The rational abelianization of the Chillingworth subgroup of the mapping class group of a surface

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1 Introduction

The Chillingworth subgroup $Ch_{g,1}$ of the mapping class group of a compact oriented connected surface of genus g with one boundary component is defined as the subgroup of the mapping class group of the surface, whose elements preserve nonsingular vector fields on the surface up to homotopy. We determined the rational abelianization (which is naturally isomorphic to the first rational homology group $H_1(Ch_{g,1};\mathbb{Q})$) of the Chillingworth subgroup as a full mapping class group module. The rational abelianization is given by the first Johnson homomorphism and the Casson–Morita homomorphism for the Chillingworth subgroup. In this paper, we present some obtained results and provide an outline of the proof. For more details, refer to [10].

2 Preliminaries

Let $\Sigma_{g,1}$ (resp. $\Sigma_{g,*}$, Σ_g) be a compact oriented connected surface of genus g with one boundary component (resp. once punctured, no boundary and no puncture), and $\mathcal{M}_{g,1}$ (resp. $\mathcal{M}_{g,*}$, $\mathcal{M}_g$) be the **mapping class group** of the surface, which is defined by isotopy classes of orientation preserving self-diffeomorphisms of the surface that are pointwise identities on the boundary and the puncture of the surface. i.e., the mapping class group is the quotient of the self-diffeomorphism group of the surface by the identity component.

$$\mathcal{M}_{g,1} = \mathrm{Diff}^{(+)}(\Sigma_{g,1}, \partial\Sigma_{g,1})/\mathrm{Diff}_0^{(+)}(\Sigma_{g,1}, \partial\Sigma_{g,1})$$

$$\mathcal{M}_{g,*} = \mathrm{Diff}^{+}(\Sigma_{g,1}, *)/\mathrm{Diff}_0^{+}(\Sigma_{g,1}, *)$$

$$\mathcal{M}_{g} = \mathrm{Diff}^{+}(\Sigma_{g,1})/\mathrm{Diff}_0^{+}(\Sigma_{g,1})$$

There exists natural homomorphisms $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to \mathcal{M}_g$ among these induced by the natural quotient maps $\Sigma_{g,1} \to \Sigma_{g,*} \to \Sigma_g$ of the surfaces. For $\Sigma_{g,1}$, the mapping class group naturally acts on the fundamental group $\pi = \pi_1(\Sigma_{g,1})$ of the surface and the first integral homology group $H = H_1(\Sigma_{g,1};\mathbb{Z})$ of the surface. The action $r\colon \mathcal{M}_{g,1} \to \mathrm{Aut}(\pi)$ of the mapping class group of the surface on the fundamental group of the surface is called the Dehn–Nielsen representation, which is known to be faithful by the Dehn–Nielsen theorem. the action $\rho\colon \mathcal{M}_{g,1} \to \mathrm{Aut}(H)$ of the mapping class group of the surface on the first homology group of the surface is called the symplectic representation because the action preserves the intersection form $\cdot\colon H \otimes H \to \mathbb{Z}$. Therefore, we have $\rho\colon \mathcal{M}_{g,1} \to \mathrm{Aut}(H, \cdot) \cong \mathrm{Sp}(2g, \mathbb{Z})$. It is known that the representation ρ is surjective classically. The kernel $\mathcal{I}_{g,1} \coloneqq \mathrm{Ker}(\rho\colon \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbb{Z}))$ of the symplectic representation is called the **Torelli group**. We summarize this in the following short exact sequence:

$$1 \to \mathcal{I}_{g,1} \to \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1.$$

Similarly, we can define Torelli groups for $\mathcal{M}_{g,*}$ and $\mathcal{M}_g$, and these are denoted by $\mathcal{I}_{g,*}$ and $\mathcal{I}_g$, respectively.

