時間遅れのある微分方程式の周期解の話題
- 黎明期の研究 -
Periodic solutions to delay differential equations and related topics
- Pioneering works in early days -

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Contents

1 Introduction

It is known that a certain class of differential equations with delayed negative feedback may cause instability of an equilibrium and emergence of oscillations, even chaotic behaviors in some models. In 1970s, various mathematical approaches to such dynamical behaviors were attempted. Specifically, theories for the existence of periodic solutions were developed in a large class of differential equations with time delays by applying fixed point theorems and the Hopf bifurcation theory. As a result, around 1980, we got a rather clear scope in this research field. Nonetheless, there still remain fundamental but hard unsolved problems.

In this article the author will survey some results from those contributions which seems to be significant, but topics are selected in his own interest. The results stated in this short article are not updated by the recent achievements. The author would be glad if the readers could add recent progresses for these topics.

2 Delay models

In this section we pick up several modes of differential equations with time delay. The models stated below, except for the last one, come from a survey paper U. an der Heiden [1], which is certainly old but still useful for learning backgrounds of the model equations. Since we don't go to the details of the models, we would like to suggest that the readers should refer to [1] for the details.

A) Model of biological control loops: Landahl [32] introduced delays in an equation of mRNA production model by Goodwin [14] as

$$\dot{S}_1 = g(S_n(t-\tau_n)) - b_1 S_1(t),$$
$$\dot{S}_i = g_{i-1} S_{i-1}(t-\tau_{i-1}) - b_i S_i(t) \quad (i = 2, 3, \dots, n),$$

$$g(S_n) = \frac{K}{1 + a S_n^p}, \quad (K > 0, a > 0),$$

where S_1 stands for the concentration of an mRNA, S_i $(i = 2, \cdots, n-1)$ are intermediate enzymes and S_n is the end product encoded by mRNA (Fig.1) .

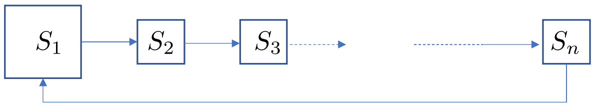

Figure 1: Loop of chain reactions

B) Control of production of red blood cells and hematopoietic disease: Lasota-Wazewska modeled the growth of blood cells by

$$\dot{N} = -\nu N(t) + \rho e^{-\gamma N(t-\tau)},$$

where $N(t)$ denotes the number of blood cells, ν is the death rate and τ is the time to produce a red blood cell. Later existence of periodic solutions was established by Chow [6].

Mackey-Glass [33] proposed the regulation of hematopoiesis (造血) by modifying g as

$$g(N) = \frac{\alpha N}{\kappa + N^r}, \qquad \alpha,\ \kappa > 0,\ r > 1.$$

In this model a chaotic behavior was numerically shown for appropriate parameter values and it looks that the behavior corresponds to an irregular change of blood cell concentration in chronic granulocytic leukemia disease (慢性骨髄性白血病).

C) Delay in neural interactions: In the theory of the compound eye (複眼) of the horseshoe crab (兜がに), optic nerve fibers leading away from the single eye (=ommatidia) are interconnected by collaterals. The activity of a fiber is decreased if activity is present in neighboring fibers. This inhibitory, however, works with a long delay. Coleman and Renninger [9] proposed the next model to describe how the rate of nerve impulses vary. Using the variables and parameters,

R : domain of ommatidia,
$r(x,t)$: rate of nerve impulses at $x \in \mathbb{R}$,
$E(x,t)$: excitation by light,
$K_L(x,y)$: lateral inhibition of other fibers at $y \in \mathbb{R}$,
K_S : strength of self-inhibition,

they modeled

$$r(x,t) = m\left[E - \frac{K_s}{\delta_s}\int_0^\infty e^{-s/\delta_s} r(t-s)ds \right.$$
$$\left. - \frac{1}{\delta_L}\int_0^\infty e^{-s/\delta_L}\int_{\mathbb{R}\setminus\{x\}} K_L(x,y) r(y, t-\tau-s))dyds\right],$$

where $m(z) := \frac{|z|+z}{2}$. For simplicity of analysis they reduce the equation to a simple one by assuming $r(x,t) = r(t), \delta = \delta_s = \delta_L$ as

$$r(t) = m\left[E - \delta\int_0^\infty e^{-s/\delta}(K_s r(t-s) + K_L r(t-\tau-s))ds\right].$$

Then one can write the equation as $r(t) = m[x(t)]$, where $x(t)$ stands for the generator potential of eccentric cell in the Limulus eye and satisfies a differential equation with delay,

$$\delta\dot{x} = E - x(t) - K_s m(x(t)) - K_L m(x(t-\tau)).$$

D) Reaction times and behavior: In the theory of balancing Schurer (1948) gave a model for balancing a stick on the top of a finger. Let $x(t)$ be the deviation from the vertical equilibrium position. Then the model is given by

$$\ddot{x} = ax(t) - h\dot{x}(t), \qquad a,\ h \geq 0,$$

where $ax(t)$ is the acceleration moment. By introducing $-bx(t-r) - c\dot{x}(t-\tau)$ acting as the balancing moment, he got to the model

$$\ddot{x} = ax(t) - h\dot{x}(t) - bx(t-r) - c\dot{x}(t-\tau).$$

The coefficients b and c can be varied by training.

E) Models of optical chaos: Ikeda [23] considered the instability of transmitted light by a ring cavity system, which is observed in the experiment device in Fig.2. Ikeda-Daido-Akioto [25] examined a reduced model given by

$$E(t) = A + BE(t-t_R)\exp(i[\varphi(t) - \varphi_0]),$$
$$\gamma^{-1}\dot{\varphi} = -\varphi(t) + \mathrm{sgn}(n_2)|E(t-t_R)|^2.$$

In addition, Ikeda-Kondo-Akimoto [26] examined some global bifurcation structure for

$$\dot{x} = -x(t) + f(x(t-t_R);\mu), \quad f(x;\mu) = \pi\mu[1 + 2B\cos(x - x_0)],$$

where μ is proportional to the power of the incident light and B stands for the dissipation of the electromagnetic field in the cavity. See also Hopf-Kaplan-Gibbs-Shoemaker [21] and Ikeda [24].

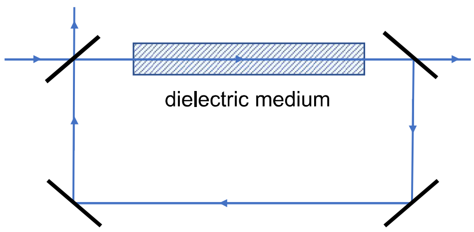

Figure 2: Optical device

3 Studies for periodic solutions

Henceforth we let $C[-r, 0]$ be the Banach space of continuous functions defined on $[-r, 0]$ with values $\mathbb{R}$ or $\mathbb{R}^n$ with supremum norm. We use the notation x_t by $x_t(\theta) = x(t+\theta)$ $(-r \le \theta \le 0)$ for a solution $x(t)$ of autonomous delay equation with initial condition in $C[-r, 0]$.

3.1 Hutchinson-Write equation

The following logistic equation with time delay is introduced by an ecologist Hutchinson [22] to show a mechanism of oscillatory phenomena observed in the growth process of a single species:

$$\frac{du}{dt} = a\left(1 - \frac{u(t-\tau)}{K}\right)u. \tag{3.1}$$

This equation is converted by the new variable

$$v(t) = \frac{u(rt)}{K} - 1,$$

into

$$\frac{dv}{dt} = -a\,(1 + v(t))\,v(t-1). \tag{3.2}$$

Write [42] examined some qualitative behavior of solutions to (3.2) in his own interest. The equation is often called the Hutchinson-Write equation.

The equation (3.2) (or (3.1)) exhibits a stable periodic solution for $a > \pi/2$. Jones [27] proved the existence of a periodic solution for $a > \pi/2$ by using a fixed point theorem. In fact, for any initial data φ in

$$K := \{\varphi \in C[-r, 0] : \varphi(\theta) \ge 0, \quad \varphi(-1) = 0\},$$

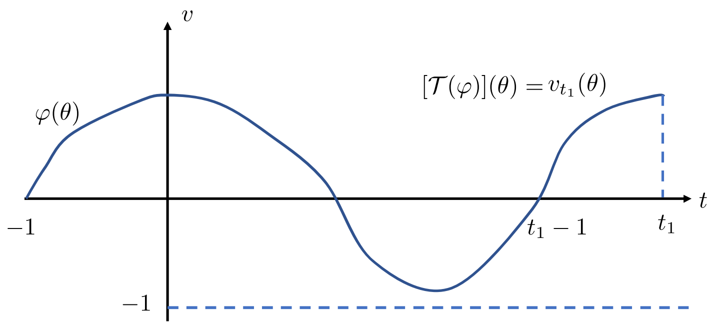

Figure 3: Map $\mathcal{T}$ defined by a solution

the solution $v(t;\varphi)$ with $v(\theta;\varphi) = \varphi(\theta)$ yields a map $\mathcal{T} : K \mapsto K$ by $[\mathcal{T}(\varphi)](\theta) := v_{t_1}(\theta) = v(t_1 + \theta;\varphi)$ $(-r \leq \theta \leq 1)$, where $v(t_1 - 1;\varphi) = 0$ (see Fig.3).

