_j$.

- $\mathcal{N}^p \cap N_i = \mathcal{N}^{p_i}$ and $\mathcal{N}^p \cap N_j = \mathcal{N}^{p_j}$. To see $\mathcal{N}^p \cap N_i = \mathcal{N}^{p_i}$, let $X \in \mathcal{N}^p \cap N_i$. Then $X \notin \{N_i, N_j, T\}$. If $X \in \mathcal{N}^{p_j}$, then $X \in N_i \cap N_j$. Hence, $X = \phi_{N_jN_i}(X) \in \phi_{N_jN_i}(\mathcal{N}^{p_j}) = \mathcal{N}^{p_i}$.
- (localized) For $X \in \mathcal{N}^{p_i}$, $\mathcal{N}^p \cap X = \mathcal{N}^{p_i} \cap X$. Similarly for $X \in \mathcal{N}^{p_j}$, $\mathcal{N}^p \cap X = \mathcal{N}^{p_j} \cap X$.
- (countable with the top) $\mathcal{N}^p \subseteq [H_\kappa]^\omega$, $|\mathcal{N}^p| < \omega_1$, and $\mathcal{N}^p = (\mathcal{N}^p \cap T) \cup \{T\}$.
- (elementary) For any $X \in \mathcal{N}^p$, we have $X \prec H_\kappa$.
- (isomorphic) Let $X_1, X_2 \in \mathcal{N}^p$ with $X_1 =_{\omega_1} X_2$. Let $X_1 \in \mathcal{N}^{p_i}$ and $X_2 \in \mathcal{N}^{p_j}$. It suffices to show that there exists an isomorphism $\phi : X_1 \longrightarrow X_2$ that satisfies $\phi(x) = x$ for all $x \in X_1 \cap X_2$. Let $Y = \phi_{N_iN_j}(X_1)$. Then $Y \in \mathcal{N}^{p_j}$ and $Y =_{\omega_1} X_1 =_{\omega_1} X_2$. Hence, there exists the isomorphism $\phi_{YX_2} : Y \longrightarrow X_2$ that satisfies $\phi_{YX_2}(x) = x$ for all $x \in Y \cap X_2$. Let $\phi = \big\{(x, y) \mid x \in X_1, y = \phi_{YX_2}\big(\phi_{N_iN_j}(x)\big)\big\}$. Then $\phi : X_1 \longrightarrow X_2$ is an isomorphism. Let $x \in X_1 \cap X_2$. Then $x \in N_i \cap N_j$ and so $\phi_{N_iN_j}(x) = x \in Y \cap X_2$. Hence, $\phi(x) = \phi_{YX_2}\big(\phi_{N_iN_j}(x)\big) = \phi_{YX_2}(x) = x$.
- (up) Let $X_3, X_2 \in \mathcal{N}^p$ with $X_3 <_{\omega_1} X_2$. We want $X_1 \in \mathcal{N}^p$ with $X_3 \in X_1 =_{\omega_1} X_2$. If $\{X_3, X_2\} \subseteq \mathcal{N}^{p_i}$ or $\{X_3, X_2\} \subseteq \mathcal{N}^{p_j}$, then there is nothing to prove. It suffices to deal with the following incomplete but essential list of cases.

Case. $X_3 \in \mathcal{N}^{p_i}$ and $X_2 \in \mathcal{N}^{p_j}$: Let $Y = \phi_{N_jN_i}(X_2)$. Then $Y \in \mathcal{N}^{p_i}$ and $X_3 <_{\omega_1} X_2 =_{\omega_1} Y$. Hence, there exists $X_1 \in \mathcal{N}^{p_i}$ with $X_3 \in X_1 =_{\omega_1} Y =_{\omega_1} X_2$.

Case. $X_3 \in \mathcal{N}^{p_i}$ and $X_2 = N_j$: Then $X_3 \in N_i =_{\omega_1} N_j =_{\omega_1} X_2$.

Case. $X_3 \in \mathcal{N}^{p_i}$ and $X_2 = N_i$: Then $X_3 \in N_i =_{\omega_1} X_2$.

Case. $X_3 \in \mathcal{N}^{p_i}$ and $X_2 = T$: Then $X_3 \in T =_{\omega_1} X_2$.

Case. $X_3 = N_i$: Then $X_2 = T$ and so $X_3 \in T =_{\omega_1} X_2$.

- (down) Let $X_1, X_2, X_3 \in \mathcal{N}^p$ with $X_3 \in X_1 =_{\omega_1} X_2$. We want to show $\phi_{X_1X_2}(X_3) \in \mathcal{N}^p$. We may assume $X_1 \neq X_2$. It suffices to deal with the following incomplete but essential list of cases.