The (first) **Johnson homomorphism** was initially defined by Johnson which is an abelian quotient of the Torelli group and is equivariant under the action of the mapping class group (see [6], [7]), it has been developed by Morita and formalized as a graded Lie algebra homomorphism using the free Lie algebra generated by H (see [12], [13], [15]). In this paper, we introduce the first Johnson homomorphism by following Johnson's original procedure. The mapping class group naturally acts on the nilpotent quotient of the fundamental group of the surface, denoted by $N_i \coloneqq \pi/\Gamma_i\pi$, where $\{\Gamma_i\pi\}_{i\geq 1}$ is the lower central series of π, i.e., $\Gamma_1 \coloneqq \pi$ and $\Gamma_{i+1} \coloneqq [\Gamma_i, \pi]$. These actions define a filtration of the mapping class group, denoted by $\mathcal{M}_{g,1}[i] \coloneqq \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{Aut}(N_i))$, called the Johnson filtration. We have $\mathcal{M}_{g,1}[1] = \mathcal{M}_{g,1}$, $\mathcal{M}_{g,1}[2] = \mathcal{I}_{g,1}$, and $\mathcal{M}_{g,1}[3] = \mathcal{K}_{g,1} \coloneqq \langle \text{Dehn twists along separating simple closed curves} \rangle$ where $\mathcal{K}_{g,1}$ is called the **Johnson kernel**; shown by Johnson in [8]. For $\varphi \in \mathcal{I}_{g,1} = \mathcal{M}_{g,1}[2]$ and $\gamma \in \pi$, we have $\varphi(\gamma)\gamma^{-1} \in \Gamma_2\pi$ by definition. Therefore, this defines a homomorphism $\mathcal{I}_{g,1} \to \mathrm{Hom}(H, \Gamma_2\pi/\Gamma_3\pi) \cong H^* \otimes \bigwedge^2 H \cong H \otimes \bigwedge^2 H$, where $\bigwedge^* H$ is the exterior power of H. This homomorphism is equivariant under the action of the mapping class group. He showed in [6] that this map is surjective to the submodule $\bigwedge^3 H = \{x \wedge y \wedge z \coloneqq x \otimes (y \wedge z) + y \otimes (z \wedge x) + z \otimes (x \wedge y) \mid x, y, z \in H\} \subset H \otimes \bigwedge^2 H$. We have $\tau_{g,1}(1)\colon \mathcal{I}_{g,1} \to \bigwedge^3 H$ and called the first Johnson homomorphism. For $\mathcal{I}_{g,*}$, we can define the first Johnson homomorphism $\tau_{g,*}(1)\colon \mathcal{I}_{g,*} \to \bigwedge^3 H$ by a similar argument. For $\mathcal{I}_g$, with a few adjustments, we can define the first Johnson homomorphism $\tau_g(1)\colon \mathcal{I}_g \to \bigwedge^3 H/H$. These Johnson homomorphisms commute with natural homomorphisms. We have the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{I}_{g,1} & \xrightarrow{\tau_{g,1}(1)} & \bigwedge^3 H \\
\downarrow & & \downarrow \\
\mathcal{I}_{g,*} & \xrightarrow{\tau_{g,*}(1)} & \bigwedge^3 H \\
\downarrow & & \downarrow \\
\mathcal{I}_g & \xrightarrow{\tau_g(1)} & \bigwedge^3 H/H
\end{array}$$

3 The action on the set of homotopy classes of nonsingular vector fields and the definition of the Chillingworth subgroup