If one can prove that there is non zero fixed point of $\mathcal{T}$ in K, then it gives a periodic solution to (3.2). The difficulty is to exclude the trivial solution $x = 0$ from the fixed point of $\mathcal{T}$. He applied Browder's fix point theorem, which is introduced below. We let

$$\begin{aligned} &X : \text{Banach space}, \\ &K : \text{infinite-dimensional closed bounded convex subset of } X, \\ &\mathcal{T} : K \rightarrow K \text{ completely continuous.} \end{aligned} \tag{3.3}$$

For a fixed point of $\mathcal{T}$, we define

x_0 is **ejective** if there exists an open neighborhood $\mathcal{U}$ of x_0 *s.t.*

$$\forall x \in \mathcal{U} \setminus \{x_0\}, \quad \exists n = n(x) \text{ s.t. } \mathcal{T}^n(x) \in K \setminus \mathcal{U}.$$

Theorem 3.1 (Browder [5]) *Let $\mathcal{T}$ be as in* (3.3)*. Then $\mathcal{T}$ has at least one fixed point which is not ejective.*

By this fixed point theorem, if the trivial solution $v = 0$ is ejective, then the map $\mathcal{T}(\varphi) = v_{t_1}(\varphi)$ has non-trivial fixed point in K.

Remark 1 By

$$x = \log(1 + v),$$

(3.2) is transformed to

$$\frac{dx}{dt} = -a(e^{x(t-1)} - 1).$$

Nussbaum significantly contributed to the existence of periodic solutions and period of them in the type of equations

$$\frac{dx}{dt} = -f(x(t-1))$$

(see [40] for further results and the references cited in [16]).

3.2 Kaplan-Yorke's contribution

In this section we introduce a work by Kaplan and Yorke. Consider

$$\frac{dx}{dt} = -f(x(t-1)), \quad f(0) = 0, \qquad f' > 0.$$

The specific function $f(u) = a(e^u - 1)$ certainly enjoys the condition. Define

$$Z(\phi) := \#\{z \in [-1, 0] : \phi(z) = 0\} \quad \text{for} \quad \phi \in C[-1, 0],$$

and

$$C_* := \{\phi \in C[-1, 0] : Z(\phi) \leq 1, \quad \phi'(z_0) \neq 0 \ (\phi(z_0) = 0)\}.$$

Definition 1 *A real-valued function $x(t)$ is slowly oscillating on $[t_0, \infty)$ if $x_t \in C_*$ $(\forall t \geq t_0 + 1)$ holds.*

Definition 2 *We call $A \subset \mathbb{R}^2$ a periodic C_*-annulus if there are periodic solutions $x(t), y(t)$ with $x_t, y_t \in C_*$ $(\forall t \in \mathbb{R})$ such that $A = \mathrm{Int}O_x \cap \mathrm{Ext}O_y$, where*

$$\begin{aligned} \mathrm{Int}O_x &: \textit{closure of the interior of } O_x := \{(x(t), -x(t-1)) : t \in \mathbb{R}\}, \\ \mathrm{Ext}O_y &: \textit{closure of the exterior of } O_y := \{(y(t), -y(t-1)) : t \in \mathbb{R}\}. \end{aligned}$$

Theorem 3.2 (Kaplan-Yorke [28]): *Assume $f : \mathbb{R} \mapsto \mathbb{R}$ is of class C^1 satisfying*

$$f(0) = 0, \quad f'(x) > 0 \ (\forall x \in \mathbb{R}) \quad \textit{and} \quad \exists B > 0 \text{ s.t. } \ f(x) > -B \quad (\forall x \in \mathbb{R}).$$

If $\alpha := 2f'(0)/\pi > 1$, then there is a periodic C_*-annulus $A \subset \mathbb{R}^2$ which is C_*-globally asymptotically stable.

The key property to prove the theorem is "trajectory crossing lemma" which holds for the trajectory of $(x(t), -x(t-1))$ in the plane. They developed this argument to a little more general equation

$$x'(t) = -f(x(t), x(t-1)),$$

under some condition for f ([29]).

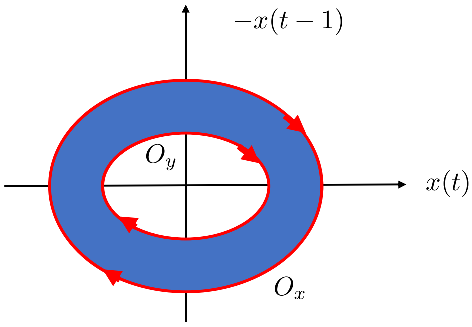

Figure 4: Periodic C_*-annulus

3.3 Stability of constant solutions

In the study for periodic solutions it is crucial to verify the stability of constant solution. In what follows we state a general result on the stability of the constant solution.

Let

$v(t) = (v_1(t), \cdots, v_n(t)),\quad F : \mathbb{R}^{n\times(m+1)} \to \mathbb{R}^n$ smooth and $F(0, \cdots, 0) = 0$,
$|\cdot|$: norm of $\mathbb{R}^n$.

Consider the following differential-difference equation

$$\dot{v} = F(v(t), v(t-\tau_1), \cdots, v(t-\tau_m)). \tag{3.4}$$

Assume $\tau := \max\{0, \tau_1, \cdots, \tau_m\} > 0$.

Form the assumption we see that $v = 0$ is a constant solution. We let

$$\dot{v} = A_0 v(t) + A_1 v_1(t-\tau_1) + \cdots + A_m v(t-\tau_m)$$

be the linearized equation around the trivial solution $v = 0$ and let

$$\det(A_0 v + A_1 e^{-\lambda\tau_1} + \cdots + A_m e^{-\lambda\tau_m}) = 0 \tag{3.5}$$

be the associated characteristic equation.

Theorem 3.3 (Bellman-Cooke [4]): *If all the roots of* (3.5) *has negative real part, then the solution* $v = 0$ *to* (3.4) *is asymptotically stable, while if one of the roots has positive real part, then* $v = 0$ *is unstable.*

3.4 Existence theorem for periodic solutions

It is expected that there exists a periodic solution if the unique constant solution is unstable as in Theorem 3.3 and all the eigenvalues with positive real part are complex.

Theorem 3.4 (Hadeler [15]): *The differential-difference equation*

$$\dot{x}(t) = f(x(t), x(t-\tau)$$

has a non-constant periodic solution with a period greater than 2τ *if*

(i) $f(x,y)$ *is continuous in* (x,y) *and uniform Lipschitz continuous with respect to* x*;*

(ii) $f(0,0) = 0$ *and there exists* $M > 0$ *such that* $f(0,y) \leq M$ $(\forall y)$ *holds;*

(iii) $f(x,y) < 0$ $(x \geq 0,\ y > 0)$ *and* $f(x,y) > 0$ $(x \leq 0, y < 0)$ *hold;*

(iv) $\nu := -(\partial f/\partial x)(0,0) < 0$ *and* $\alpha := (\partial f/\partial y)(0,0) < 0$ *hold;*

(v) $\tau^2(\alpha^2 - \nu^2) > \gamma^2$ *holds, where* γ *is determined by the condition* $\gamma \cot \gamma = \tau\nu,\ \pi/2 < \gamma < \pi$*.*

As an application of Theorem 3.4 consider

$$\dot{N} = -\nu N(t) + \rho e^{-\gamma N(t-\tau)}. \tag{3.6}$$

Let N^* be the positive equilibrium of (3.6), that is,

$$-\nu N^* + \rho e^{-\gamma N^*} = 0.$$

Put $x = N - N^*$. Then (3.6) is transformed to

$$\dot{x} = f(x(t), x(t-r)), \qquad f(x,y) := -\nu x + \rho e^{-\gamma N^*}(e^{-\gamma y} - 1).$$

In addition to

$$f_x(0,0) = -\nu < 0, \quad f_y(0,0) = -\gamma\rho e^{-\gamma N^*} < 0,$$

we can verify the all the conditions of Theorem 3.4, though we need to cut off the function f out side a bounded region.

The above equation is nothing but the model equation stated in §2 B). As stated there, the existence of a periodic solution was first proved in [6], though Theorem (3.4) slightly generalized the assumptions in [6].

