Quantum invariants based on ideal triangulations

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1 Introduction

This is a survey of four papers [13, 14, 17, 18], with the last one being an upcoming paper.

The universal invariant [11, 12, 15] associated with a ribbon Hopf algebra is an invariant of framed links. It possesses a universal property over the Reshetikhin-Turaev invariant [16], which means that we can recover the Reshetikhin-Turaev invariant when a finite-dimensional representation of the ribbon Hopf algebra is specified. Note that the Jones polynomial and the colored Jones polynomial are Reshetikhin-Turaev invariants associated with the quantum group of $\mathfrak{sl}_2$. The universal invariant also provides important quantum invariants of closed 3-manifolds such as the Witten-Reshetikhin-Turaev invariant [16] and the Hennings-Kauffman-Radford invariant [5, 9]. These invariants are obtained from the universal invariant of framed links, where closed 3-manifolds are obtained by performing surgery on framed links in S^3.

The relationship between the universal invariant and 3-dimensional, global, topological properties of links is not well understood, mainly because of the 2-dimensional definition using link diagrams. In this article, we give three types of reconstructions of the universal invariant using ideal triangulation of link complements, and give an extension of the universal invariant to an invariant of integral normal o-graphs, which represent framed 3-manifolds. We expect that our framework will become a new method to study quantum invariants in a 3-dimensional way.

Sections 2, 3 are based on [17]. In these sections, we explain the definition of the universal invariant and the initial reconstruction of it. Sections 4, 5 are primarily based on [13, 14]. In Section 4 we describe how integral normal o-graphs can be used to represent framed 3-manifolds, with a specific focus on their application to knot complements. In Section 5 we define the invariant Z of closed framed 3-manifolds with the vanishing first Betti number. We also discuss some connections between Z and the $SO(3)$ WRT invariant. Section 6 will be part of the upcoming paper [18], where we revisit the universal invariant and present two alternative reconstructions using integral normal o-graphs. The final section provides a summary of the main points covered in the survey.

2 Drinfeld double and Heisenberg double

In this section we follow the notation in [17].

2.1 Quasi-triangular Hopf algebra

Let $(A, \eta_A, m_A, \varepsilon_A, \Delta_A, \gamma_A)$ be a finite dimensional Hopf algebra over a field k, with k-linear maps

$$\begin{aligned} \eta_A\colon\ & k \to A, \\ \varepsilon_A\colon\ & A \to k, \\ m_A\colon\ & A \otimes A \to A, \\ \Delta_A\colon\ & A \to A \otimes A, \\ \gamma_A\colon\ & A \to A, \end{aligned}$$

which are called *unit*, *counit*, *multiplication*, *comultiplication*, and *antipode*, respectively. For simplicity we will omit the subscript A of each map above when there is no confusion. Set $\Delta^{\mathrm{op}} = \tau \circ \Delta$ and $m^{\mathrm{op}} = m \circ \tau$, where τ is the symmetry map such that $\tau(a \otimes b) = b \otimes a$ for $a, b \in A$.

For distinct integers $1 \le j_1, \ldots, j_m \le l$ and $x = \sum x_1 \otimes \cdots \otimes x_m \in A^{\otimes m}$, we use the notation

$$x^{(l)}_{j_1 \cdots j_m} = \sum (x_1)_{j_1} \cdots (x_m)_{j_m} \in A^{\otimes l}, \tag{1}$$

where $(x_i)_{j_i}$ represents the element in $A^{\otimes l}$ obtained by placing x_i on the j_ith tensorand, i.e.,

$$(x_i)_{j_i} = 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1,$$

where x_i is at the j_ith position. For example, for $x = \sum x_1 \otimes x_2 \otimes x_3$, we have $x^{(3)}_{312} = \sum x_2 \otimes x_3 \otimes x_1$. Abusing the notation, we will omit the superscript of $x^{(l)}_{j_1 \cdots j_m}$ as $x_{j_1 \cdots j_m}$.

A *quasi-triangular Hopf algebra* $(A, \eta, m, \varepsilon, \Delta, \gamma, R)$ is a Hopf algebra $(A, \eta, m, \varepsilon, \Delta, \gamma)$ with an invertible element $R = \sum \alpha \otimes \beta \in A^{\otimes 2}$, called *the universal R-matrix*, such that

$$\begin{aligned} &\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1} \quad \text{for } x \in A, \\ &(\Delta \otimes 1)(R) = R_{13}R_{23}, \quad (1 \otimes \Delta)(R) = R_{13}R_{12}. \end{aligned}$$

A quasi-triangular Hopf algebra has a special element

$$u = \sum \gamma(\beta)\alpha \in A,$$

which gives the square of the antipode by conjugation

$$\gamma^{\otimes 2}(x) = uxu^{-1} \tag{2}$$

for $x \in A$.

2.2 Drinfeld double and Yang-Baxter equation

For any finite dimensional Hopf algebra with invertible antipode, the Drinfeld quantum double construction gives a quasi-triangular Hopf algebra [4]. In this section, we follow the notation in [7].

Let $(A, \eta, m, \varepsilon, \Delta, \gamma,)$ be a finite dimensional Hopf algebra with invertible antipode, $A^{\rm op} = (A, \eta, m^{\rm op}, \varepsilon, \Delta, \gamma^{-1})$ the opposite Hopf algebra and $(A^{\rm op})^* = (A^*, \varepsilon^*, \Delta^*, \eta^*, (m^{\rm op})^*, (\gamma^{-1})^*)$ the dual of the opposite Hopf algebra. For simplicity, we set

$$\bar{\gamma} = \gamma^{-1}.$$

In what follows, for $x \in A$ or $x \in A^*$, we use the notation

$$\Delta(x) = \sum x' \otimes x'',$$
$$(\Delta \otimes 1)\Delta(x) = \sum x' \otimes x'' \otimes x'''.$$

For $f \in A^*$, we have

$$(m^{\rm op})^*(f) = \Delta^{\rm op}(f) = \sum f'' \otimes f'.$$ [1]

There is a unique left action

$$A \otimes (A^{\rm op})^* \to (A^{\rm op})^*, \quad a \otimes f \mapsto a \cdot f,$$

such that

$$\langle a \cdot f, x\rangle = \sum \langle f, \bar{\gamma}(a'')xa'\rangle,$$

for $a, x \in A$ and $f \in (A^{\rm op})^*$, which induces the left A-module coalgebra structure on $(A^{\rm op})^*$. Also, there is a unique right action

$$A \otimes (A^{\rm op})^* \to A, \quad a \otimes f \mapsto a^f,$$

such that

$$a^f = \sum f(\bar{\gamma}(a''')a')a''$$

for $a \in A$ and $f \in (A^{\rm op})^*$, which induces the right $(A^{\rm op})^*$-module coalgebra structure on A.

