The knot quandles of oriented 2-knots

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1 Introduction

The purpose of this note is to give a summery of the results given in the papers [18] and [20]. In this note, an *oriented 1-knot* is an oriented circle smoothly embedded in the 3-sphere S^3, an *oriented 2-knot* is an oriented 2-sphere smoothly embedded in the 4-sphere S^4 and they are correctively called *oriented knots*.

A quandle [9, 11] is an algebra whose axioms correspond to the Reidemeister moves. It is a useful tool to define the oriented knot invariants. A typical example of knot invariants using quandles is the *knot quandle*. The knot quandle of an oriented knot $\mathcal{K}$ is defined as the homotopy classes of pathes in the knot exterior with a certain operation. Since then, some properties of the knot quandle of an oriented 1-knot have been studied.

In this note, we focus on the knot quandles of oriented 2-knots. First, we discussed the differences between knot quandles and knot groups as invariants for oriented 2-knots. By [9, 11], if two oriented knots have the same knot quandle, then their knot groups are isomorphic as groups. For oriented 1-knots, it is known that there is a pair of oriented 1-knots with the same knot group but different knot quandles. In Section 3, we showed that there are infinitely many triples of oriented 2-knots with the same knot group but different knot quandles. This result implies that the knot quandle is really stronger invariant than the knot group for oriented 2-knots. Second, we study the *quandle homology groups* of knot quandles of oriented 2-knots. In [1], Carter, Jelsovsky, Kamada, Langford and Saito introduced the quandle homology group and defined invariants of oriented knots using them, which are called the *quandle cocycle invariants*. In general, it is difficult to determine the quandle homology group of a quandle. We show that the second quandle homology group of the knot quandle of an oriented 2-knot is trivial in Section 4. As a consequence of this result, we see that the knot quandle of a non-trivial oriented 1-knot can not be realized by the knot quandle of 2-knots.

2 Definitions

A *quandle* X [9, 11] is a non-empty set equipped with a binary operation $*$ satisfying the following conditions:

- For any $x \in X$, we have $x * x = x$.
- For any $y \in X$, the map $S_y : X \to X; x \mapsto x * y$ is a bijection.

- For any $x, y, z \in X$, we have $(x * y) * z = (x * z) * (y * z)$.

Here are examples of quandles:

Example 2.1. Let G be a group and $f : G \to G$ a group automorphism. We define the operation $*$ on G by $x * y := f(xy^{-1})y$. Then, $\mathrm{GAlex}(G, f) = (G, *)$ is a quandle, which is called the *generalized Alexander quandle*.

Example 2.2. Let k be an oriented 1-knot. Let $N(k)$ be a tubular neighborhood of k and $E(k) = S^3 \backslash \mathrm{int} N(k)$ an exterior of k. We fix a point $p \in E(k)$. Let $Q(k, p)$ be the set of homotopy classes of all pathes in $E(k)$ from a point in $\partial E(k)$ to p. The set $Q(k, p)$ is a quandle with an operation defined by $[\alpha] * [\beta] := [\alpha \cdot \beta^{-1} \cdot m_{\beta(0)} \cdot \beta)]$, where $m_{\beta(0)}$ is a meridian loop starting from $\beta(0)$ and going along in the positive direction. We call $Q(k, p)$ the *knot quandle* of k. The isomorphism class of the knot quandle does not depend on the base point p. Thus, we denote the knot quandle simply by $Q(k)$. For an oriented 2-knot F, the *knot quandle* $Q(F)$ of F is defined in the same way as for oriented 1-knots.

The *associated group* of X, denoted by $\mathrm{As}(X)$, is the group defined as

$$\langle x \ (x \in X) \mid x * y = y^{-1}xy \ (x, y \in X) \rangle.$$

The associated group $\mathrm{As}(X)$ acts on X from the right by $x \cdot y := x * y$ for any $x, y \in X$. A quandle X is *connected* if the action of $\mathrm{As}(X)$ on X is transitive.

A map $f : X \to Y$ between quandles is a *quandle isomorphism* if $f(x*y) = f(x)*f(y)$ for any $x, y \in X$ and f is a bijection. When there is a quandle isomorphism $f : X \to Y$, we say that X and Y are *quandle isomorphic*.

3 Knot quandles vs Knot groups

In this section, we compare the knot group and the knot quandle. In Subsection 3.1, we review a relation between the knot group and the knot quandle, and introduce our result. In Subsection 3.2, we explain the outline of the proof. This section is a joint work with Kokoro Tanaka.