4 Fundamental theory

For the linear ODE

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n,$$

its adjoint equation is defined by

$$\dot{y} = -A^T y, \quad y \in \mathbb{R}^n.$$

Then the inner product $(x(t), y(t))$ in $\mathbb{R}^n$ is constant in t since

$$\begin{aligned}\frac{d}{dt}(x(t), y(t)) &= (\dot{x}(t), y(t)) + (x(t), \dot{y}(t)) \\ &= (A(t)x(t), y(t)) + (x(t), -A(t)^T y(t)) \\ &= (A(t)x(t), y(t)) - (A(t)x(t), y(t)) = 0.\end{aligned}$$

The idea to consider the adjoint equation can be extended to retarded functional differential equations.

For a bounded linear functional

$$L : C[-r, 0] \to \mathbb{R}^n,$$

there exists a bounded variational $n \times n$ matrix function $\eta(\theta)$ such that

$$L\phi = \int_{-r}^{0} [d\eta(\theta)]\phi(\theta).$$

Consider a retarded linear functional differential equation

$$\begin{aligned}\dot{x}(t) = Lx_t = \int_{-r}^{0} [d\eta(\theta)]x(t+\theta), \qquad (4.1)\\ x(\theta) = \phi(\theta)\ (-r \le \theta \le 0), \quad \phi \in C[-r, 0].\end{aligned}$$

Put $C^* := C[0, r]$ and consider

$$\begin{aligned}\dot{y}(t) = -L^* y^t := -\int_{0}^{r} [d\eta(-\theta)]^T y(t+\theta) = -\int_{-r}^{0} [d\eta(\theta)]^T y(t-\theta), \qquad (4.2)\\ y(\theta) = \psi(\theta)\ (0 \le \theta \le r), \quad \psi \in C^*,\end{aligned}$$

which is called a formal adjoint equation to (4.1). We define a bilinear form

$$\langle \phi, \psi \rangle := (\phi(0), \psi(0)) - \int_{-r}^{0} \int_{0}^{\theta} ([d\eta(\theta)]\phi(\xi), \psi(\xi - \theta))d\xi, \qquad (4.3)$$

for $\phi \in C[-r, 0]$, $\psi \in C^* := C[0, r]$. Then

$$\begin{aligned}\langle x_t, y^t \rangle &= (x(t), y(t)) - \int_{-r}^{0} \int_{0}^{\theta} ([d\eta(\theta)]x(t+\xi), y(t+\xi-\theta))d\xi \\ &= (x(t), y(t)) - \int_{-r}^{0} \int_{t}^{t+\theta} ([d\eta(\theta)]x(s), y(s-\theta))ds.\end{aligned}$$

We compute

$$\begin{aligned}\frac{d}{dt}\langle x_t, y^t \rangle &= (\dot{x}(t), y(t)) + (x(t), \dot{y}(t)) \\ &\quad - \int_{-r}^{0} ([d\eta(\theta)]x(t+\theta), y(t)) + \int_{-r}^{0} ([d\eta(\theta)]x(t), y(t-\theta)) \\ &= (\dot{x}(t) - \int_{-r}^{0} [d\eta(\theta)]x(t+\theta), y(t)) \\ &\quad + (x(t), \dot{y}(t) + \int_{-r}^{0} [d\eta(\theta)]^T y(t-\theta)) = 0.\end{aligned}$$

The next inhomogeneous linear equation plays a crucial role in perturbation problems.

$$\dot{x} = Lx_t + h(t), \tag{4.4}$$

where $h(t)$ is a continuous function. By a simple computation we can verify the next result.

Lemma 4.1 *Let $x(t)$ be a solution to (4.4) for $t \geq \sigma$ and $y(t)$ be a solution to (4.2) in $(-\infty, \infty)$. Then*

$$\frac{d}{dt}\langle x_t, y^t \rangle = (h(t), y(t)), \qquad t \geq \sigma$$

holds, namely,

$$\langle x_t, y^t \rangle = \langle x_\sigma, y^\sigma \rangle + \int_{\sigma}^{t} (h(t), y(t))ds$$

holds.

The next lemma is crucial to verify if the system with a periodic forcing term could have a periodic solution.

Lemma 4.2 *Let $h(t)$ be a continuous periodic function with period $T > 0$. Then (4.4) has a T-periodic solution if and only if*

$$\int_{0}^{T} (h(t), y(t))ds = 0$$

holds for any T-periodic solution $y(t)$ of (4.2).

For the proof see Chapter 6 of [16].

5 Hopf bifurcation theory for RFDEs

In this section we formulate the Hopf bifurcation theory for retarded functional differential equations (RFDEs) by using the Lyapunov-Schmidt reduction method.

We introduce the following Banach spaces:

$$\begin{aligned}
&\mathbb{P}_{2\pi} := \{x(\cdot) : x(\cdot) \in C(\mathbb{R}), x(s+2\pi) = x(s), \forall s \in \mathbb{R}\},\\
&\|x\| := \max_{0\le s\le 2\pi} |x(s)| \quad (x \in \mathbb{P}_{2\pi}),\\
&\mathbb{P}^1_{2\pi} := \{x \in \mathbb{P}_{2\pi} : dx/ds \in \mathbb{P}_{2\pi}\},\\
&\|x\|_1 := \max\{\|x\|, \|dx/ds\|\}.
\end{aligned}$$

Define $J_0 : \mathcal{D}(J) := \mathbb{P}^1_{2\pi} \ \to \ \mathbb{P}_{2\pi}$ by

$$[J_0 x](t) := \omega_0 \frac{dx}{ds}(s) - Lx_s.$$

We also define the formal adjoint operator $J_0^* : \mathcal{D}(J_0^*) := \mathbb{P}^1_{2\pi} \ \to \ \mathbb{P}_{2\pi}$ by

$$[J^* y](s) := \omega_0 \frac{dy}{ds}(s) + L^* y^s.$$

Define the projections

$$P_N : \ \mathbb{P}_{2\pi} \ \to \ \mathcal{N}(J^*), \qquad \Pi : \ \mathbb{P}_{2\pi} \ \to \ \mathcal{N}(J).$$

Then we have $\mathcal{R}(J_0) = (I - P_N)\mathbb{P}_{2\pi}$. Moreover, we can define the linear operator

$$K : \ (I - P_N)\mathbb{P}_{2\pi} \ \to \ \mathcal{D}(J_0) \cap (I - \Pi)\mathbb{P}_{2\pi}$$

by $K := (J_{|(I-\Pi)\mathbb{P}_{2\pi}})^{-1}$ ([17]).

We let $F : I_\mu \times C[-r, 0] \to \mathbb{R}^n$ be class of C^k and satisfy $F(\mu; 0) = 0$. Consider

$$\frac{dx}{dt} = F(\mu; x_t) = Lx_t + N(\mu; x_t), \qquad L := F_x(0; 0). \tag{5.1}$$

Assume

$$\det(i\omega_0 I - L[e^{i\omega_0 \cdot}]) = 0,$$

namely, L has a pair of complex conjugate eigenvalues $\pm i\omega_0$. We let ζ_0 be an eigenvector

$$L[e^{i\omega_0 \cdot}]\zeta_0 = \left(\int_{-r}^0 e^{i\omega_0 \theta} [d\eta(\theta)] d\theta \right) \zeta_0 = i\omega_0 \zeta_0.$$

Then

$$z(t) = e^{i\omega_0 t}\zeta_0, \quad \overline{z(t)} = e^{-i\omega_0 t}\overline{\zeta_0}$$

are solutions to the linear equation

$$\frac{dz}{dt}(t) = Lz_t.$$

To fix the period of solutions to (5.1), we change the variables as

$$s := \omega t, \qquad u(s) := x(t).$$

Define

$$u_{s,\omega}(\theta) := u(s + \omega\theta), \quad -r \le \theta \le 0,$$

and we have

$$\frac{d}{dt}x(t) = \omega\frac{d}{ds}u(s).$$

We write the equation (5.1) as

$$\omega\frac{d}{ds}u(s) = Lu_{s,\omega} + N(\mu; u_{s,\omega}).$$

By the operator $J_0 : \mathbb{P}^1_{2\pi} \to \mathbb{P}_{2\pi}$ defined by

$$[J_0 u](s) := \omega_0\frac{d}{ds}u(s) - \int_{-r}^{0}[d\eta(\theta)]u(s + \omega_0\theta),$$

we write the equation as

$$J_0 u(s) = -(\omega - \omega_0)\frac{d}{ds}u(s) + N(\mu, u_{s,\omega}) =: N_1(\mu, \omega, u). \tag{5.2}$$

We are only concerned with 2π-periodic solution to (5.2).