The Drinfeld double

$$D(A) = ((A^{\rm op})^* \otimes A, \eta_{D(A)}, m_{D(A)}, \varepsilon_{D(A)}, \Delta_{D(A)}, \gamma_{D(A)}, R)$$

is the quasi-triangular Hopf algebra defined as the bicrossed product of A and $(A^{\rm op})^*$. Its unit, counit, and comultiplication are given by these of $(A^{\rm op})^* \otimes A$, i.e., we have

$$\eta_{D(A)}(1) = \eta_{(A^{\rm op})^* \otimes A}(1) = 1 \otimes 1,$$
$$\varepsilon_{D(A)}(f \otimes a) = \varepsilon_{(A^{\rm op})^* \otimes A}(f \otimes a) = f(1)\varepsilon(a),$$
$$\Delta_{D(A)}(f \otimes a) = \Delta_{(A^{\rm op})^* \otimes A}(f \otimes a) = \sum f'' \otimes a' \otimes f' \otimes a'',$$

for $a \in A$ and $f \in (A^{\rm op})^*$. The multiplication is given by

$$m_{D(A)}\left((f \otimes a) \otimes (g \otimes b)\right) = \sum f(a' \cdot g'') \otimes a''^{g'} b \quad (3)$$
$$= \sum fg(\bar{\gamma}(a''')?a') \otimes a''b, \quad (4)$$

[1] In [7], he uses the notation $\Delta^{\rm op}(f) = \sum f' \otimes f''$.

for $a, b \in A$ and $f, g \in (A^{\mathrm{op}})^*$, where the question mark ? denotes a place of the variable. Its antipode is given by

$$\gamma_{D(A)}(f \otimes a) = \sum \gamma(a'') \cdot \bar{\gamma}^*(f') \otimes \gamma(a')^{\bar{\gamma}^*(f'')},$$

for $a \in A$ and $f \in (A^{\mathrm{op}})^*$.

Fix a basis $\{e_a\}_{a \in \mathcal{I}}$ of A and its dual basis $\{e^a\}_{a \in \mathcal{I}}$ of A^*. The universal R-matrix is defined as the canonical element

$$R = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in D(A) \otimes D(A).$$

2.3 Heisenberg double and pentagon relation

Let A be a finite dimensional Hopf algebra with an invertible antipode as in the previous section. The Heisenberg double

$$\mathcal{H}(A) = (A^* \otimes A, \eta_{\mathcal{H}(A)}, m_{\mathcal{H}(A)})$$

is the algebra with the unit $\eta_{\mathcal{H}(A)}(1) = \eta_{A^* \otimes A}(1) = 1 \otimes 1$ and the multiplication

$$m_{\mathcal{H}(A)}\left((f \otimes a) \otimes (g \otimes b)\right) = \sum fg(?a') \otimes a''b, \tag{5}$$

for $a, b \in A$ and $f, g \in (A^{\mathrm{op}})^*$.

Proposition 2.1 ([6, Theorem 1]). *The canonical element*

$$S = \sum_a (1 \otimes e_a) \otimes (e^a \otimes 1) \in \mathcal{H}(A) \otimes \mathcal{H}(A)$$

satisfies the pentagon relation

$$S_{12}S_{13}S_{23} = S_{23}S_{12} \quad \in \mathcal{H}(A)^{\otimes 3}. \tag{6}$$

2.4 Drinfeld double and Heisenberg double

The Drinfeld double $D(A)$ can be realized as a subalgebra in the tensor product $\mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}}$ of the Heisenberg double $\mathcal{H}(A)$ and its opposite algebra $\mathcal{H}(A)^{\mathrm{op}}$ as follows. [2]

Proposition 2.2 ([6, Theorem 2] [17, Theorem 3.3]). *There is an algebra homomorphism*

$$\phi\colon\ D(A) \to \mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}} \tag{7}$$

defined by

$$\phi = m_{\mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}}} \circ ((1 \otimes \eta)^{\otimes 2} \otimes (\eta \otimes 1)^{\otimes 2}) \circ (1 \otimes \bar{\gamma}^* \otimes 1 \otimes \gamma) \circ (\Delta^{\mathrm{op}} \otimes \Delta),$$

i.e., we have

$$\begin{aligned}\phi(f \otimes x) &= \sum \langle \bar{\gamma}^*\left(f'\right)'', \gamma(x'')' \rangle f'' \otimes x' \otimes \bar{\gamma}^*\left(f'\right)' \otimes \gamma(x'')'' \\ &= \sum \langle f', x''' \rangle f''' \otimes x' \otimes \bar{\gamma}^*(f'') \otimes \gamma(x''),\end{aligned}$$

for $f \in A^*$ and $x \in A$.

[2] In [6] he uses $H(A^*)$ instead of $\mathcal{H}(A)^{\mathrm{op}}$.