3.1 Back ground and Main result

Let $\mathcal{K}$ and $\mathcal{K}'$ be oriented knots. We consider the following conditions:

(i) The knot groups $G(\mathcal{K}')$ and $G(\mathcal{K})$ are group isomorphic.

(ii) The knot quandles $Q(\mathcal{K}')$ and $Q(\mathcal{K})$ are quandle isomorphic.

Since the associated group $\mathrm{As}(Q(\mathcal{K}))$ is group isomorphic to the knot group $G(\mathcal{K})$ for an oriented knot $\mathcal{K}$ [9, 11], we have (ii) $\Rightarrow$ (i). For oriented 1-knots, the converse does not hold, that is, there are oriented 1-knots with the same knot group but different knot quandles. For example, the square knot $3_1 \# 3_1^*$ and the granny knot $3_1 \# 3_1$ satisfy the condition (i) but does not satisfy the condition (ii). In this section, we consider the case of oriented 2-knots. More precisely, we consider the following question:

Question 3.1. *Are there oriented 2-knots such that these oriented 2-knots satisfy the condition (i) but do not satisfy the condition (ii)?*

In this note, we give an affirmative answer to this question.

Theorem 3.2. *There exist infinitely many triples $\{F_1, F_2, F_3\}$ of oriented 2-knots such that*

(1) *the knot groups $G(F_1), G(F_2)$ and $G(F_3)$ are mutually group isomorphic, and*

(2) *no two of knot quandles $Q(F_1), Q(F_2)$ and $Q(F_2)$ are quandle isomorphic.*

The 2-knots F_1, F_2 and F_3 are obtained from the *twist spinning construction*, which is introduced by Zeeman [21]. We review the some properties of a twist spun knot. Let k be an oriented 1-knot in S^3 and n a non-negative integer. We denote by $\tau^n(k)$ the *n-twist spun knot* of k. Zeeman introduced the n-twist spun knot and showed that if $n \neq 0$, the n-twist spun knot $\tau^n(k)$ is a fibered 2-knot whose fiber is the once punctured M_k^n, where M_k^n is the n-fold cyclic branched covering space of S^3 branched along k. In particular, the 1-twist spun knot $\tau^1(k)$ is a trivial 2-knot for any oriented 1-knot k.

Remark 3.3. (1) It is known that for any oriented 1-knot k, the knot group of $\tau^n(k)$ is a quotient group of the knot group of k. Thus, the knot groups of $\tau^n(3_1\#3_1)$ and $\tau^n(3_1\#3_1^*)$ are group isomorphic for any non-negative integer n. However, since $\tau^0(3_1\#3_1)$ and $\tau^0(3_1\#3_1^*)$ are equivalent [6, 16], we see that $\tau^n(3_1\#3_1)$ and $\tau^n(3_1\#3_1^*)$ satisfy the condition (ii) for any non-negative integer n (cf. [17]).
(2) Let $S(p, q)$ be the 2-bridge knot of type (p, q). It is known that $\tau^2(S(p, q))$ and $\tau^2(S(p, q'))$ satisfy the condition (i) for any q, q' (see [15]). In [7], Inoue showed that for any q, the knot quandle $Q(\tau^2(S(p, q)))$ is quandle isomorphic to the quandle $\mathrm{GAlex}(\mathbb{Z}/p\mathbb{Z}, \mathrm{Inv})$, where $\mathrm{Inv} : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is the group automorphism defined by $\mathrm{Inv}(x) = -x$. This implies that $\tau^2(S(p, q))$ and $\tau^2(S(p, q'))$ satisfy the condition (ii) for any q, q'. Hence, 2-twist spun 2-bridge knots do not give an example of such 2-knots.

3.2 Outline of proof of Theorem

Let p, q, r be coprime positive integers. We denote the torus knot of type (m, n) by $t_{m,n}$.

Theorem 3.4. [5] *The knot group $G(\tau^p(t_{q,r}))$ is group isomorphic to $\pi_1(M^p_{t_{q,r}}) \times \mathbb{Z}$.*

Since $M^p_{t_{q,r}}, M^q_{t_{r,p}}$ and $M^r_{t_{p,q}}$ are homeomorphic ([12]), the oriented 2-knots $F_1 := \tau^p(t_{q,r}), F_2 := \tau^q(t_{r,p})$ and $F_3 := \tau^r(t_{p,q})$ satisfy the condition (1).

To show the oriented 2-knots F_1, F_2 and F_2 satisfy the condition (2), we focus on the notion of *type* of a quandle. The *type* of a quandle X, denoted by $\mathrm{type}(X)$, is a minimum positive integer n such that S_x^n is the identity map id_X for any $x \in X$. If there is no such n, we define $\mathrm{type}(X) = \infty$. In general, it is difficult to determine $\mathrm{type}(X)$ for a given quandle X. However, it is easy to compute the type of a generalized Alexander quandle.