We look for a solution in $\mathbb{P}^1_{2\pi}$ by the anzatz

$$\begin{aligned}
&u = \varepsilon u_0 + \hat{u},\\
&\quad u_0(s) := \zeta_0 e^{is} + \overline{\zeta_0 e^{is}} \in (\Pi\mathbb{P}_{2\pi}) \cap \mathbb{P}^1_{2\pi},\\
&\quad \hat{u} = \hat{u}(\mu, \omega, \varepsilon) \in (I - \Pi)\mathbb{P}_{2\pi} \cap \mathbb{P}^1_{2\pi}.
\end{aligned}$$

Then (5.2) is decomposed as

$$\begin{aligned}
&\hat{u} = K(I - P_N)\tilde{N}(\mu, \omega, \varepsilon, \hat{u}), \quad \tilde{N}(\mu, \omega, \varepsilon, \hat{u}) := N_1(\mu, \omega, \varepsilon u_0 + \hat{u}),\\
&P_N\tilde{N}(\mu, \omega, \varepsilon, \hat{u}) = 0,\\
&\tilde{N} : I_\mu \times \mathbb{R} \times \mathbb{R} \times \mathbb{P}^1_{2\pi} \ \to \ \mathbb{P}_{2\pi}.
\end{aligned}$$

For fixed ω, $\tilde{N}$ is class of C^k $(k \geq 2)$ in $(\mu, \varepsilon, \hat{u})$ but not C^k in ω. Hence, the solution $\hat{u}(\mu, \omega, \varepsilon)$ obtained by applying the implicit function theorem is only C^1 in ω.

In order to prove that $\hat{u}(\mu, \omega, \varepsilon)$ is C^k in $(\mu, \omega, \varepsilon)$, one can utilize a bootstrap argument as follows: Consider the equation

$$w = K(I - P_N)\left[\frac{\partial \tilde{N}}{\partial \omega} + \frac{\partial \tilde{N}}{\partial \tilde{u}} w\right].$$

The solution w obtained by the implicit function theorem is C^1 in $(\mu, \omega, \varepsilon)$, so $w = \partial \hat{u}/\partial \omega$ is C^1 in $(\mu, \omega, \varepsilon)$. Repeat this argument up to C^k. In the sequel we reduced the problem to the finite-dimensional problem

$$P_N \tilde{N}(\mu, \omega, \varepsilon, \hat{u}(\mu, \omega, \varepsilon)) = 0,$$

namely,

$$H(\mu, \omega, \varepsilon) := \int_0^{2\pi} ([\tilde{N}(\mu, \omega, \varepsilon, \hat{u}(\mu, \omega, \varepsilon)](s), \overline{z^*(s)}) ds = 0,$$

where $H(\mu, \omega, \varepsilon)$ are C^k for $(\mu, \omega, \varepsilon)$ and $z^* \in P_N \mathbb{P}_{2\pi}$ is given by

$$z^* = e^{is} \zeta_0^*, \qquad \left(i\omega_0 I - \int_0^r e^{i\omega_0 \theta} [d\eta(-\theta)]^T\right) \zeta_0^* = 0.$$

We expand H in ε as

$$H(\mu, \omega, \varepsilon) = (\Lambda(\varepsilon) + B\varepsilon^2)\varepsilon + O(|(\mu, \omega - \omega_0)|^2 + \varepsilon^4),$$

where

$$\begin{aligned}
B &:= (F_{uu}(0,0)(\zeta_0, \hat{\zeta}_2), \overline{\zeta_0}) + (F_{uu}(0,0)(\overline{\zeta_0}, \zeta_2), \overline{\zeta_0}) + \frac{1}{2}(F_{uuu}(0,0)(\zeta_0, \zeta_0, \overline{\zeta_0}),\ \overline{\zeta_0^*}),\\
\zeta_2 &:= \frac{1}{2}(2i\omega_0 - L)^{-1}(F_{uu}(0,0), (\zeta_1, \zeta_1),\ \overline{\zeta_0^*}),\\
\hat{\zeta}_2 &:= -L^{-1}(F_{uu}(0,0)(\zeta_1, \overline{\zeta_1}),\ \overline{\zeta_0^*}),
\end{aligned}$$

while

$$\begin{aligned}
\Lambda(\varepsilon) &:= -i(\omega - \omega_0) + \mu \frac{d\lambda}{d\mu}(0)\\
&= \left(-i\omega_2 + \mu_2 \frac{d\lambda}{d\mu}(0)\right)\varepsilon^2 + O(\varepsilon^3),
\end{aligned}$$

where μ_2 and ω_2 are determined by

$$\mathrm{Re}\left(\mu_2 \frac{d\lambda}{d\mu}(0) + B\right) = 0, \quad \omega_2 = \mathrm{Im}\left(\mu_2 \frac{d\lambda}{d\mu}(0) + B\right).$$

Consequently, a bifurcating periodic solution is given by

$$u(s) = \varepsilon(\zeta_0 e^{is} + \overline{\zeta_0 e^{is}}) + \varepsilon^2(\zeta_2 e^{2is} + \overline{\zeta_2 e^{2is}} + \hat{\zeta}_2) + \varepsilon^2 \hat{u}(s;\varepsilon),$$

with

$$\mu = \mu(\varepsilon) = \mu_2\varepsilon^2 + O(\varepsilon^3), \quad \omega = \omega(\varepsilon) = \omega_0 + \omega_2\varepsilon^2 + O(\varepsilon^3).$$

Remark 2 In Chapter 11 of the book [16] the Hopf bifurcation theorem is provided. However, there is a wrong computation. The correction is given in [17].

We apply the above result to the Hutchinson-Write equation

$$\dot{x} = -\left(\frac{\pi}{2} + \mu\right)(1 + x(t))x(t-1). \tag{5.3}$$

The linear part of the equation is

$$\dot{x} = Lx_t := -\frac{\pi}{2}x(t-1),$$

and we can compute $\omega_0 = \pi/2$,

$$\begin{aligned} &\zeta_0 = 1, \qquad \zeta_0^* = \kappa := 1/(1 - i\pi/2), \\ &\zeta_2 = \frac{1}{5}(2 - i), \quad \hat{\zeta}_2 = 0, \\ &B = \frac{\pi}{10}(1 - 3i)\overline{\kappa} = -\frac{\pi(3\pi - 2)}{5(\pi^2 + 4)} - i\frac{\pi(\pi + 6)}{5(\pi^2 + 4)}. \end{aligned}$$

Remark 3 We give a remark on the stability of the bifurcation solution. Let $p_\varepsilon(t)$ be the bifurcating periodic solution to (5.3) with period $T_\varepsilon = \omega(\varepsilon)/2\pi$. Consider the linearized equation around $v = p_\varepsilon$ given by

$$\begin{aligned} \dot{z} = &-\left(\frac{\pi}{2} + \mu(\varepsilon)\right)z(t-1) \\ &- \left(\frac{\pi}{2} + \mu(\varepsilon)\right)(p_\varepsilon(t-1)z(t) + p_\varepsilon(t)z(t-1))). \end{aligned} \tag{5.4}$$

If this linear periodic system has a solution

$$z = e^{\gamma t}q(t), \qquad q(t) : \text{periodic function with period } T_\varepsilon,$$

then γ is called a Floquet exponent. Since $z = \dot{p}_\varepsilon$ is a periodic solution with period T_ε, the system has 0 Floquet exponent. Moreover, it can be shown that the other Floquet exponents have negative real part near the bifurcation point. As for the Floquet theory for retarded functional differential equation, see Chaper 8 of [16].

We next consider the following two compartment model:

$$\dot{x}_1 = a(1 - x_1(t-1))x_1(t) + \nu(x_2(t) - x_1(t)),$$
$$\dot{x}_2 = a(1 - x_2(t-1))x_2(t) + \nu(x_1(t) - x_2(t)),$$

that is,

$$\dot{u} = -\left(\frac{\pi}{2} + \mu\right)(1 + u(t))u(t-1) + \nu(v(t) - u(t)),$$
$$\dot{v} = -\left(\frac{\pi}{2} + \mu\right)(1 + v(t))v(t-1) + \nu(u(t) - v(t)).$$

Then this system has a periodic solution $u(t) = v(t) = p_\varepsilon(t)$ (called an in-phase periodic solution) near the bifurcation point. We examine the linearized system around the in-phase solution $(u, v) = (p_\varepsilon(t), p_\varepsilon(t))$, which is given in the decomposed form

$$\dot{z} = -\left(\frac{\pi}{2} + \mu\right)z(t-1) - \left(\frac{\pi}{2} + \mu\right)(p_\varepsilon(t-1)z(t) + p_\varepsilon(t)z(t-1))),$$
$$\dot{w} = -2\nu w(t) - \left(\frac{\pi}{2} + \mu\right)w(t-1) - \left(\frac{\pi}{2} + \mu\right)(p_\varepsilon(t-1)w(t) + p_\varepsilon(t)w(t-1))).$$

The first equation is same as (5.4). To investigate the Floquet exponent of the second equation, we look for a solution with the form $w = q(t)e^{\gamma t}$, where $q(t)$ is periodic with the same period as the one of $p(t)$. Making use of

$$\mu = \mu_2\varepsilon^2 + \cdots$$
$$p_\varepsilon = \varepsilon(\zeta_0 e^{i\omega_0 t} + \overline{\zeta_0 e^{i\omega_0 t}}) + \varepsilon^2(\zeta_2 e^{2i\omega_0 t} + \overline{\zeta_2 e^{2i\omega_0 t}}) + \cdots,$$

we expand

$$2\nu = \nu_2\varepsilon^2 + \cdots, \quad \gamma = \gamma_2\varepsilon^2 + \cdots$$
$$q = q_0(t) + q_1(t)\varepsilon + q_2(t)\varepsilon^2 + \cdots, \quad q_0(t) = \dot{p}_\varepsilon.$$

Then by the solvability condition we obtain

$$\left(1 + \left(\frac{\pi}{2}\right)^2\right)\gamma_2^2 + 2\left(\nu_2 - \frac{\pi}{10} + \frac{3\pi^2}{20}\right)\gamma_2 + \nu_2^2 - \frac{\pi}{5}\nu_2 = 0.$$

This implies that for $\nu_2 < \pi/5$ there is a Floquet exponent with positive real part for sufficiently small ε and the in-phase periodic solution is unstable ([38]).