Proposition 2.3 ([6, Proposition 5]). *We have*

$$\phi^{\otimes 2}(R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23} \quad \in (\mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}})^{\otimes 2}. \tag{8}$$

Here we set

$$\begin{aligned} S' &= \sum (1 \otimes \tilde{e}_a) \otimes (e^a \otimes 1) \quad &&\in \mathcal{H}(A)^{\mathrm{op}} \otimes \mathcal{H}(A), \\ S'' &= \sum (1 \otimes e_a) \otimes (\tilde{e}^a \otimes 1) \quad &&\in \mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}}, \\ \tilde{S} &= \sum (1 \otimes \tilde{e}_a) \otimes (\tilde{e}^a \otimes 1) \quad &&\in \mathcal{H}(A)^{\mathrm{op}} \otimes \mathcal{H}(A)^{\mathrm{op}}, \end{aligned}$$

where $\tilde{e}_a = \gamma(e_a), \tilde{e}^b = \bar{\gamma}^*(e^b)$.

3 Universal invariant

We continue to follow the notation in [17]. A *tangle* means a proper embedding of a compact, oriented 1-manifold in a cube $[0,1]^3$, whose boundary points are on the two parallel lines $[0,1] \times \{0,1\} \times \{1/2\}$. A *tangle diagram* is a diagram of a tangle obtained from the projection $p\colon (x,y,z) \mapsto (x,y,0)$ to the (x,y)-plane. A *framed* tangle is a tangle equipped with a trivialization of its normal tangent bundle, which is presented in a diagram by the blackboard framing.

3.1 Regular isotopy

In what follows we will consider regular isotopy refinement of the universal invariant. Regular isotopy [8] is the equivalence relation of tangle diagrams generated by only Reidemeister II and III (and planar isotopy). The winding number and framing are invariants under regular isotopy, and conversely, the regular isotopy class of a diagram is determined by its winding number and framing [19]. In particular, for braid diagrams, regular isotopy classes correspond to isotopy classes.

3.2 Universal invariant

The universal invariant [11, 12, 15] is originally defined as an invariant of framed tangles using a ribbon Hopf algebra. In this article, we consider a regular isotopy version of the universal invariant, which enables us to use a quasi-triangular Hopf algebra instead of strictly requiring a ribbon Hopf algebra. To obtain the original definition from the modified one, we can adjust the differences by multiplying the appropriate powers of the ribbon element, provided that the quasi-triangular Hopf algebra satisfies the additional condition of being a ribbon Hopf algebra.

For the sake of simplicity, we define the invariant for $(1,1)$-tangle diagrams and the Drinfeld double $D(A)$. We can extend the definition to more general cases in a similar way to the original definition of the universal invariant.

Let T be a $(1,1)$-tangle diagram. We define the regular isotopy version of the universal invariant $J_R(T, D(A)) \in D(A)$ in three steps as follows.

Step 1. Choose a slice diagram. We assume that the tangle diagram T is obtained by pasting, horizontally and then vertically, copies of the fundamental tangles depicted in Figure 1.

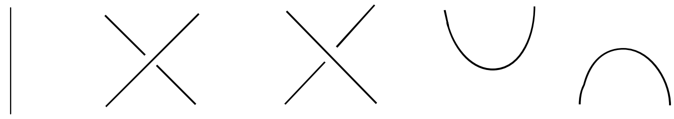


Figure 1: Fundamental tangles, where the orientation of each strand is arbitrary.

Step 2. Attach labels. We attach labels on the copies of the fundamental tangles in T, following the rule described in Figure 2, where each γ' should be replaced with γ if the string is oriented upwards, and with the identity otherwise. We do not attach any labels to the other copies of fundamental tangles, such as a straight strand or a local maximum or minimum oriented from right to left.

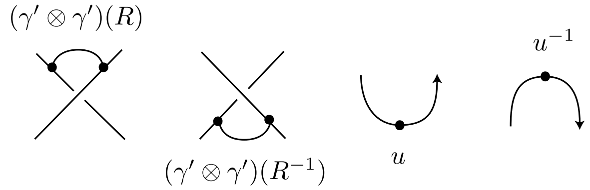


Figure 2: How to place labels on the fundamental tangles.

Step 3. Read the labels. We define $J_R(T, D(A))$ as the product of the labels on T, where the labels are read off along T reversing the orientation, and written from left to right. The labels on the crossings are read as in Figure 3.

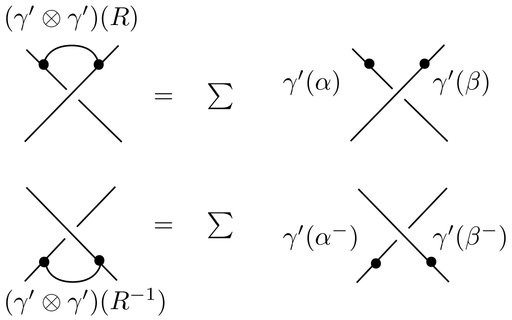


Figure 3: How to read the labels on crossings, where $R^{-1} = \sum \alpha^- \otimes \beta^-$.

Then $J_R(T, D(A))$ is an invariant under Reidemeister II and III, and thus defines a regular isotopy invariant of tangle diagrams.

For example, for the $(1,1)$-tangle diagram T_{41} shown in Figure 4 whose closure is the figure eight knot, we have

$$J_R(T_{41}) = \sum \gamma(\bar{\beta}_i)\gamma(\bar{\alpha}_j)\beta_l\alpha_k\bar{\beta}_j\bar{\alpha}_i\gamma(\beta_k)\gamma(\alpha_l), \tag{9}$$

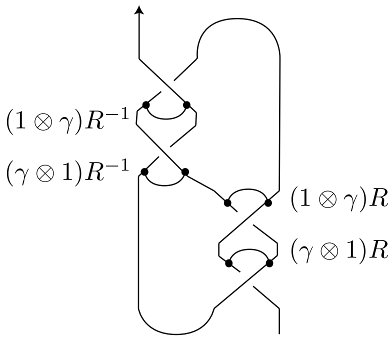


Figure 4: T_{41}.

where we use notations $R = \sum \alpha_i \otimes \beta_i$ and $R^{-1} = \sum \bar{\alpha}_j \otimes \bar{\beta}_j$.