Proposition 3.5. *Let G be a group and φ a group automorphism of G. The type of $\mathrm{GAlex}(G, \varphi)$ is equal to the order of φ.*

As we mentioned above, the n-twist spun knot $\tau^n(k)$ is a fibered 2-knot for any positive integer n. Inoue studied the structure of the knot quandle of a fibered 2-knot and showed the following theorem.

Theorem 3.6. [7] *Let F be an oriented fibered 2-knot, M the fiber of $S^4\backslash F$ and φ the monodromy of $S^4\backslash F$. Then, the knot quandle $Q(F)$ is quandle isomorphic to* $\mathrm{GAlex}(\pi_1(M), \varphi_*)$*, where $\varphi_* : \pi_1(M) \to \pi_1(M)$ is the group automorphism induced by φ.*

By [21], the monodromy φ of $S^4\backslash\tau^n(k)$ is the canonical covering homeomorphism of M_k^n. In particular, it holds that the order of the induced group automorphism $\varphi_* : \pi_1(M_k^n) \to \pi_1(M_k^n)$ is n. Thus, we obtain the following theorem.

Theorem 3.7. *Let k be an oriented 1-knot and n a positive integer. Then, we have* $\mathrm{type}(Q(\tau^n(k))) = n$.

Since $F_1 = \tau^p(t_{q,r}), F_2 = \tau^q(t_{r,p})$ and $F_3 = \tau^r(t_{p,q})$, we have $\mathrm{type}(Q(F_1)) = p, \mathrm{type}(Q(F_2)) = q$ and $\mathrm{type}(Q(F_3)) = r$. This implies that F_1, F_2 and F_3 satisfy the condition (2).

4 Quandle homology groups of knot quandles

In this section, we discuss the quandle homology group of the knot quandle $Q(\mathcal{K})$. We review the quandle homology group [1] in Subsection 4.1 and the f-twisted Alexander matrix [8] in Subsection 4.2. We give the outline of the proof in Subsection 4.3.

4.1 Definition and Main result.

Let X be a quandle. For each positive integer n, we denote by $C_n^R(X)$ the free abelian group whose basis is X^n. We set $C_0^R(X) = 0$. For each $(x_1, \ldots, x_n) \in X^n$, let us define an element $\partial(x_1, \ldots, x_n) \in C_{n-1}^R(X)$ by

$$\begin{aligned}\partial(x_1, \ldots, x_n) \; &:= \; \sum_{i=2}^{n}(-1)^i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &\quad - \sum_{i=2}^{n}(-1)^i(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n).\end{aligned}$$

Using this, we have a group homomorphism $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ for $n \geq 2$. We define $\partial_1 : C_1^R(X) \to C_0^R(X)$ by the zero map. Then, we see that $\partial_{n-1} \circ \partial_n$ is the zero map. Hence, $(C_n^R(X), \partial_n)$ is a chain complex.

Let $C_n^D(X)$ be the subgroup of $C_n^R(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some i. We can see that $\partial_n(C_n^D(X)) \subset C_{n-1}^D(X)$ for any n. Thus, setting $C_n^Q(X) = C_n^R(X)/C_n^D(X)$, we have a chain complex $(C_n^Q(X), \partial_n)$. The *n-th quandle homology group* $H_n^Q(X)$ [1] is the n-th homology group of the chain complex $(C_n^Q(X), \partial_n)$.

Problem 4.1. *Determine the quandle homology group $H_n^Q(X)$ for a quandle X.*

We consider this problem for knot quandles. Since a knot quandle is connected, we see that $H_1^Q(Q(\mathcal{K})) = \mathbb{Z}$ for an oriented knot $\mathcal{K}$ (cf. [2]). For a 1-knot k, the second and third quandle homology groups of the knot quandle $Q(k)$ have been calculated.

Theorem 4.2. [3] *Let k be a non-trivial oriented 1-knot. Then we have $H_2^Q(Q(k)) = \mathbb{Z}$.*

Theorem 4.3. [14] *Let k be a non-trivial oriented 1-knot. Then we have $H_3^Q(Q(k)) = \mathbb{Z}$.*

In contrast to oriented 1-knots, there are few knot quandles of oriented 2-knots whose second quandle homology groups are computed.