Remark 4 We consider the delay-diffusion equation

$$u_t = d\Delta u - \left(\frac{\pi}{2} + \mu\right)(1 + u(x,t))u(x,t-1) \quad \text{in} \quad \Omega,$$
$$\partial u/\partial \mathrm{n} = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is bounded domain in $\mathbb{R}^n$ and $\partial/\partial \mathrm{n}$ stands for the normal outer derivative on the smooth boundary $\partial\Omega$. It is clear that $u = p_\varepsilon(t)$ is a spatially homogeneous periodic solution. Let $\{\sigma_j\}$ and $\{\varphi_j\}$ be the eigenvalues and the corresponding eigenfunctions of $-\Delta$ with the Neumann boundary condition

$$-\Delta\varphi_j = \sigma_j\varphi \quad + \quad \text{(Neumann B.C.)}, \quad \sigma_1 = 0 < \sigma_2 \leq \cdots,$$

and decompose the linearized equation around the spatially homogeneous periodic solution by the Fourier expansion. Then we obtain the linearized equations

$$\begin{aligned}\dot{w}_j = &-d\sigma_j w_j(t) - \left(\frac{\pi}{2} + \mu\right) w_j(t-1) \\ &- \left(\frac{\pi}{2} + \mu\right)(p_\varepsilon(t-1)w_j(t) + p_\varepsilon(t)w_j(t-1))), \quad j = 1, 2, \cdots.\end{aligned}$$

Thus, we see that for sufficiently small $d > 0$ there is a Floquet exponent with positive real part and the spatially homogeneous periodic solution is unstable ([37]). As for the partial functional differential equations including the above delay diffusion equation, see Wu [43].

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6 Center manifold reduction

In 1970's the center manifold theory was developed and it made possible to reduce infinite-dimensional flows of a class of evolution equations into lower-dimensional ones on invariant manifolds (center manifolds). However, some of articles or books used a wrong phase space when they handled retarded functional differential equations. Indeed, the abstract ODE on the standard phase space $C[-r, 0]$ does not make sense. In order to overcome this difficulty, we need to take a broader phase space. In this section we explain it by introducing the argument in the work of Chow-Mallet-Paret [8].

We begin with the linear equation

$$\begin{aligned}&\dot{x}(t) = Lx_t = \int_{-r}^{0} [d\eta(\theta)]x(t+\theta), \quad L : C[-r, 0] \to \mathbb{R}^n, \\ &x(\theta) = \phi(\theta) \ (-r \leq \theta \leq 0), \quad \phi \in C[-r, 0].\end{aligned} \tag{6.1}$$

Define the semiflow $\{T(t)\}_{t\geq 0}$ on $C[-r,0]$ by the solution to (6.1) as

$$[T(t)\phi](\theta) := x(t+\theta) \quad (-r \leq \theta \leq 0).$$

Then the infinitesimal generator $\mathcal{A}$, $\mathcal{A} := \lim_{h\downarrow 0} \dfrac{T(h)-I}{h}$, is obtained as

$$[\mathcal{A}\phi](\theta) = \begin{cases} \dfrac{d\phi}{d\theta} & (-r \leq \theta < 0), \\ L\phi & (\theta = 0), \end{cases}$$

where the domain and the range are given by

$$\mathcal{D}(\mathcal{A}) = \{\phi \in C^1[-r,0] : (d\phi/d\theta)(0) = L\phi\} \text{ and } \mathcal{R}(\mathcal{A}) = C[-r,0],$$

respectively.

Consider the following equation with nonlinear term:

$$\dot{x}(t) = Lx_t + F(\mu, x_t). \tag{6.2}$$

Then by the variation-of-constant formula ([16]) we have the integral form

$$\begin{aligned} x(t+\theta) &= [T(t)x_0](\theta) + \int_0^t [T(t-s)X_0](\theta)F(\mu, x_s)ds \quad (-r \leq \theta \leq 0), \\ X_0(\theta) &:= \begin{cases} 0 & (-r \leq \theta < 0), \\ I & (\theta = 0). \end{cases} \end{aligned} \tag{6.3}$$

Although the integral in (6.3) is taken in $\mathbb{R}^n$ for each θ, we have a formal expression of the abstract integral equation as

$$x_t = T(t)x_0 + \int_0^t T(t-s)X_0F(\mu, x_s)ds,$$

so the formal abstract ODE on the Banach space $C[-r,0]$ has the form

$$\frac{d}{dt}x_t = \mathcal{A}x_t + X_0F(\mu, x_t).$$

However, in general, x_t does not belong to the domain of $\mathcal{A}$, though belong to $C^1[-r,0]$ at least for $t > r$. In addition, X_0 does not belong to $C[-r,0]$. In order to make sense of the abstract ODE, we have to extend the domain of $\mathcal{A}$ to $C^1[-r,0]$.

Solve $\mathcal{A}\phi = \psi$ to obtain the formula for $\mathcal{A}^{-1}$. $\mathcal{A}\phi = \psi$ implies

$$\phi'(\theta) = \psi(\theta) \quad (-r \leq \theta < 0), \quad \phi'(0) = L\phi = \int_{-r}^0 [d\eta(\theta)]\phi(\theta) = \psi(0).$$

Integrating this yields

$$\phi(\theta) = \phi(0) + \int_0^\theta \psi(s)ds,$$
$$\psi(0) = \int_{-r}^0 [d\eta(\theta)]\phi(\theta) = \int_{-r}^0 [d\eta(\theta)] \left(\phi(0) + \int_0^\theta \psi(s)ds\right).$$

Hence, as long as $\det \left[\int_{-r}^0 [d\eta(\theta)]\right] \neq 0$, $\phi(0)$ is uniquely determined and $\mathcal{A}$ has a bounded inverse. Extend the domain of $\mathcal{A}^{-1}$ as follows. For $\psi = X_0$, define the inverse by $\mathcal{A}$ by

$$\mathcal{A}^{-1}X_0 = \left[\int_{-r}^0 [d\eta(\theta)]\right]^{-1} \quad (= \text{constant value}).$$

Then any $\phi \in C^1[-r, 0]$ is decomposed as

$$\phi = \phi_1 + \phi_2, \qquad \phi_1'(0) = L\phi_1, \quad \phi_2 : \text{constant function}.$$

Indeed, set ϕ_2 satisfying $L\phi_2 = L\phi - \phi'(0)$, i.e.,

$$\phi_2 = \left[\int_{-r}^0 [d\eta(\theta)]\right]^{-1} (L\phi - \phi'(0)).$$

Then

$$\phi'(0) = \phi_1'(0) = L(\phi - \phi_2) = L\phi_1.$$

We define $\mathcal{A}$ in $C^1[-r, 0]$ as

$$\mathcal{A}\phi = \phi' + X_0[L\phi - \phi'(0)] \in C \oplus \langle X_0 \rangle,$$

namely,

$$[\mathcal{A}\phi](\theta) = \begin{cases} \phi'(\theta) = \phi_1'(\theta) & (-r \leq \theta < 0), \\ L\phi & (\theta = 0), \end{cases}$$

where $\langle X_0 \rangle = \{X_0\boldsymbol{c} : \boldsymbol{c} \in \mathbb{R}^n\}$.

Eventually, we have

Theorem 6.1 (Chow-Mallet-Paret [8]): *Consider the retarded functional equation*

$$\dot{x}(t) = Lx_t + F(\mu, x_t), \quad L\phi = \int_{-r}^0 [d\eta(\theta)]\phi(\theta).$$

Let the operator $\mathcal{A}$ map $C^1[-r,0]$ into $BC := C[-r,0] \oplus \langle X_0 \rangle$ by

$$\mathcal{A}\phi = \phi' + X_0[L\phi - \phi'(0)].$$

Then any solution for $t \geq t_0$ satisfies

$$\frac{d}{dt}x_t = \mathcal{A}x_t + X_0 F(\mu, x_t) \tag{6.4}$$

as long as $t \geq t_0 + r$ (or, as long as $x_t \in C^1[-r,0]$).