3.3 Reconstruction of universal invariant version 1

We reconstruct the image of Kashaev's embedding ϕ of the universal invariant J_R using the Heisenberg double.

In what follows, for simplicity, we use the notation

$$fx = f \otimes x \in A^* \otimes A,$$

for $f \in A^*$ and $x \in A$. In particular we have

$$S = \sum_a e^a \otimes e_a, \quad S' = \sum_a \tilde{e}^a \otimes e_a, \quad S'' = \sum_a e^a \otimes \tilde{e}_a, \quad \tilde{S} = \sum_a \tilde{e}^a \otimes \tilde{e}_a.$$

We denote their inverses by

$$\sum_a u_a \otimes u^a = S^{-1} = \sum_a \gamma(e_a) \otimes e^a, \quad \sum_a \tilde{u}_a \otimes u^a = (S')^{-1} = \sum_a \gamma(\tilde{e}_a) \otimes e^a,$$
$$\sum_a u_a \otimes \tilde{u}^a = (S'')^{-1} = \sum_a \gamma(e_a) \otimes \tilde{e}^a, \quad \sum_a \tilde{u}_a \otimes \tilde{u}^a = (\tilde{S})^{-1} = \sum_a \gamma(\tilde{e}_a) \otimes \tilde{e}^a.$$

Let T be a $(1,1)$-tangle diagram. We define an element $J'_R(T, \mathcal{H}(A)) \in \mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}}$ by modifying the definition of $J_R(T, D(A))$ as follows.

We duplicate T and thicken the left strands following the orientation, and denote the result by $d(T)$. See (a), (b) in Figure 5 for an example. Then we attach labels on crossings as in Figure 6, where each γ' and each $(\bar{\gamma}^*)'$ should be replaced with γ and $\bar{\gamma}^*$, respectively, if the string is oriented upwards, and with the identities otherwise. Then we define the first and the second tensorands of $J'_R(T, \mathcal{H}(A))$ as the product of the labels on the thin and the thick strands, respectively.

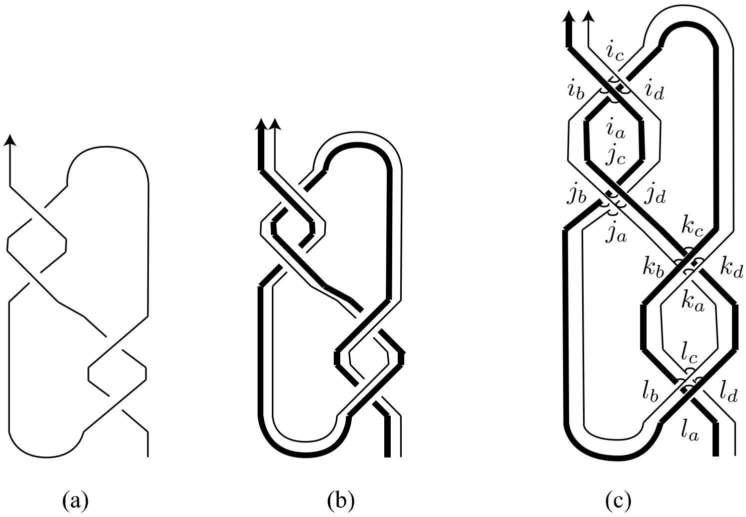


Figure 5: (a) T_{41}, (b) $d(T_{41})$, (c) Parameters for $d(T_{41})$.

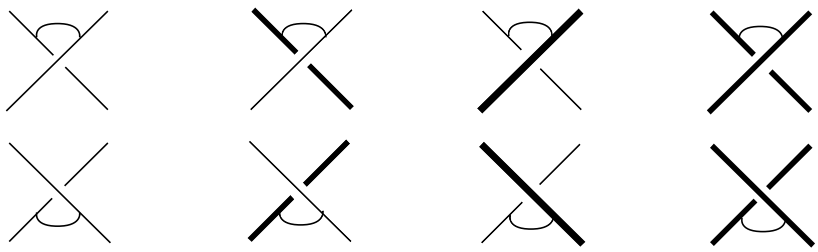


Figure 6: Labels on crossings.

For the example with T_{41}, with the parameters as in Figure 5 (c), we have

$$
\begin{aligned}
J'_R(T_{41}, \mathcal{H}(A)) = & \sum_{i_a,i_b,i_c,i_d,j_a,j_b,j_c,j_d,k_a,k_b,k_c,k_d,l_a,l_b,l_c,l_d} \bar{\gamma}^*(u^{i_c})\bar{\gamma}^*(u^{i_d})\gamma(u_{j_d})\gamma(u_{j_a})e^{l_b}e^{l_c}e_{k_a}e_{k_b} \\
& \times u^{j_a}u^{j_b}u_{i_b}u_{i_c}\bar{\gamma}^*(e^{k_d})\bar{\gamma}^*(e^{k_a})\gamma(e_{l_c})\gamma(e_{l_d}) \\
& \otimes \bar{\gamma}^*(\tilde{u}^{i_b})\bar{\gamma}^*(\tilde{u}^{i_a})\gamma(\tilde{u}_{j_c})\gamma(\tilde{u}_{j_b})\tilde{e}^{l_a}\tilde{e}^{l_d}\tilde{e}_{k_d}\tilde{e}_{k_c} \\
& \times \tilde{u}^{j_d}\tilde{u}^{j_c}\tilde{u}_{i_a}\tilde{u}_{i_d}\bar{\gamma}^*(\tilde{e}^{k_c})\bar{\gamma}^*(\tilde{e}^{k_b})\gamma(\tilde{e}_{l_b})\gamma(\tilde{e}_{l_a}).
\end{aligned}
$$

Observe that in the definition of J_R, we attach u and u^{-1} on $\cup$ and $\cap$, respectively, while in the definition of J'_R we attach no elements on the corresponding image in $d(T)$. Because we would like to make correspondence between J_R and J'_R at each fundamental tangle, we adjust the difference by creating corresponding elements to $u^{\pm}$ as follows; Let $T_{(\leftarrow)}$ be the diagram obtained from T by replacing each of $\cap$ and $\cup$ with and , respectively. The following result is equivalent to the restriction of [17, Theorem 4.1] to

(1, 1)-tangles, which was stated with the universal invariant of framed tangles.