Let X be a nonsingular vector field on the surface $\Sigma_{g,1}$ and $\Xi(\Sigma_{g,1})$ be the set of homotopy classes of nonsingular vector fields on the surface. The mapping class group acts on $\Xi(\Sigma_{g,1})$ naturally. Let γ be an oriented regular closed curve on the surface. The winding number of γ with respect to X denoted by $\omega_X(\gamma)$ is defined by the number of times its tangent transversely intersects with the section of the unit tangent bundle $UT\Sigma_{g,1} \to \Sigma_{g,1}$ induced by X. The winding number function ω_X is regarded as an element of the first integral cohomology group $H^1(UT\Sigma_{g,1};\mathbb{Z})$ of the unit tangent bundle of the surface and these elements are characterized by the preimage of $1 \in H^1(S^1;\mathbb{Z})$ under $H^1(UT\Sigma_{g,1};\mathbb{Z}) \to H^1(S^1;\mathbb{Z})$. Especially, the mapping $e_X \colon \mathcal{M}_{g,1} \to H^1(\Sigma_{g,1};\mathbb{Z}) = H^*, f \mapsto ([\gamma] \mapsto \omega_X(f\circ\gamma) - \omega_X(\gamma))$ is well-defined, and the map $e_X \colon \mathcal{M}_{g,1} \to H^*$ is called the **Chillingworth homomorphism**. The Chillingworth homomorphism is not a homomorphism but a crossed homomorphism, i.e., $e_X(fg) = e_X(g) + (g^{-1})^* e_X(f)$. The kernel of the Chillingworth homomorphism $\mathrm{Ker}(e_X) := {e_X}^{-1}(0)$ is the subgroup of the mapping class group whose elements preserve X up to homotopy. In particular, the Chillingworth homomorphism e_X depends on the choice of a vector field X. Let us consider the restriction of the Chillingworth homomorphism to the Torelli group. The restricted Chillingworth homomorphism $e_X|_{\mathcal{I}_{g,1}}$ is a homomorphism in the usual sense. Moreover, the restricted Chillingworth homomorphism does not depend on the choice of a nonsingular vector field on the surface. The Chillingworth subgroup $Ch_{g,1}$ is defined by the kernel $\mathrm{Ker}(e_X|_{\mathcal{I}_{g,1}})$ of the restricted Chillingworth homomorphism, the Chillingworth subgroup of the once punctured surface $Ch_{g,*}$ is defined similarly, and we *define* the Chillingworth subgroup of the closed surface as the image of the Chillingworth subgroup under the natural homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to \mathcal{M}_g$. Morita proved in [14] that $H^1(\mathcal{M}_{g,1};H^*) \cong H^1(\mathcal{M}_{g,1};H)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$ and the twisted 1-cocycle e_X is a generator of $H^1(\mathcal{M}_{g,1};H)$. Hence, the Chillingworth subgroup is characterized by the kernel $\mathrm{Ker}(\mathcal{M}_{g,1} \curvearrowright \Xi(\Sigma_{g,1}))$ of the action on the set of homotopy classes of nonsingular vector fields on the surface.

Although the proof is omitted, we have considered a generating system and obtained the following proposition.

Proposition 1 *For $g \geq 3$, Chillingworth subgroup $Ch_{g,1}$ is normally generated by one element $B_0 := T_{\gamma_2'}{T_{\gamma_3'}}^{-1}$ as shown in the* Figure 1 *and the Johnson kernel $\mathcal{K}_{g,1}$ in the full mapping class group, where T_γ is the Dehn twist along γ. We have $Ch_{g,1} = \langle\langle B_0 \rangle\rangle \mathcal{K}_{g,1}$.*

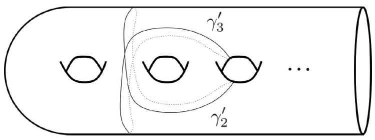

Figure 1: Simple closed curves γ_2', γ_3' defining $B_0 := T_{\gamma_2'}{T_{\gamma_3'}}^{-1}$

4 Representation theory of the symplectic group

If we take the tensor product with $\mathbb{Q}$ (we use the notation subscripted with $\mathbb{Q}$), then we can handle $\mathrm{Sp}(2g,\mathbb{Z})$-modules as representations of the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$. By general representation theory, every finite dimensional polynomial representation of the rational Symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$ is in one-to-one correspondence with that of $\mathrm{Sp}(2g;\mathbb{C})$ and $\mathfrak{sp}(2g;\mathbb{C})$, and these are parametrized by Young diagrams.

As examples, we summarize the results of the irreducible decompositions of several representations of the rational symplectic group that appear in the subsequent discussions.

Proposition 2 *For $g \geq 3$, we have the following irreducible decompositions as $\mathrm{Sp}(2g,\mathbb{Q})$-modules*

1. $H_\mathbb{Q} \cong H^*_\mathbb{Q} = [1]_{\mathrm{Sp}}$
2. $\bigwedge^3 H_\mathbb{Q} = [1^3]_{\mathrm{Sp}} \oplus [1]_{\mathrm{Sp}}$
3. $U_\mathbb{Q} := \mathrm{Im}(\tau_{g,1}(1)\colon Ch_{g,1} \to \bigwedge^3 H)_\mathbb{Q} = [1^3]_{\mathrm{Sp}}$
4. $(\bigwedge^3 H/H)_\mathbb{Q} = [1^3]_{\mathrm{Sp}}$
5. $H^2(\bigwedge^3 H/H;\mathbb{Q}) \cong H_2(\bigwedge^3 H/H;\mathbb{Q}) \cong H^2(U;\mathbb{Q}) \cong H_2(U;\mathbb{Q}) \cong \bigwedge^2[1^3]_{\mathrm{Sp}}$

$$= \begin{cases} [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}} & (g \geq 6) \\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} & (g = 5) \\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} & (g = 4) \\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} & (g = 3) \end{cases} \quad \text{(Hain [5])}$$