Remark 5 In view of (6.2) we can write

$$\begin{cases} \dfrac{\partial}{\partial t}x(t+\theta) = \dfrac{\partial}{\partial \theta}x(t+\theta) & (0 \leq \theta < 0), \\ \dfrac{d}{dt}x(t) = Lx_t + F(\mu, x_t) & (\theta = 0), \end{cases}$$

and formally have (6.4). This made the confusion that this allows the abstract form (6.4). As stated above we cannot directly have the evolution equation on $C^1[-r,0]$ from this expression.

When we apply the center manifold theory to the Hopf bifurcation, we decompose

$BC = P \oplus Q$,
2-dimensional eigenspace $P \subset C^1[-r,0]$ and the complement Q,

according to the assumption for the spectrum of $\mathcal{A}$. For $\phi \in C[-r,0]$ and $\psi \in C^* := C[0,r]$, define

$$\langle \phi, \psi \rangle := (\phi(0), \psi(0)) - \int_{-r}^{0}\int_{0}^{\theta} ([d\eta(\theta)]\phi(\xi), \psi(\xi - \theta))d\xi.$$

Let Φ be a basis of P and let Ψ be a dual basis in C^* satisfying $\langle \Phi, \Psi \rangle = I$. Put

$$\phi^P = \Phi\langle \phi, \Psi \rangle, \quad \phi^Q = \phi - \phi^P,$$

and

$$\mathcal{A}_P := \mathcal{A}_{|P}, \quad \mathcal{A}_Q := \mathcal{A}_{|Q}.$$

Then there is a matrix A_P

$$\mathcal{A}\Phi = \frac{d}{d\theta}\Phi = \Phi A_P.$$

By the decomposition

$$x_t = z_t^P + z_t^Q = \Phi\langle x_t, \Psi\rangle + z_t^Q,$$

$\xi(t) := \langle x_t, \Psi\rangle$ and $\zeta_t := z_t^Q$ enjoy the equations

$$\begin{aligned}\dot{\xi}(t) &= A_P\xi(t) + (F(\mu, \Phi\xi(t) + \zeta_t), \Psi(0)),\\ \frac{d}{dt}\zeta_t &= A_Q\zeta_t + X_0^Q F(\mu, \Phi\xi(t) + \zeta_t),\end{aligned}$$

respectively. Making use of this decomposition, we can apply the center manifold theory for the infinite-dimensional evolution equation.

Remark 6 In order to obtain the bifurcating solution depending on some parameters, we need an appropriate smooth dependence on the parameters for the center manifold. About this topic the readers refer to Faria-Magalhaes [12].

7 Global structure

7.1 Morse decomposition

We are concerned with

$$\sigma\dot{x}(t) = -x(t) + f(x(t-1)),$$

where

$$\begin{aligned}&f(0) = 0, \qquad a = f'(0) < -1, \qquad xf(x) < 0 \quad (\forall x \neq 0),\\ &|f(x)| < |x| \qquad (\text{for large } |x|), \qquad f \text{ is of class } C^\infty.\end{aligned}$$

In the paper [34] Mallet-Paret surveyed the progress for

(1) Global continuation of periodic solutions from Hopf bifurcations;

(2) Asymptotic form of such solutions as $\sigma \downarrow 0$;

(3) Existence of a Morse decomposition.

As for (1), there exists a sequence $\{\sigma_{2n-1}(a)\}$,

$$s_1(a) > s_3(a) > s_5(a) > \cdots \to 0,$$

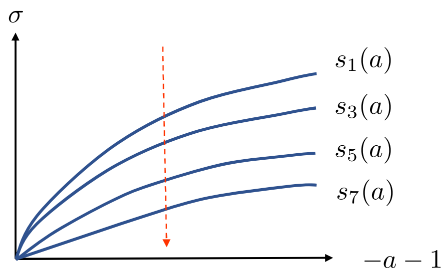

Figure 5: Bifurcation curves

such that a Hopf bifurcation takes place from the trivial solution $x = 0$ at each $s_{2n-1}(a)$ as σ decreases. In the range $s_{2n+1}(a) < \sigma < s_{2n-1}(a)$, the trivial solution $x = 0$ has a $2n$-dimensional unstable manifold ([36]).

For the Morse decomposition in (3) we introduce some result in [35]. Consider

$$\dot{x}(t) = -f(x(t), x(t-1)), \quad \eta f(0, \eta) > 0 \ \ (\forall \eta \neq 0), \quad \frac{\partial f(\xi, \eta)}{\partial \eta}_{|(0,0)} > 0.$$

This equation has the global attractor (the maximal compact attractor), say Ψ. Define an integer valued Lyapunov function V on $\Psi \setminus \{0\}$ as follows: Let $x \cdot t \in C(\mathbb{R})$ by $(x \cdot t)(\theta) = x(t + \theta)$ $(\theta \in \mathbb{R})$ for $x \in \Psi$. Define

$$\sigma := \inf\{t \geq 0 : x(t) = 0\},$$

and

$$V(x) := \begin{cases} \#\{t \in (\sigma - 1, \sigma] : x(t) = 0 \ \} \ \text{(counting multiplicity)} & \text{if } \ \sigma \text{ is bounded,} \\ 1 & \text{if } \ \sigma = -\infty. \end{cases}$$

Then $V(x \cdot t)$ is nonincreasing in t, that is,

$$V(x \cdot t_1) \geq V(x \cdot t_2) \qquad (t_1 \leq t_2)$$

holds, and

$$V: \ \Psi \setminus \{0\} \ \to \ \{1, 3, 5, \ldots, 2M + 1\}$$

(see Fig.6). Define

$$S_N := \{x \in \Psi \setminus \{0\} : V(x \cdot t) = N \ \ (\forall t \in \mathbb{R}) \ \text{ and } \ 0 \notin \alpha(x) \cup \omega(x)\},$$

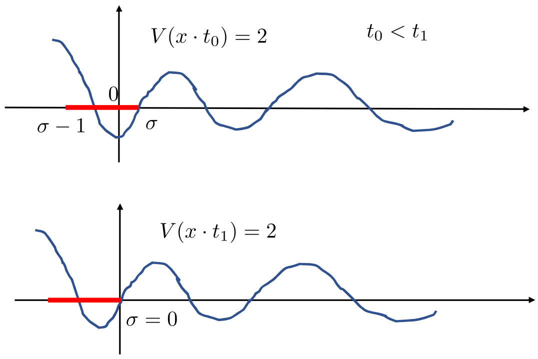

Figure 6: Example of $V(x \cdot t)$

and $S_{N^*} = \{0\}$, where N^* is an even integer with the property

$$\begin{aligned} x \in W^s \quad &\Rightarrow \quad V(x \cdot t) > N^* \quad (\forall t \in \mathbb{R}), \\ x \in W^u \quad &\Rightarrow \quad V(x \cdot t) < N^* \quad (\forall t \in \mathbb{R}) \end{aligned}$$

(N^* is nothing but the dimension of the unstable manifold of $x = 0$). It was proved that S_N contains a periodic orbit if $N < N^*$.

Then [35] tells that the attractor Ψ consists of $\{S_j\}_{j=1}^M$ and the family of connecting orbits

$$C_K^N := \{x \in \Psi : \alpha(x) \subset S_N, \quad \omega(x) \subset S_K\},$$

where $\alpha(x)$ and $\omega(x)$ are respectively the alpha and omega limit sets through x. We note $C_N^{N^*} \neq \emptyset$ for $N < N^*$.