Theorem 3.1 ([17]). *We have*

$$\phi \circ J_R(T, D(A)) = J'_R(T_{(\leftarrow)}, \mathcal{H}(A)) \quad \in \mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}}.$$

Proof. It is enough to show

(1) $u = J_R($$, D(A))$, $u^{-1} = J_R($$, D(A))$,

(2) $\phi^{\otimes 2} \circ J_R(c, D(A)) = J'_R(c, \mathcal{H}(A))$ for each crossing c.

(1) is straightforward.

We prove (2). Let c_+ (resp. c_-) denote a positive (resp. negative) crossing with strands oriented downwards. We have

$$\begin{aligned}
\phi^{\otimes 2} \circ J_R(c_+, D(A)) &= \phi^{\otimes 2}(R)_{1234} = S''_{14} S_{13} \tilde{S}_{24} S'_{23} \\
&= \sum_{a,b,c,d} e_a e_b \otimes \tilde{e}_d \tilde{e}_c \otimes e^b e^c \otimes \tilde{e}^a \tilde{e}^d = J'_R(c_+, \mathcal{H}(A)), \\
\phi^{\otimes 2} \circ J_R(c_-, D(A)) &= \phi^{\otimes 2}(R^{-1})_{1234} = (S'_{23})^{-1} (\tilde{S}_{24})^{-1} (S_{13})^{-1} (S''_{14})^{-1} \\
&= \sum_{a,b,c,d} u_b u_c \otimes \tilde{u}_a \tilde{u}_d \otimes u^a u^b \otimes \tilde{u}^d \tilde{u}^c = J'_R(c_-, \mathcal{H}(A)),
\end{aligned}$$

see Figure 7. For other crossings, the assertion follows similarly from

$$\begin{aligned}
\phi^{\otimes 2} \circ (\gamma_{D(A)} \otimes 1)(R) &= \sum_{a,b,c,d} \gamma(e_c)\gamma(e_d) \otimes \gamma(\tilde{e}_b)\gamma(\tilde{e}_a) \otimes e^b e^c \otimes \tilde{e}^a \tilde{e}^d, \\
\phi^{\otimes 2} \circ (1 \otimes \gamma_{D(A)})(R) &= \sum_{a,b,c,d} e_a e_b \otimes \tilde{e}_d \tilde{e}_c \otimes \bar{\gamma}^*(e^d)\bar{\gamma}^*(e^a) \otimes \bar{\gamma}^*(\tilde{e}^c)\bar{\gamma}^*(\tilde{e}^b), \\
\phi^{\otimes 2} \circ (\gamma_{D(A)} \otimes 1)(R^{-1}) &= \sum_{a,b,c,d} \gamma(u_d)\gamma(u_a) \otimes \gamma(\tilde{u}_c)\gamma(\tilde{u}_b) \otimes u^a u^b \otimes \tilde{u}^d \tilde{u}^c, \\
\phi^{\otimes 2} \circ (1 \otimes \gamma_{D(A)})(R^{-1}) &= \sum_{a,b,c,d} u_b u_d \otimes \tilde{u}_a \tilde{u}_d \otimes \bar{\gamma}^*(u^c)\bar{\gamma}^*(u^d) \otimes \bar{\gamma}^*(\tilde{u}^b)\bar{\gamma}^*(\tilde{u}^a),
\end{aligned}$$

which completes the proof. □

4 Integral normal o-graphs and framed 3-manifolds

In this section we follow the notation in [14]. The content of Section 4.4 will be included in the upcoming paper [18].

4.1 Integral normal o-graphs

A *normal o-graph* is an oriented virtual link diagram. When we refer to a crossing, we mean a real crossing.

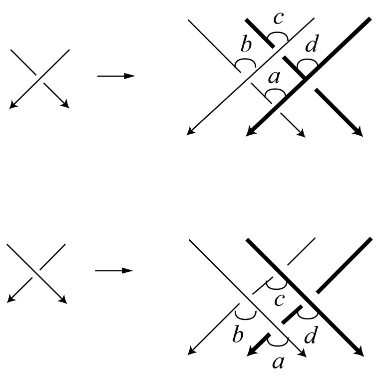


Figure 7: Labels on the colored diagrams $d(c_{\pm})$ associated to positive and negative crossings $c_{\pm}$.

An *integral normal o-graph* is a normal o-graph with an integer weight attached on each edge. Here, an egde means a path between real crossings. We consider integral normal o-graphs modulo Reidemeister type moves in Figure 8, and denote the set of equivalent classes by $\mathcal{IG}$.

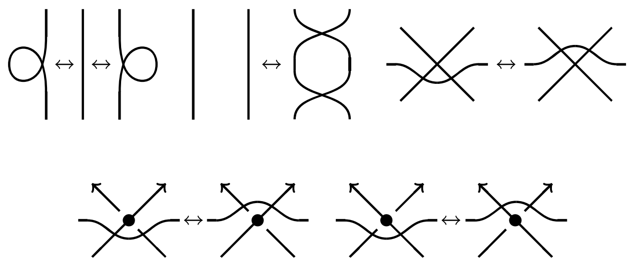


Figure 8: Reidemeister type moves.

4.2 Integral 0-2 move, integral MP-move and H-move

We define the following three types of moves on $\mathcal{IG}$ and denote by $\sim$ the generated equivalence relation.