5 Casson Morita homomorphism

Morita introduced a certain *map* $d\colon \mathcal{M}_{g,1} \to \mathbb{Z}$ related to the Casson invariant in [13]. Here, we only present the construction of d without delving into its relationship with the Casson invariant. To define the **Casson–Morita homomorphism** d, we introduce some 2-cocycles of the full mapping class group $\mathcal{M}_{g,1}$. Let $\tau\colon \mathcal{M}_{g,1} \times \mathcal{M}_{g,1} \to \mathbb{Z}$ be the *Meyer cocycle* characterized by the signature of the 4-manifold defined by the surface Σ_g bundle over a pair of pants $\Sigma_{0,3}$ with corresponding monodromies (see [11]). Next, let $k\colon \mathcal{M}_{g,1} \to H^{(*)}$ be a crossed homomorphism representing a generator of $H^1(\mathcal{M}_{g,1};H^{(*)}) \cong \mathbb{Z}$, for example the Chillingworth homomorphism $k = e_X$. We define the 2-cocycle $c\colon \mathcal{M}_{g,1} \times \mathcal{M}_{g,1} \to \mathbb{Z}$ by $c(\varphi,\psi) := k(\varphi)\cdot k(\psi^{-1})$ called the *intersection cocycle*. These 2-cocycle are related by $[-3\tau] = [c] \in H^2(\mathcal{M}_{g,1};\mathbb{Z})$, and for $g \geq 3$, $H^1(\mathcal{M}_{g,1};\mathbb{Z}) = 0$ holds. Therefore, there exist a unique map $d\colon \mathcal{M}_{g,1} \to \mathbb{Z}$ such that the coboundary δd coincides with $c + 3\tau$ as 2-cocycles. We have the following by definition.

Proposition 3 *For φ, $\psi \in \mathcal{M}_{g,1}$, we have*

$$d(\varphi\psi) = d(\varphi) + d(\psi) - k(\varphi)\cdot k(\psi^{-1}) - 3\tau(\varphi,\psi).$$

By this equality, $d = d|_{Ch_{g,1}} \colon Ch_{g,1} \to \mathbb{Z}$ is a homomorphism on the Chillingworth subgroup because the Meyer cocycle τ is vanish on the Torelli group $\mathcal{I}_{g,1}$ and the crossed homomorphism k is trivial on the Chillingworth subgroup $Ch_{g,1}$.

We studied the Casson–Morita homomorphism for the Chillingworth subgroup $Ch_{g,1}$ and determined the image and the kernel.

Proposition 4 *The Casson–Morita homomorphism for the Chillingworth subgroup $Ch_{g,1}$ is an $\mathcal{M}_{g,1}$-invariant homomorphism and the image* $\mathrm{Im}(d\colon Ch_{g,1} \to \mathbb{Z})$ *coincides with* $8\mathbb{Z}$.

For the kernel $\mathrm{Ker}(d\colon Ch_{g,1} \to \mathbb{Z})$, we omit the proof, a result of Faes in the case of the Johnson kernel inspires our study (see [3]).

Theorem 1 *The kernel* $\mathrm{Ker}(d\colon Ch_{g,1} \to \mathbb{Z})$ *of the Casson–Morita homomorphism for the Chillingworth subgroup is given by the subgroup $\langle T_{\gamma_1'} \rangle$ generated by Dehn twists along the boundary of a genus one subsurface with one boundary of the surface as shown in* Figure 2, *the normal subgroup $\langle\langle B_0 \rangle\rangle \lhd \mathcal{M}_{g,1}$ generated by a certain element $B_0 := T_{\gamma_2'} T_{\gamma_3'}{}^{-1}$ as shown in* Figure 3, *and the commutator subgroup $[\mathcal{K}_{g,1}, \mathcal{M}_{g,1}]$ of the Johnson kernel and the full mapping class group as follows:*

$$\mathrm{Ker}(d\colon Ch_{g,1} \to \mathbb{Z}) = \langle\langle B_0 \rangle\rangle \langle T_{\gamma_1'} \rangle [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}].$$