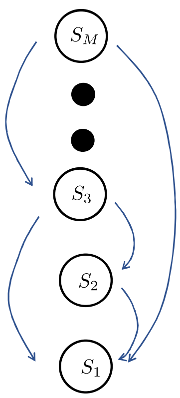

Figure 7: Morse decomposition

7.2 Specific example exhibiting a global structure

Chow-Diekmann-Mallet-Paret [7] studied the next integral equation

$$x(t) = \frac{1}{2\varepsilon} \int_{1-\varepsilon}^{1+\varepsilon} f(x(t-\tau))d\tau. \tag{7.1}$$

This is equivalent to the delay difference equation

$$\dot{x}(t) = \frac{1}{2\varepsilon}[f(x(t-1-\varepsilon) - f(x(t-1+\varepsilon)],$$

and as $\varepsilon \downarrow 0$, the integral equation of (7.1) formally converges to the map

$$x(t) = f(x(t-1)).$$

Assume

$$H_1 : f(-x) = -f(x) \quad (-\infty < x < \infty), \qquad H_2 : f(1) = -1,$$
$$H_3 : f'(x) < 0 \quad (-\infty < x < \infty), \qquad H_4 : f''(x) > 0 \quad (0 < x < \infty).$$

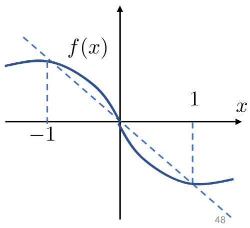

Figure 8: Profile of $f(x)$

They proved that there exists a periodic solution $x_\varepsilon(t)$ with period 2 in a range $\varepsilon \in (0, \varepsilon^*)$ satisfying

$$x_\varepsilon(-t) = -x_\varepsilon(t), \quad x_\varepsilon(t+1) = -x_\varepsilon(t),$$
$$x'_\varepsilon(t) > 0 \qquad (-1/2 < t < 1/2), \quad x''_\varepsilon(t) < 0 \qquad (0 < t < 1),$$
$$x_{\varepsilon_1}(t) < x_{\varepsilon_2}(t) \quad (0 < t < 1) \quad \text{for} \quad \varepsilon_1 > \varepsilon_2,$$

where ε^* is a Hopf bifurcation point. The convergence

$$x_\varepsilon \to \text{sqw}(t) := \begin{cases} 1 & (t \ (\text{mod}2) \in (0,1)), \\ -1 & (t \ (\text{mod}2) \in (-1,0)), \end{cases} \qquad \text{as } \varepsilon \downarrow 0$$

was also shown.

They first examined the linearized equation around $x = 0$

$$z(t) = \frac{f'(0)}{2\varepsilon} \int_{1-\varepsilon}^{1+\varepsilon} z(t-\tau) d\tau.$$

Inserting $z = e^{\mu t}$ implies

$$1 = f'(0) e^{-\mu} \frac{e^{\varepsilon\mu} - e^{-\varepsilon\mu}}{2\varepsilon\mu}.$$

If $0 < \varepsilon \leq 1/3$, then the characteristic equation has at least two roots in $|\mathrm{Im}\mu| \leq \frac{\pi}{1-\varepsilon}$.

Since the Hopf bifurcation theory for a general class of Volterra convolution integral equations

$$x(t) = \int_0^\infty B(\tau) f(x(t-\tau)) d\tau$$

was established in [10], the theory can apply to the present case by taking

$$B(\tau) = \begin{cases} \dfrac{1}{2\varepsilon} & (1-\tau \leq \tau \leq 1+\varepsilon), \\ 0 & (\text{otherwise}). \end{cases}$$

Hence, the existence of a bifurcating periodic solution near the bifurcation point can be confirmed.

In order to obtain the periodic solution globally in ε, introduce the function spaces as

$$\begin{aligned} P_2 &:= \{x \in C(\mathbb{R}) \ : \ x(t+2) = x(t), \quad \forall t \in \mathbb{R}\}, \\ P_{\pm 1} &:= \{x \in P_2 : x(t+1) = \pm x(t), \quad \forall t \in \mathbb{R}\}, \\ P^s &:= \{x \in P_2 : x(-t) = x(t), \quad \forall t \in \mathbb{R}\}, \\ P^a &:= \{x \in P_2 : x(-t) = -x(t), \quad \forall t \in \mathbb{R}\}. \end{aligned}$$

Then the map $T : P_2 \to P_2$ defined by

$$[Tx](t) := \frac{1}{2\varepsilon} \int_{1-\varepsilon}^{1+\varepsilon} f(x(t-\tau)) d\tau$$

leaves $P_{\pm}^{a,s}$ invariant. Moreover, T leaves

$$C^a := \{x \in P_{-1}^a : x(t) \geq 0 \ \ (t \in [0,1])\},$$

invariant.

Theorem 7.1 (Chow-Diekman-Mallet-Paret [7]):

(i) *T has no nontrivial fixed point in C^a if $\sin \pi\varepsilon \le \pi\varepsilon/g'(0)$, where $g(x) := -f(x)$.*

(ii) *If $\sin \pi\varepsilon > \pi\varepsilon/g'(0)$, then T has a unique nontrivial fixed point $x_\varepsilon \in C^a$. Moreover,*

$$|x_\varepsilon(t)| < 1 \ \ (\forall t), \ \ x'_\varepsilon(t) > 0 \ \ (t \in (-1/2, \ 1/2)), \ \ x''_\varepsilon(t) < 0 \ \ (t \in (0,1)).$$

The existence of the solution in the above theorem is proved by applying the fixed point theorem in Krasnosel'skii [30]. Indeed, one can prove that that there is $\delta < 1$ such that $T(\delta\phi) \ge \delta\phi$, where $\phi := \sin \pi t$, and that $T^m(\delta\phi)$ monotonically increases in m and converges to a limit $x_\varepsilon \in C^a$. We emphasize that in this case the map T generates a monotone discrete flow in the cone.

On the other hand, in order to prove the uniqueness define

$$h(t) := \begin{cases} \dfrac{x_\varepsilon}{y(t)} & (0 < t < 1), \\ \dfrac{x'_\varepsilon(0)}{y'(0)} & (t = 0, 1), \end{cases}$$

where $y(t)$ is any other nontrivial fixed point. Prove $1 \le \alpha := \inf\{h(t) : 0 \le t \le 1\}$ by contradiction. Then $\alpha = 1$ follows from reversing the roles of x_ε and y.

In [7] they also prove that if $\varepsilon \le 1/3$, then the solution x_ε is asymptotically stable. This is done by showing that the linearized equation has the simple zero Floquet exponent and other exponents has negative real part by using a homotopy method.

8 Other topics

8.1 Effect of small delay

Kurzweil [31] says "Small delays don't matter" in [31], where the next delay-difference equation is studied as a specific case,

$$\dot{x}(t) = f(x(t), x(t-\varepsilon)), \qquad f : \mathbb{R}^n \times \mathbb{R}^n \ \to \ \mathbb{R}^n. \tag{8.1}$$

Putting $t = \varepsilon\tau$ yields

$$\frac{dx(\tau)}{d\tau} = \varepsilon f(x(\tau), x(\tau - 1)).$$

It is shown that if ε is sufficiently small, then there is a map $p : \mathbb{R}^n \to C([-1,0];\mathbb{R}^n)$ such that

$$\frac{dx(t)}{dt} = \varepsilon f([p(x(\tau))](0),[p(x(\tau))](-1)) = f([p(x(t))](0),[p(x(t))](-1)),$$

and $[p(x(t))(0)]$ is C^1 close $[p(x(t))](-1)$, so (8.1) is close to the equation $\dot{x} = f(x(t), x(t))$, where the latter "close" means that any solution to (8.1) on $\mathbb{R}$ is close to a solution to $\dot{x} = f(x(t), x(t))$ on $\mathbb{R}$ and vice versa. Therefore, the recent result of Eremin-Ishiwata-Ishiwata-Nakata [11] does not contradicts to the result by Kurzweil since he only cared the time global solution.

It is interesting that the following paragraph in "Part 3. Small Delays Can Make a Difference", by Hale [18]:

> "It is possible to prove that the limiting dynamics is determined by the ordinary differential equation obtained by putting all of the delays equal to zero (see Kruzweil (1970, 1971) and a more complete discussion in Hale-Magalhaes-Oliva (1984)). In such a situation, it is fair to say that small delays are unimportant."

Here the reference of Hale-Magalhaes-Oliva (1984) is the book [19].

It seems that in those days they didn't care about the small delay in the usual delay-differential equation. Thus, [11] gave a counter example showing an importance of small delay. On the other hand, in [18] there is a sentence

> "Small change of delay in neutral differential difference equations can be bad."

and the following example is given:

$$\begin{aligned} &\frac{d}{dt}[x(t) + \frac{1}{2}x(t-r_1) + \frac{1}{2}x(t-r_2)] = -\gamma x(t), \\ &r_1 = 1 - \frac{1}{2k+1}, \qquad r_2 = 2 \end{aligned}$$

(see [20] for the details).

8.2 Existence of chaos

Since a chaotic behavior was found in a delay difference equation by Mackey-Glass [33], some mathematicians have challenged to prove the existence of chaotic dynamics for delay differential equations of the form

$$\dot{x} = -\alpha x(t) + f(x(t-1)). \tag{8.2}$$

Since it is extremely difficult to prove it for the original model equation of [33] or [26], Walther [41] proved it in the case $\alpha = 0$ by modifying $f(u)$ by a stepwise constant function. Later, an der Heiden-Walther also proved it by a slightly different modification of $f(u)$ with $\alpha > 0$. Those are far from the model of [33] or [26] but help us to understand a mechanism of emergence of chaos in the type of (8.2). The readers can refer to Chapter 4 in [39] for the idea of [2].

8.3 Approximation in delay equation

As the last topic we introduce an approximation to the delay-differential equation by Banks [3].