- integral 0-2 move (Figure 9)
- integral MP-move (Figure 10)
- H-move (Figure 11)

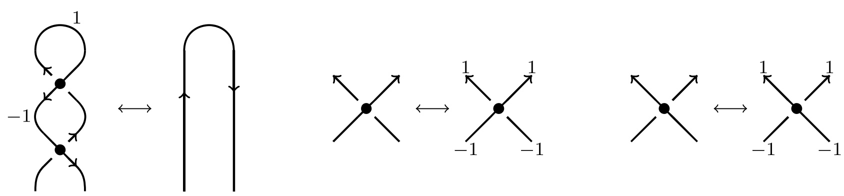


Figure 9: Integral 0-2 move.

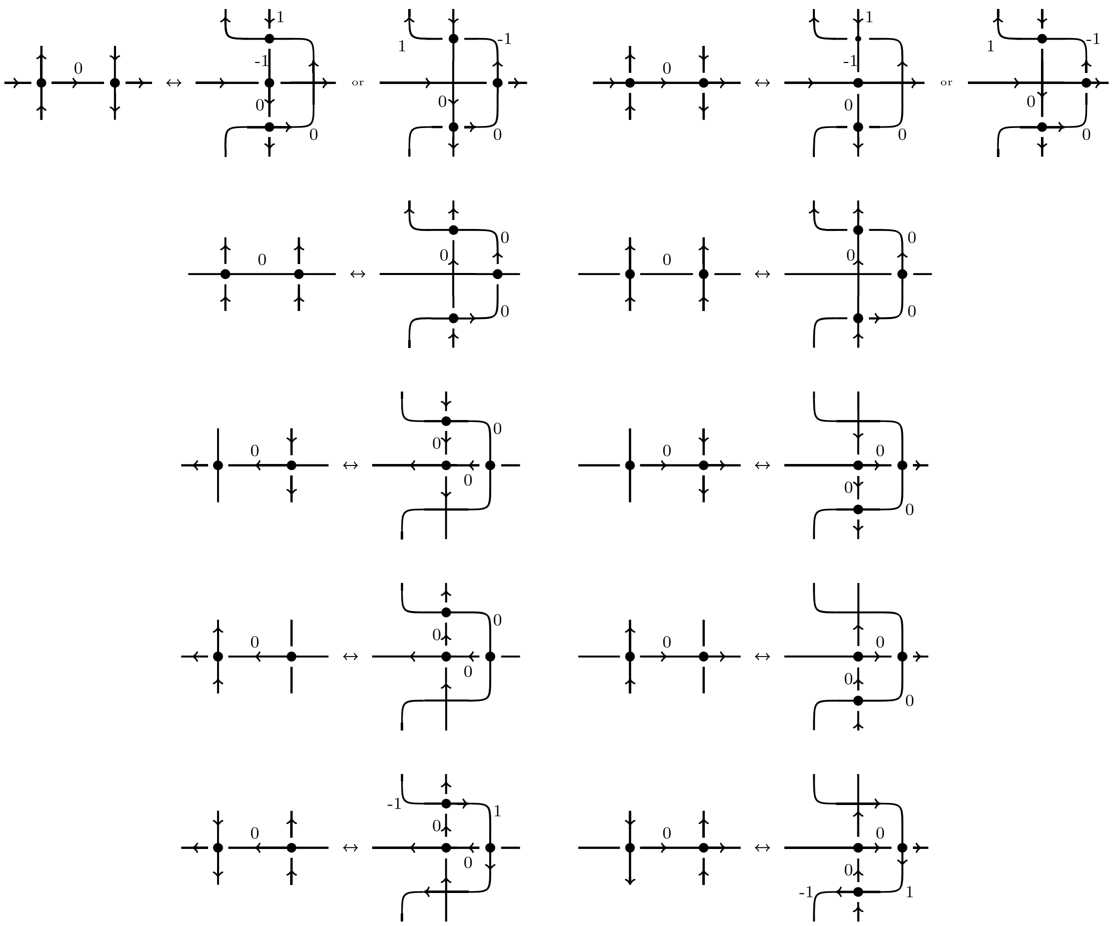


Figure 10: Integral MP-move.

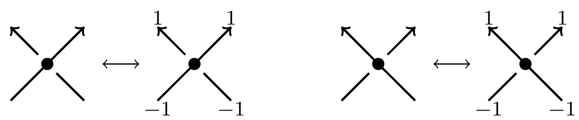


Figure 11: H-move.

Here, in Figure 10, the orientations of the non-oriented edges are arbitrary if they match before and after the move. If there are multiple weights on an edge after the move, the weights should be added in the additive group $\mathbb{Z}$.

4.3 Integral normal o-graphs and framed 3-manifolds

Roughly speaking there is a good subset $\mathcal{FIG} \subset \mathcal{IG}$ consisting of *framed integral normal o-graphs*, which represents equivalent classes of closed framed 3-manifolds, i.e., we can construct a surjective map

$$\Phi_{\mathrm{fram}} \colon \mathcal{FIG} \to \mathcal{M}_{\mathrm{fram}},$$

where $\mathcal{M}_{fram}$ is the set of equivalent classes of closed framed 3-manifolds.

We give a rough explanation of the construction of Φ_{fram}. Please refer to [2, 14] for the details. Each crossing of a normal o-graph corresponds to a ideal tetrahedron with ordered vertices, which we call a *branched ideal tetrahedron.* The order of the vertices gives a non-vanishing vector field inside the ideal tetrahedron, and these ideal tetrahedra are attached along edges of a normal o-graph so that the vector fields are attached continuously. If the boundary of the resulting 3-manifold is a single S^2 and the vector field at the S^2 is trivial (this condition is included in "good" property of $\mathcal{FIG}$), we can cap this S^2 by a 3-ball with a trivial vector field. This results in a closed 3-manifold with a non-vanishing vector field, which is referred to as a closed *combed* 3-manifold.