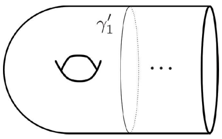

Figure 2: the boundary curve γ_1' of a genus one subsurface with one boundary of the surface defining $T_{\gamma_1'}$

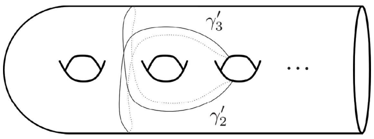

Figure 3: Simple closed curves γ_2', γ_3' defining $B_0 := T_{\gamma_2'} T_{\gamma_3'}{}^{-1}$

6 Main theorem

For the Torelli group, the rational abelianization is obtained from the first Johnson homomorphism as a mapping class group module. More precisely, Johnson showed in [9] that the abelianization of the Torelli group is isomorphic to the direct sum of the target space of the Johnson homomorphism and some 2-torsion parts: the target space of the Birman–Craggs homomorphism which is related to spin structures and the Rokhlin invariant (see [1]). The first Johnson homomorphism for the Chillingworth subgroup is one of the abelian quotients of the Chillingworth subgroup. However, the Casson–Morita homomorphism exists which is a homomorphism on the Chillingworth subgroup and nontrivial on the kernel of the first Johnson homomorphism. Therefore, $d \oplus \tau_{g,1}(1) \colon Ch_{g,1} \to (8\mathbb{Z} \oplus U) \otimes \mathbb{Q} \cong [0]_{\mathrm{Sp}} \oplus [1^3]_{\mathrm{Sp}}$ is a better lower bound for the rational abelianization $H_1(Ch_{g,1}; \mathbb{Q}) \cong (Ch_{g,1})^{ab} \otimes \mathbb{Q}$ of the Chillingworth subgroup.

To determine the rational abelianization of the Chillingworth subgroup, we consider the *inflation-restriction exact sequence* of the rational homology for the short exact sequence

$1 \to \mathcal{K}_{g,1} \to Ch_{g,1} \to U \to 0$ induced by the first Johnson homomorphism for the Chillingworth subgroup is as follows:

$$H_2(Ch_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q}) \to H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \to H_1(Ch_{g,1};\mathbb{Q}) \to H_1(U;\mathbb{Q}) \cong U_{\mathbb{Q}} \to 0,$$

and we determine the structure of the image $\mathrm{Im}((\tau_{g,1}(1))_*\colon H_2(Ch_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q}))$ of the homomorphism between second rational homology group induced by the first Johnson homomorphism and the U-coinvariant $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U$ of the rational abelianization of the Johnson kernel $\mathcal{K}_{g,1}$ as $\mathcal{M}_{g,1}$-modules. This exact sequence is equivariant under the natural action of the mapping class group $\mathcal{M}_{g,1}$.

Hain studied the homomorphism $(\tau_g(1))^*\colon H^2(\bigwedge^3 H/H;\mathbb{Q}) \to H^2(\mathcal{I}_g;\mathbb{Q})$ between the second rational cohomology group induced by the first Johnson homomorphism and determined the kernel of this map as an $\mathrm{Sp}(2g,\mathbb{Q})$-module using representation theory.

Theorem 2 (Hain [5]) *For $g \geq 3$, we have*

$$\mathrm{Ker}\left((\tau_g(1))^*\colon H^2(\textstyle\bigwedge^3 H/H;\mathbb{Q}) \to H^2(\mathcal{I}_g;\mathbb{Q})\right) = [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}}$$

as an $\mathrm{Sp}(2g,\mathbb{Q})$*-module.*

Moreover, the dual of the preceding implies that the image $(\tau_g(1))_*\colon H_2(\mathcal{I}_g;\mathbb{Q}) \to H_2(\bigwedge^3 H/H;\mathbb{Q})$ of the homomorphism between the second rational homology induced by the first Johnson homomorphism is decomposed as $\mathrm{Sp}(2g,\mathbb{Q})$-modules as follows:

Theorem 3 (Hain [5]) *For $g \geq 3$, we have*

$$\mathrm{Im}\left((\tau_g(1))_*\colon H_2(\mathcal{I}_g;\mathbb{Q}) \to H_2(\textstyle\bigwedge^3 H/H;\mathbb{Q})\right) = \begin{cases} [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}} & (g \geq 6) \\ [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} & (g = 5) \\ [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} & (g = 4) \\ \{0\} & (g = 3) \end{cases}$$

as an $\mathrm{Sp}(2g,\mathbb{Q})$*-module.*

We determined the problem of the Chillingworth subgroup version regarding the problems mentioned above. By the above, we can deduce $[0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \subset \mathrm{Ker}((\tau_{g,1})^*\colon H^2(U;\mathbb{Q}) \to H^2(Ch_{g,1};\mathbb{Q}))$. We showed that, in fact, nontrivial summand $[1^2]_{\mathrm{Sp}}$ that does not comply with the above further appears in the kernel for $g \geq 3$.

Theorem 4 *For $g \geq 3$, we have*

$$\mathrm{Ker}\left((\tau_{g,1}(1))^*\colon H^2(U;\mathbb{Q}) \to H^2(Ch_{g,1};\mathbb{Q})\right) = \begin{cases} [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} & (g \geq 4) \\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} & (g = 3) \end{cases}$$

and

$$\mathrm{Im}\left((\tau_{g,1}(1))_*\colon H_2(Ch_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\right) = \begin{cases} [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}} & (g \geq 6) \\ [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} & (g = 5) \\ [2^2 1^2]_{\mathrm{Sp}} & (g = 4) \\ \{0\} & (g = 3) \end{cases}$$

as $\mathrm{Sp}(2g,\mathbb{Q})$*-modules.*

These are proven by constructing cycles called abelian cycles for the summands $[2^21^2]_{\mathrm{Sp}}$, $[1^4]_{\mathrm{Sp}}$, and $[1^6]_{\mathrm{Sp}}$ and studying the bracket of the graded Lie algebra defined from the lower central series of the Chillingworth subgroup for the summand $[1^2]_{\mathrm{Sp}}$.

Next, to determine the structure of the U-coinvariant $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U$ of the rational abelianization of the Johnson kernel, we use the rational abelianization by Faes and Massuyeau [4]. Here, we do not carry out the construction, but it is described by the Casson–Morita homomorphism for the Johnson kernel $d = d|_{\mathcal{K}_{g,1}}\colon \mathcal{K}_{g,1} \to 8\mathbb{Z}$ and a certain homomorphism $(r_2^\theta, r_3^\theta)\colon \mathcal{K}_{g,1} \to \mathcal{T}_2(H_\mathbb{Q}) \times \mathrm{Ker}(\mathrm{Tr}_3)$ known as the truncations of the *infinitesimal Dehn-Nielsen representation* for $g \geq 6$.

Theorem 5 (Faes–Massuyeau [4]) *For $g \geq 6$, the rational abelianization of the Johnson kernel $H_1(\mathcal{K}_{g,1};\mathbb{Q})$ as an $\mathcal{M}_{g,1}$-module is given by the Casson–Morita homomorphism d and the truncations of the infinitesimal Dehn–Nielsen representation (r_2^θ, r_3^θ) as follows:*

$$d \oplus (r_2^\theta, r_3^\theta)\colon \mathcal{K}_{g,1} \to \mathbb{Q} \oplus (\mathcal{T}_2(H_\mathbb{Q}) \times \mathrm{Ker}(\mathrm{Tr}_3)) \subset \mathbb{Q} \oplus (\mathcal{T}_2(H_\mathbb{Q}) \times \mathcal{T}_3(H_\mathbb{Q})).$$

Remark 1 *The action of the mapping class group $\mathcal{M}_{g,1}$ on $\mathbb{Q}\oplus(\mathcal{T}_2(H_\mathbb{Q})\times\mathrm{Ker}(\mathrm{Tr}_3))$ does not factor through the integral symplectic group $\mathrm{Sp}(2g;\mathbb{Z})$.*

From the above and direct computation, we obtain the following.