Consider

$$\dot{x} = Lx_t + F(x_t),$$

and its abstract ODE

$$\frac{d}{dt}x_t = \mathcal{A}x_t + X_0 F(x_t),$$

or

$$\frac{d}{dt}x(t+\theta) = \begin{cases} \dfrac{\partial x(t+\theta)}{\partial \theta} & (-r \le \theta < 0), \\ Lx_t + F(x_t) & (\theta = 0). \end{cases}$$

Define

$$\begin{aligned} &Z := \mathbb{R}^n \times L^2(-r,0), \quad (L^2(-r,0) := L^2([-r,0];\mathbb{R}^n)), \\ &z(t;\phi) := (x(t;\phi), x_t(\phi)), \qquad t \in [0, t_1]. \end{aligned}$$

For ease in exposition, take

$$L(\phi) = A_0\phi(0) + A_1\phi(-r), \qquad A_0,\ A_1\ :\ n \times n \ \text{ matrices.}$$

Take a partition of the interval $[-r, 0]$ by $\{t_j^N\}_{j=0}^N$, $t_j^N := -\dfrac{jr}{N}$, and let χ_j^N $(j = 2, 3, \dots, N)$ and χ_1^N be the characteristic functions of

$$[t_j^N, t_{j-1}^N) \ \ (j = 2, 3, \dots, N) \ \text{ and } \ [t_1^N, t_0^N] = [-r/N, 0],$$

respectively. Define $S^N(t) = e^{\mathcal{A}^N t}$, where $\mathcal{A}^N : Z \to Z$ are given by

$$\mathcal{A}^N(\eta, \phi) = \left(A_0\eta + A_1\phi_N^N,\ \sum_{j=1}^{N} \frac{N}{r}[\phi_{j-1}^N - \phi_j^N]\chi_j^N \right).$$

Let

$$z^N(t) = \sum_{j=0}^{N} w_j^N(t) e_j^N, \qquad e_0^N := (1,0), \quad e_j^N := (0, \chi_j^N) \quad (j = 1, 2, \ldots, N).$$

Then the system of the ordinary differential equations

$$\dot{w}_0^N(t) = A_0 w_0^N(t) + A_1 w_N^N(t) + F(\sum_{j=1}^{N} w_j^N(t) e_j^N),$$
$$\dot{w}_j^N(t) = \frac{N}{r}[w_{j-1}^N - w_j^N(t)] \quad (j = 1, 2, \ldots, N)$$

approximate the delay equation ([3]).

References

[1] U. an der Heiden, Delay in physiological systems, J. Math. Biology **8** (1979), 345-364.

[2] U. an der Heiden and H. O. Walther Existence of chaos in control systems with delayed feedback, J. Differential Equations **47** (1983), 273-295.

[3] H. T. Banks, Delay systems in biological models: approximation techniques. Nonlinear Systems and Applications, An International Conference, Academic Press, 1977, 21-38.

[4] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York & London, 1963.

[5] F. Browder, A further generalization of the Schauder fixed-point theorem, Duke Math. J. **32** (1965), 575-578.

[6] S.-N. Chow, Existence of periodic solutions of autonomous functional differential equations, J. Differential Equations **15** (1974), 350-378.

[7] S.-N. Chow, O. Diekmann and J. Mallet-Paret, Stability, multiplicity and global continuation of symmetric periodic solutions of a nonlinear Volterra integral equation, Japan, J. Appl. Math. **2** (1985), 433-469.

[8] S.-N. Chow and J. Mallet-Paret, Integral averaging and bifurcation, J. Differential Equations **26** (1977), 112-159.

[9] B. D. Coleman and G. Renninger, Periodic solutions of certain nonlinear integral equations with a time lag, SIAM J. Appl. Math. **31** (1976), 111-120.

[10] O. Diekmann and S. A. van Gils, Invariant manifolds for Volterra integral equations of convolution type. J. Differential Equations **54** (1984), 139-180.

[11] A. Eremin, E. Ishiwata, T. Ishiwata, and Y. Nakata, Delay-induced blow-up in a planar oscillation model, Japan J. Indust. Appl. Math. **38** (2021), 1037-1061.

[12] T. Faria and L. T. Magalhaes, Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation, J. Differential Equations **122** (1995), 181-200.

[13] L. Glass and M. Mackey, Oscillation and chaos in physiological control systems. Science **197** (1977), 287-289.

[14] B. C. Goodwin, "Advances in Enzyme Regulation", (G. Weber, ed.), **3**, p. 425, Oxford: Pergamon Press, 1965.

[15] K. P. Hadeler, Periodic solutions of the equation $x'(t) = -f(x(t), x(t-1))$, Math. Methods in the Applied Sciences **1** (1979), 62-69.

[16] J. K. Hale, *Theory of Functional Differential Equations*, Springer, 1977.

[17] J. K. Hale, Nonlinear oscillations in equations with delays, Nonlinear Oscillations in Biology (ed. F. C. Hoppensteadt), AMS Lectures in Applied Math. **17** (1978), 157-186.

[18] J. K. Hale, Effects of delays on dynamics, Granas, A., Frigon, M., Sabidussi, G. (eds), Topological Methods in Differential Equations and Inclusions, NATO ASI Series, vol 472 (1995), Springer, Dordrecht.

[19] J. K. Hale, L. T. Magalhaes and W. M. Oliva, *An Introduction to Infinite Dimensional Dynamical Systems*, Springer, 2013.

[20] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, 1993

[21] F. A. Hopf, D. L. Kaplan, H. M. Gibbs, and R. L. Shoemaker (1982), Bifurcations to chaos in optical bistability, Phys. Rev. A **25** (1982), 2172.

[22] C. E. Hutchinson, Circular causal systems in ecology, Ann. N. Y. Acad. Sci. **50** (1948), 221-246.

[23] K. Ikeda, Opt. Commun. 30. Multiple-valued stationary states and its instability of transmitted light by a ring cavity system, Opt. Commun. **30** (1979), 257-261.

[24] K. Ikeda (池田 研介), Is optical chaos applicable? (光カオスは応用可能か?), 光学, 第 17 巻 第 10 号 (1988), 508-515.

[25] K. Ikeda, H. Daido, and O. Akimoto, Optical turbulence: Chaotic behavior of transmitted light from a ring cavity, Phys. Rev. Lett. **45** (1980), 709.

[26] K. Ikeda, K. Kondo, and O. Akimoto, Successive higher-harmonic bifurcations in systems with delayed feedback, Phys. Rev. Lett. **49** (1982), 1467.

[27] G. Jones, The existence of periodic solutions $f'(x) = -\alpha f(x-1)[1+f(x)]$, J. Math. Ana. Appl. **5** (1962), 435-450.

[28] J. L. Kaplan and J. A. Yorke, On the stability of a periodic solution of a differential delay equation, SIAM J. Math. Anal., **6** (1975), 268-282.

[29] J. L. Kaplan and J. A. Yorke, On the nonlinear differential delay equation $x'(t) = f(x(t), x(t-1)$, J. Differential Equations **23** (1977), 293-314.

[30] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff Ltd., Gronigen, 1964.

[31] J. Kurzweil, Small delays don't matter. In: Chillingworth, D. (eds) Proceedings of the Symposium on Differential Equations and Dynamical Systems. Lecture Notes in Mathematics, vol 206 (1971), Springer, Berlin, Heidelberg.

[32] H. D. Landahl, Some conditions for sustained oscillations in biochemical chains with feedback inhibition, Bull. Math. Biophys. **31** (1969), 775-787.

[33] M. C. Mackey and L. Glass, Oscillation and chaos in physiological control system, Scinece **197** (1977), 287-289.

[34] J. Mallet-Paret, Morse decompositions and global continuation of periodic solutions for singularly perturbed delay equations, Systems of Nonlinear Partial Differential Equations (ed. J. M. Ball), Reidel, Dordrecht, 1983, 351-365.

[35] J. Mallet-Paret, Morse decompositions for delay differential equations, J. Differential Equations **72** (1988), 270-315.

[36] J. Mallet-Paret and R. D. Nussbaum, Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation, Annali di Matematica Pura ed Applicata **145** (1986), 33-128.

[37] Y. Morita, Destabilization of periodic solutions arising in delay-diffusion systems in several space dimensions, Japan J. Appl. Math. **39** (1984), 39-65.

[38] Y. Morita, Stability changes of periodic solutions to a coupled nonlinear equation with time delay, Publ. RIMS Kyoto Univ., **21** (1985), 47-74.

[39] 森田善久著,「生物モデルのカオス」朝倉書店 1996 年.

[40] R. D. Nussbaum, A global bifurcation theorem with applications to functional differential equations, J. Funct. Anal. **19** (1975), 319-338.

[41] H.-O. Walther, Homoclinic solution and chaos in $x'(t) = f(x(t-1)))$, Nonlinear Anal. Theory Methods Appl. **5** (1981), 775-788.

[42] E. M. Wright, A nonlinear differential equation, J. Reine Angew Math. **194** (1955), 66-87.

[43] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, 1996, Springer.

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