A *framing* of 3-manifold is a trivialization of the tangent bundle, which is a triple (v_1, v_2, v_3) of independent vector fields. The first vector field is given naturally by the combing structure induced by normal o-graphs, and the second vector field is constructed by the integer weights on edges; we construct the second vector fields as a section of the plane bundle which is orthogonal to v_1. From now, we use the terminology of dual spines of ideal triangulations. On each 0-cell, which in dual corresponds to a ideal tetrahedron, we can take a kind of "standard" section, and on each 1-cell we take the section specified by the rotation number between two 0-cells which bound the 1-cell. To extend it to 2-cells we meet an obstruction since $\pi_1(S^1) \neq 0$, and the "good" property of $\mathcal{FIG}$ ensures that the obstruction vanishes. Thus we can extend the section to 2-cells and the resuld is unique since $\pi_2(S^1) = 0$. By $\pi_2(S^1) = \pi_3(S^1) = 0$, we can extend it uniquely to 3-cells, which is the 3-ball we cap on the single S^2 boundary.

Especially if a manifold has the vanishing first Betti number, the correspondence is one to one.

Proposition 4.1 ([14, Proposition 4.1])**.** *Let $\mathcal{M}^0_{fram} \subset \mathcal{M}_{fram}$ be the subset consisting of closed framed 3-manifolds with the vanishing first Betti number, and $\mathcal{FIG}^0 \subset \mathcal{FIG}$ the inverse image of $\mathcal{M}^0_{fram}$ by Φ_{fram}. The restriction of Φ_{fram} induces a bijection*

$$\Phi^0_{fram}\colon\ \mathcal{FIG}^0/\sim\ \to\ \mathcal{M}^0_{fram}.$$

Example 4.2. *Figure 12 shows S^3 with the framing f which extends the combing induced by the Hopf fibering, and lens space $L(2,1)$ with the framing f' which extends the canonical combing induced by its Seifeld fibered structure.*

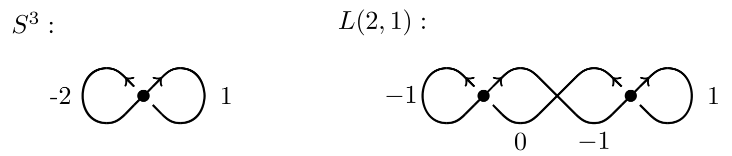


Figure 12: Framed integral normal o-graphs representing (S^3, f) and $(L(2,1), f')$.

Remark 4.3. *Benedetti and Petronio* [2] *showed that "good" $\mathbb{Z}_2$-weighted normal o-graphs, modulo a specific equivalence relation, can represent framed closed 3-manifolds in a bijective manner, without imposing any conditions on the first Betti number. However, we still require Proposition 4.1 because we intend to use integer weights in the construction* *of the invariant. These integer weights are closely related to the powers of the antipode of the Hopf algebra, as explained in Section 5.*

4.4 Knot complements as framed 3-manifolds

A combed 3*-manifolds with concave boundary* [3] is a 3-manifold M with a non-vanishing vector field v on M such that if v is tangent to ∂M it is in a concave fashion. We can extend the construction of closed framed 3-manifolds in the previous section to any framed 3-manifolds which come from combed 3-manifolds with concave boundary. In particular, we can consider knot complements within this framework. In this section we construct an integral normal o-graph from a knot diagram, which presents a framing structure on the knot complement in $\mathbb{R}^2 \times I$. By using this framework, we can study new aspects of quantum invariants.

Let K be a slice diagram of a knot. We construct an integral normal o-graph $\iota \circ \delta(K)$ in two steps as follows.

Step 1. Duplicate and reverse the diagram. Let $\delta(K)$ be the diagram obtained by duplicating K and thickening the left strand following the orientation, and then reversing the orientation of the thickened strand.

Step 2. Attach integral weights. We define the integral normal o-graph $\iota \circ \delta(K)$ by attaching integral weights to the edges of $\delta(K)$ as shown in Figure 13, where the sign (positive or negative) of each crossings is arbitrary. For other fundamental tangle diagrams, the weights are uniquely determined in such a way that the result is invariant under planar isotopy.

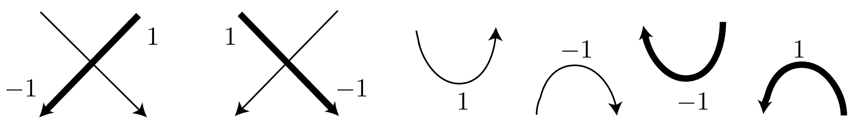


Figure 13: Integral weight put on the fundamental tangles.

Theorem 4.4. *$\iota \circ \delta(T)$ modulo $\sim$ is a regular isotopy invariant.*

Proof. We can show the invariance under RII by performing a sequence of four *pure sliding moves* and their inverses [2, Figure 4.7], while taking the framing into account. The invariance under RIII can be shown by a sequence of eight integral MP moves and some pure sliding moves. □

Note that the distinction between thin and thick strands in $\delta(K)$ is only used to define the integral weight of $\iota \circ \delta(K)$. The underlying diagram of $\delta(K)$ is a normal o-graph, which provides a combing structure on the knot complement in S^3 minus two points. To construct the framing, we introduce a second vector field that is tangent to the 2-cells, following the procedure described in Section 4.3. However, in this case, we do not cap the boundary as we did for closed 3-manifolds. In fact, the 2-cells in this context correspond to the crossings, edges, and regions of the knot diagram K. With the exception of the outermost region, these 2-cells admit a framing. So, we puncture the outermost region and obtain the framing in the knot complement in $\mathbb{R}^2 \times I$. Indeed, this setting is consistent with the fact that the integral normal o-graph $\iota \circ \delta(K)$ is a regular isotopy invariant of K. In this context, strands of the knot diagram in $\mathbb{R}^2$ are not allowed to go through infinity, ensuring that the construction remains well-defined.