- It is known that for any oriented 1-knot k, the knot quandle of $\tau^0(k)$ is quandle isomorphic to the knot quandle of the *long knot* corresponding to k. Eisermann showed that $H_2^Q(Q(\hat{k}))$ is trivial for any long knot $\hat{k}$. Thus, we have $H_2^Q(Q(\tau^0(k))) = 0$.

- Let $S(p,q)$ be the 2-bridge knot of type (p,q). As we mentioned above, $Q(\tau^2(S(p,q)))$ is quandle isomorphic to $\mathrm{GAlex}(\mathbb{Z}/p\mathbb{Z}, \mathrm{Inv})$. In [13], Mochizuki showed that the second quandle homology group $H_2^Q(\mathrm{GAlex}(\mathbb{Z}/p\mathbb{Z}, \mathrm{Inv}))$ is trivial for any odd prime integer p. Thus, we have $H_2^Q(Q(\tau^2(S(p,q)))) = 0$.

Main result of this section is the following theorem.

Theorem 4.4. *Let F be am oriented 2-knot. Then, we have $H_2^Q(Q(F)) = 0$.*

The proof of Theorem 4.4 is based on the *f-twisted Alexander matrix* introduced by Ishii and Oshiro. The outline of the proof is explained later.

At the end of this section, we discuss a difference between knot quandles of oriented 1-knots and knot quandles of oriented 2-knots. Since the knot group of an oriented 1-knot k is group isomorphic to the knot group of $\tau^0(k)$, we see that

$$\{G(k) \mid k : \text{a 1-knot}\} \subset \{G(F) \mid F : \text{a 2-knot}\}.$$

On the other hand, by Theorem 4.2 and Theorem 4.4, the knot quandle of a non-trivial oriented 1-knot can not be realized by the knot quandle of oriented 2-knots, that is, it holds that

$$\{Q(k) \mid k : \text{a non-trivial 1-knot}\} \cap \{Q(F) \mid F : \text{a 2-knot}\} = \emptyset.$$

4.2 f-twisted Alexander matrix

Let X be a quandle, and R a ring with the unity 1. A pair (f_1, f_2) of maps $f_1, f_2 : X \times X \to R$ is called an *Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $x \in X$, we have $f_1(x,x) + f_2(x,x) = 1$.

- For any $x, y \in X$, $f_1(x,y)$ is invertible.

- For any $x, y, z \in X$, we have

$$\begin{aligned}
&f_1(x*y, z) f_1(x,y) = f_1(x*z, y*z) f_1(x,z),\\
&f_1(x*y, z) f_2(x,y) = f_2(x*z, y*z) f_1(y,z), \text{ and}\\
&f_2(x*y, z) = f_1(x*z, y*z) f_2(x,z) + f_2(x*z, y*z) f_2(y,z).
\end{aligned}$$

Example 4.5. Let X be a quandle and $\mathbb{Z}[t^{\pm 1}]$ the ring of Laurent polynomials with integer coefficients. The maps $f_1, f_2 : X \times X \to \mathbb{Z}[t^{\pm 1}]$ defined by $f_1(x, y) := t$ and $f_2(x, y) := 1 - t$ give an Alexander pair.

Example 4.6. Let X be a quandle and A an abelian group. A map $\theta : X \times X \to A$ is a *quandle 2-cocycle* [1] if it satisfies the following conditions:

- For any $x \in X$, we have $\theta(x, x) = 0_A$, where 0_A is the identity element.
- For any $x, y, z \in X$, we have $\theta(x * y, z) + \theta(x, y) = \theta(x * z, y * z) + \theta(x, z)$.

Let $\theta : X \times X \to A$ be a quandle 2-cocycle and $\mathbb{Z}[A]$ the group ring. We set maps $f_\theta, 0 : X \times X \to \mathbb{Z}[A]$ by $f_\theta(x, y) := 1 \cdot \theta(x, y)$ and $0(x, y) = 0$. Then, the pair $(f_\theta, 0)$ is an Alexander pair. We call this Alexander pair $(f_\theta, 0)$ the *Alexander pair associated with a quandle 2-cocycle* θ [19].