Case. $\{X_1, X_2\} \subseteq \mathcal{N}^{p_i}$: Then $X_3 \in \mathcal{N}^p \cap N_i = \mathcal{N}^{p_i}$. Hence, $\phi_{X_1X_2}(X_3) \in \mathcal{N}^{p_i}$.

Case. $X_1 \in \mathcal{N}^{p_i}$ and $X_2 \in \mathcal{N}^{p_j}$: Let $Y = \phi_{N_iN_j}(X_1)$. Since $X_3 \in \mathcal{N}^p \cap X_1 = \mathcal{N}^{p_i} \cap X_1$, we have $X_3 \in \mathcal{N}^{p_i} \cap X_1$. Hence, $Y = \phi_{N_iN_j}(X_1) \in \phi_{N_iN_j}(\mathcal{N}^{p_i}) = \mathcal{N}^{p_j}$ and $\phi_{N_iN_j}(X_3) \in \phi_{N_iN_j}(\mathcal{N}^{p_i} \cap X_1) = \phi_{N_iN_j}(\mathcal{N}^{p_i}) \cap \phi_{N_iN_j}(X_1) = \mathcal{N}^{p_j} \cap Y$. Since $Y, X_2 \in \mathcal{N}^{p_j}$, $Y =_{\omega_1} X_1 =_{\omega_1} X_2$, and $\phi_{N_iN_j}(X_3) \in \mathcal{N}^{p_j} \cap Y$, we have $\phi_{X_1X_2}(X_3) = \phi_{YX_2}\big(\phi_{N_iN_j}(X_3)\big) \in \mathcal{N}^{p_j}$.

Case. $X_1 = N_i$ and $X_2 = N_j$: Then $X_3 \in \mathcal{N}^p \cap N_i = \mathcal{N}^{p_i}$ and so $\phi_{N_iN_j}(X_3) \in \mathcal{N}^{p_j}$.

- (partition) Need to observe $\mathcal{N}^p = \mathrm{zero}(\mathcal{N}^p) \cup \mathrm{suc}(\mathcal{N}^p) \cup \lim(\mathcal{N}^p)$.

$*$ $\mathrm{zero}(\mathcal{N}^{p_i}) \subseteq \mathrm{zero}(\mathcal{N}^p)$: Let $X \in \mathrm{zero}(\mathcal{N}^{p_i})$. Then $\mathcal{N}^p \cap X = \mathcal{N}^{p_i} \cap X = \emptyset$.

$*$ $\mathrm{suc}_1(\mathcal{N}^{p_i}) \subseteq \mathrm{suc}_1(\mathcal{N}^p)$: Let $X \in \mathrm{suc}_1(\mathcal{N}^{p_i})$. Then there exists X_0 s.t. $\mathcal{N}^{p_i} \cap X = (\mathcal{N}^{p_i} \cap X_0) \cup \{X_0\}$. But $\mathcal{N}^p \cap X = \mathcal{N}^{p_i} \cap X$ and $\mathcal{N}^p \cap X_0 = \mathcal{N}^{p_i} \cap X_0$. Hence, $\mathcal{N}^p \cap X = (\mathcal{N}^p \cap X_0) \cup \{X_0\}$.

$*$ $\mathrm{suc}_2(\mathcal{N}^{p_i}) \subseteq \mathrm{suc}_2(\mathcal{N}^p)$: Similar with $X_1, X_2 \in \mathcal{N}^{p_i} \cap X$.

$*$ $\lim(\mathcal{N}^{p_i}) \subseteq \lim(\mathcal{N}^p)$: Let $X \in \lim(\mathcal{N}^{p_i})$. Then $X = \bigcup(\mathcal{N}^{p_i} \cap X) = \bigcup(\mathcal{N}^p \cap X)$.

$*$ $N_i, N_j \in \mathrm{suc}_1(\mathcal{N}^p)$: To show $N_i \in \mathrm{suc}_1(\mathcal{N}^p)$, just calculate $\mathcal{N}^p \cap N_i = \mathcal{N}^{p_i} = (\mathcal{N}^{p_i} \cap T^{p_i}) \cup \{T^{p_i}\} = (\mathcal{N}^p \cap T^{p_i}) \cup \{T^{p_i}\}$.

$*$ $T \in \mathrm{suc}_2(\mathcal{N}^p)$: Remember that $N_i, N_j \in \mathcal{N}^p \cap T$, $N_i =_{\omega_1} N_j$, and $\Delta := (\omega_2 \cap N_i) \cap (\omega_2 \cap N_j) < (\omega_2 \cap N_i) \setminus \Delta < (\omega_2 \cap N_j) \setminus \Delta \neq \emptyset$. Just calculate $\mathcal{N}^p \cap T = (\mathcal{N}^{p_i} \cup \{N_i\}) \cup (\mathcal{N}^{p_j} \cup \{N_j\}) = \big((\mathcal{N}^p \cap N_i) \cup \{N_i\}\big) \cup \big((\mathcal{N}^p \cap N_j) \cup \{N_j\}\big) = (\mathcal{N}^p \cap N_i) \cup (\mathcal{N}^p \cap N_j) \cup \{N_i, N_j\}$.