Proposition 5 *The U-coinvariant $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U$ of the rational abelianization of the Johnson kernel is isomorphic to $\mathbb{Q}\oplus\mathcal{T}_2(H_\mathbb{Q}) \cong [0]_{\mathrm{Sp}}\oplus([0]_{\mathrm{Sp}}\oplus[2^2]_{\mathrm{Sp}}\oplus[1^2]_{\mathrm{Sp}})$ as an $\mathcal{M}_{g,1}$-module, and the action of the mapping class group on it factors through the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$.*

Now, we handle the inflation-restriction exact sequence of the rational homology for the short exact sequence $1 \to \mathcal{K}_{g,1} \to Ch_{g,1} \to U \to 0$ to determine the rational abelianization $H_1(Ch_{g,1};\mathbb{Q})$ of the Chillingworth subgroup. For $g \geq 6$, we have determined the image $\mathrm{Im}(H_2(Ch_{g,1};\mathbb{Q}) \to \bigwedge^2 U) \cong [2^21^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}}$ of the homomorphism between the second rational homology group induced by the first Johnson homomorphism for the Chillingworth subgroup, the U-coinvariant $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \cong [0]_{\mathrm{Sp}} \oplus ([0]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}})$ of the rational abelianization of the Johnson kernel and $U_\mathbb{Q} \cong [1^3]_{\mathrm{Sp}}$. By adding the information obtained from the above to the long exact sequence, we obtain

$$\begin{array}{ccccccccc} H_2(Ch_{g,1};\mathbb{Q}) & \xrightarrow{(\tau_{g,1}(1))_*} & \underset{\substack{([2^21^2]_{\mathrm{Sp}}\oplus[1^4]_{\mathrm{Sp}}\oplus[1^6]_{\mathrm{Sp}})\\ \oplus([0]_{\mathrm{Sp}}\oplus[2^2]_{\mathrm{Sp}}\oplus[1^2]_{\mathrm{Sp}})}}{H_2(U;\mathbb{Q})} & \to & \underset{\substack{([0]_{\mathrm{Sp}}\oplus[2^2]_{\mathrm{Sp}}\oplus[1^2]_{\mathrm{Sp}})\\ \oplus[0]_{\mathrm{Sp}}}}{H_1(\mathcal{K}_{g,1};\mathbb{Q})_U} & \to & \underset{\substack{\text{“}[0]_{\mathrm{Sp}}\\ \oplus[1^3]_{\mathrm{Sp}}\text{”}}}{H_1(Ch_{g,1};\mathbb{Q})} & \to & \underset{[1^3]_{\mathrm{Sp}}}{U_\mathbb{Q}} \to 0. \end{array}$$

We note that the preceding argument alone does not determine whether $H_1(Ch_{g,1};\mathbb{Q})$ decompose into a direct sum of two summands $[0]_{\mathrm{Sp}}$ and $[1^3]_{\mathrm{Sp}}$ as an $\mathcal{M}_{g,1}$-module. However, we do have $\dim_\mathbb{Q} H_1(Ch_{g,1};\mathbb{Q}) = \dim_\mathbb{Q}([0]_{\mathrm{Sp}}\oplus[1^3]_{\mathrm{Sp}})$. Combining this with the lower bound of the rational abelianization of the Chillingworth subgroup already obtained, $d \oplus \tau_{g,1}(1)\colon Ch_{g,1} \to \mathbb{Q} \oplus U_\mathbb{Q} \cong [0]_{\mathrm{Sp}} \oplus [1^3]$ gives the rational abelianization of the Chillingworth subgroup. Therefore, this long exact sequence splits at the $H_1(Ch_{g,1};\mathbb{Q})$ as an $\mathcal{M}_{g,1}$-module.

Thus, we conclude.

Theorem 6 *For $g \geq 6$, the first rational homology (rational abelianization) and the first rational cohomology of the Chillingworth subgroup $Ch_{g,1}$ for the genus g surface with one boundary are induced by the Casson–Morita homomorphism and the first Johnson homomorphism for the Chillingworth subgroup $d \oplus \tau_{g,1}(1)\colon Ch_{g,1} \to 8\mathbb{Z} \oplus U$, and are as follows:*

$$(Ch_{g,1})^{ab} \otimes \mathbb{Q} \cong H_1(Ch_{g,1};\mathbb{Q}) \cong [1^3]_{\mathrm{Sp}} \oplus [0]_{\mathrm{Sp}}$$
$$((Ch_{g,1})^{ab} \otimes \mathbb{Q})^* \cong H^1(Ch_{g,1};\mathbb{Q}) \cong [1^3]_{\mathrm{Sp}} \oplus [0]_{\mathrm{Sp}}$$

as $\mathcal{M}_{g,1}$-modules.

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