Next, we review the definition of the f-twisted Alexander matrix. Refer to [8] for more details. Let $FQ(S)$ the free quandle on a finite set $S = \{x_1, \ldots, x_n\}$, Q a quandle with a finite presentation of $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ and $\mathrm{pr} : FQ(S) \to Q$ the canonical projection. In this note, we omit pr to present $\mathrm{pr}(a)$ as a. Let R be a ring with the unity 1 and $f = (f_1, f_2)$ an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$. For $x_j \in S$, the *f-derivative with respect to x_j* [8] is a map $\dfrac{\partial_f}{\partial x_j} : FQ(S) \to R$ satisfies the following conditions:

- For any $x_i \in S$, we have $\dfrac{\partial_f}{\partial x_j}(x_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
- For any $x, y \in FQ(S)$, we have
$$\frac{\partial_f}{\partial x_j}(x * y) = f_1(x, y)\frac{\partial_f}{\partial x_j}(x) + f_2(x, y)\frac{\partial_f}{\partial x_j}(y).$$

For a relator $r = (r_1, r_2)$, we set $\dfrac{\partial_f}{\partial x_j}(r) := \dfrac{\partial_f}{\partial x_j}(r_1) - \dfrac{\partial_f}{\partial x_j}(r_2)$.

Let A be an $m \times n$ matrix over a commutative ring R. The *d-th elementary ideal* of A, denoted by $E_d(A)$, is the ideal generated by all $(n - d)$-minors of A if $n - m \leq d < n$, and $E_d(A) = \begin{cases} 0 & \text{if } d < n - m, \\ R & \text{if } n \leq d. \end{cases}$

Let Q be a quandle with a finite presentation of $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$, X a quandle, $\rho : Q \to X$ a quandle homomorphism, R a ring with the unity 1 and $f = (f_1, f_2)$ an Alexander pair of maps $f_1, f_2 : X \times X \to R$. We set $f \circ (\rho \times \rho) = (f_1 \circ (\rho \times \rho),) f_2 \circ (\rho \times \rho)$. Then the pair $f \circ (\rho \times \rho)$ is also an Alexander pair. The *f-twisted Alexander matrix of* (Q, ρ) [8], which is denoted by $A(Q, \rho; f_1, f_2)$, is the $m \times n$ matrix defined by
$$A(Q, \rho; f_1, f_2) := \begin{pmatrix} \dfrac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(r_1) & \cdots & \dfrac{\partial_{f\circ(\rho\times\rho)}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial_{f\circ(\rho\times\rho)}}{\partial x_1}(r_m) & \cdots & \dfrac{\partial_{f\circ(\rho\times\rho)}}{\partial x_n}(r_m) \end{pmatrix}.$$

Suppose that R is a commutative ring. Let Q' be a finitely presented quandle and $\rho' : Q' \to X$ a quandle homomorphism. Ishii and Oshiro [8] showed that if there is a quandle isomorphism $\varphi : Q \to Q'$ such that $\rho = \rho' \circ \varphi$, then we have $E_d(A(Q, \rho; f_1, f_2)) = E_d(A(Q', \rho'; f_1, f_2))$ for all d.

4.3 Outline of proof of Theorem

Let $\theta : X \times X \to A$ be a quandle 2-cocycle. By the definition, the linear extension $\theta : \mathbb{Z}[X \times X] \to A$ is a 2-cocycle of $C^2_Q(X; A)$. Thus, we can regard θ as a group homomorphism from $H^Q_2(X)$ to A. In [20], we proved the following theorems:

Theorem 4.7. *Let Q be a connected quandle with a finite presentation, X a quandle, A an abelian group and $\theta : X \times X \to A$ a quandle 2-cocycle. For any quandle homomorphism $\rho : Q \to X$, we have*

$$E_0(A(Q, \rho; f_\theta, 0)) = (\{1 \cdot a - 1 \cdot 0_A \mid a \in \mathrm{Im}(\theta \circ \rho_*)\}) \subset \mathbb{Z}[A],$$

where $\rho_* : H^Q_2(Q) \to H^Q_2(X)$ is the group homomorphism induced by ρ.

Theorem 4.8. *Let F be an oriented 2-knot, X a quandle, A an abelian group and $\theta : X \times X \to A$ a quandle 2-cocycle. For any quandle homomorphism $\rho : Q \to X$, we have*

$$E_0(A(Q(F), \rho; f_\theta, 0)) = (0).$$

Let F be an oriented 2-knot. Suppose that $H^Q_2(Q(F))$ is non-trivial. We set $A := H^Q_2(Q(F)), X := Q(F)$ and $\rho := \mathrm{id} : Q(F) \to Q(F) = X$. By the universal coefficient theorem, there is a quandle 2-cocycle $\theta : X \times X \to A$ such that the group homomorphism $\theta \circ \rho_* : H^Q_2(Q(F)) \to A = H^Q_2(Q(F))$ coincides with the identity map on $H^Q_2(Q(F))$. By the assumption and Theorem 4.7, we have $E_0(A(Q, \rho; f_\theta, 0)) \neq (0)$. This contradicts to Theorem 4.8.

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