□

Lemma. $P \subseteq H_\kappa$ preserves the cofinalities, cardinalities, and GCH.

Proof. Since P is σ-closed, P preserves CH. For $\lambda \geq \omega_1$, we have $(|P|^{\omega_1})^\lambda = ((2^{\omega_1})^{\omega_1})^\lambda = 2^\lambda = \lambda^+$. Hence, $2^\lambda = \lambda^+$ in the generic extensions.

□

Definition. Let G be P-generic over the ground model V. In the generic extension $V[G]$, let

$$\mathcal{N} = \bigcup\{\mathcal{N}^p \mid p \in G\}.$$

Lemma. $\mathcal{N}$ satisfies (elementary), (isomorphic), (up), (down), (partition), and (cofinal).

Proof. We check item by item.

- (elementary) Let $N \in \mathcal{N}$. Then there exists $p \in G$ with $N \in \mathcal{N}^p$. Hence, N is a countable elementary substructure of $(H_\kappa)^V$.
- (isomorphic) Let $N_1, N_2 \in \mathcal{N}$ with $N_1 =_{\omega_1} N_2$. Then there exists $p \in G$ with $\{N_1, N_2\} \subseteq \mathcal{N}^p$. Hence, there exists the unique isomorphism $\phi_{N_1N_2} : N_1 \longrightarrow N_2$ that satisfies $\phi_{N_1N_2}(x) = x$ for all $x \in N_1 \cap N_2$.
- (up) Let $N_3, N_2 \in \mathcal{N}$ with $N_3 <_{\omega_1} N_2$. Then there exists $p \in G$ with $\{N_3, N_2\} \subseteq \mathcal{N}^p$. Hence, there exists $N_1 \in \mathcal{N}^p$ with $N_3 \in N_1 =_{\omega_1} N_2$.
- (down) Let $N_1, N_2, N_3 \in \mathcal{N}$ with $N_3 \in N_1 =_{\omega_1} N_2$. Then there exists $p \in G$ with $\{N_1, N_2, N_3\} \subseteq \mathcal{N}^p$. Hence $\phi_{N_1N_2}(N_3) \in \mathcal{N}^p$.
- (partition) $\mathcal{N} = \mathrm{zero}(\mathcal{N}) \cup \mathrm{suc}(\mathcal{N}) \cup \lim(\mathcal{N})$. To show this observe
- (localized) If $p \in G$ and $N \in \mathcal{N}^p$, then $\mathcal{N} \cap N = \mathcal{N}^p \cap N$. To show this, let $X \in \mathcal{N} \cap N$. Then $X \in N \subseteq T^p$. Take $q \in G$ s.t. $q \leq p$ in P and $X \in \mathcal{N}^q$. Since $\mathcal{N}^q \cap T^p = \mathcal{N}^p \cap T^p$, we have $X \in \mathcal{N}^p$.

  $*$ For $p \in G$, $\mathrm{zero}(\mathcal{N}^p) \subseteq \mathrm{zero}(\mathcal{N})$: Let $N \in \mathrm{zero}(\mathcal{N}^p)$. Then $\mathcal{N} \cap N = \mathcal{N}^p \cap N = \emptyset$.

  $*$ For $p \in G$, $\mathrm{suc}_1(\mathcal{N}^p) \subseteq \mathrm{suc}_1(\mathcal{N})$: Let $N \in \mathrm{suc}_1(\mathcal{N}^p)$. Then there exists N_0 s.t. $\mathcal{N}^p \cap N = (\mathcal{N}^p \cap N_0) \cup \{N_0\}$. Since $\mathcal{N} \cap N = \mathcal{N}^p \cap N$ and $\mathcal{N} \cap N_0 = \mathcal{N}^p \cap N_0$, conclude $\mathcal{N} \cap N = (\mathcal{N} \cap N_0) \cup \{N_0\}$.

  $*$ For $p \in G$, $\mathrm{suc}_2(\mathcal{N}^p) \subseteq \mathrm{suc}_2(\mathcal{N})$: Similarly with $N_1, N_2 \in \mathcal{N}^p$.

  $*$ For $p \in G$, $\lim(\mathcal{N}^p) \subseteq \lim(\mathcal{N})$: Let $N \in \lim(\mathcal{N}^p)$. Then $N = \bigcup(\mathcal{N}^p \cap N) = \bigcup(\mathcal{N} \cap N)$.
- (cofinal) $\bigcup \mathcal{N} = (H_\kappa)^V$. For any $x \in (H_\kappa)^V$, there exists $p \in G$ with $x \in T^p$.

□

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