Remark 4.5. *We can extend δ and ι to functors*

$$\delta\colon \mathcal{D} \to \mathcal{D}',$$
$$\iota\colon \mathcal{D}' \to \mathcal{ID},$$

where $\mathcal{D}$ is the category of tangle diagrams, $\mathcal{D}'$ is the category of 2-colored (by thin and thick) tangle diagrams, and $\mathcal{ID}$ is the category of integral normal o-tangles. This means that we can extend ι to 2-colored diagrams which does not necessarly come from knot diagrams. There is the forgetful functor from $\mathcal{D}'$ to the category of normal o-tangles by forgetting the distinction between thick and non-thick strands. Conversely, the category $\mathcal{D}'$ contains the category of normal o-tangles as a subcategory in two ways depending on the choice of thin or thick.

5 Invariant of framed 3-manifolds

We continue to follow the notation in [14].

5.1 Pivotal like element

Set

$$G = \sum_{i,j} e^i e^j \otimes S^{-1}(e_j) S^2(e_i) \in \mathcal{H}(A),$$

which is an analog of a pivotal element in a Hopf algebra, i.e., we have

$$\Theta^2(x) = GxG^{-1}, \quad x \in \mathcal{H}(A) \tag{10}$$

for $\Theta = \gamma \otimes \gamma^{-1}\colon \mathcal{H}(A) \to \mathcal{H}(A)$ being an analog of the antipode.

Lemma 5.1 ([18])**.** *We have*

$$\phi^{\otimes 2}((u^n \otimes 1) \triangleright R^{\pm 1}) = ((G^{-n})^{\otimes 2} \otimes 1^{\otimes 2}) \triangleright \phi^{\otimes 2}(R^{\pm 1}),$$
$$\phi^{\otimes 2}((1 \otimes u^n) \triangleright R^{\pm 1}) = (1^{\otimes 2} \otimes (G^{-n})^{\otimes 2}) \triangleright \phi^{\otimes 2}(R^{\pm 1}).$$

where $\triangleright$ is the action by conjugation.

Proof. We have the assertion by (2), (10), and

$$\begin{aligned}
\phi^{\otimes 2}(\gamma^{2n}_{D(A)} \otimes 1)(R^{\pm}) &= \left((1 \otimes \gamma^{2n})^{\otimes 2} \otimes 1^{\otimes 4}\right) \phi^{\otimes 2}(R^{\pm}) \\
&= ((\Theta^{-2n})^{\otimes 2} \otimes 1^{\otimes 2}) \phi^{\otimes 2}(R^{\pm 1}), \\
\phi^{\otimes 2}(1 \otimes \gamma^{2n}_{D(A)})(R^{\pm 1}) &= \left(1^{\otimes 4} \otimes (\gamma^{-2n} \otimes 1)\right) \phi^{\otimes 2}(R^{\pm 1}) \\
&= (1^{\otimes 2} \otimes (\Theta^{-2n})^{\otimes 2}) \phi^{\otimes 2}(R^{\pm 1}).
\end{aligned}$$

□

5.2 Invariant of integral normal o-graphs and framed 3-manifolds

For simplicity we define the invariant $Z(\Gamma, \mathcal{H}(A)) \in \mathcal{H}(A)$ for a $(1,1)$-integral normal o-tangle Γ as follows. We attach labels on crossings and edges of the diagram as depicted in Figure 14. It is important to note that we can rotate the diagram while attaching the labels. Once the labels are attached, we read them in the same manner as described in Section 3.2, and obtain an element $Z(\Gamma, \mathcal{H}(A)) \in \mathcal{H}(A)$. This element is invariant under the equivalence relation $\sim$. This construction is easily extended to general integral normal o-graphs.

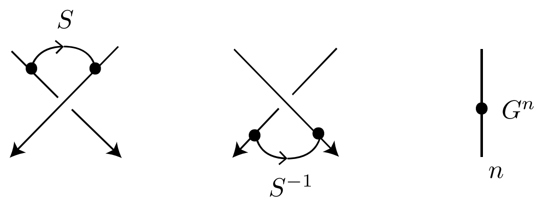


Figure 14: How to place labels on the diagram.

Remark 5.2. *Note that the definition of $Z(-, \mathcal{H}(A))$ automatically guarantees its invariance under planar isotopy. This is a notable difference from the construction of the universal invariant, where slice diagrams are used, and the invariance under planar isotopy is not trivial and requires careful considerations.*

For the closure $\overline{\Gamma}$ of Γ we define $Z(\overline{\Gamma}, \mathcal{H}(A)) := \overline{Z(\Gamma, \mathcal{H}(A))} \in \mathcal{H}(A)/[\mathcal{H}(A), \mathcal{H}(A)]$, where $[\mathcal{H}(A), \mathcal{H}(A)]$ is the vector space spanned by $ab - ba$, $a, b \in \mathcal{H}(A)$. It is known that the $\mathcal{H}(A)/[\mathcal{H}(A), \mathcal{H}(A)]$ is one dimensional vector space, i.e., isomotphic to the base field k. Note that the value $\overline{Z(\Gamma, \mathcal{H}(A))}$ does not depend on the choice of $(1,1)$-tangle Γ whose closure is $\overline{\Gamma}$.

Theorem 5.3 ([14])**.** *The map $Z(-, \mathcal{H}(A)) \colon \mathcal{IG} \to k$ is invariant under $\sim$.*

As a result, the restriction of the invariant $Z(-, \mathcal{H}(A))$ to $\mathcal{FIG}^0$ gives an invariant

$$Z(-, \mathcal{H}(A)) \colon \mathcal{M}^0_{\text{fram}} \to k.$$

When A is an involutory Hopf algebra, the restriction of the Betti number is no longer necessary, and the invariant $Z(-, \mathcal{H}(A))$ becomes an invariant of closed combed 3-manifolds. Furthermore, if A is additionally unimodular and counimodular, the invariant becomes a topological invariant of closed 3-manifolds [13].

5.3 Invariant for small quantum Borel subalgebra of $\mathfrak{sl}_2$ and $\mathrm{SO}(3)$ WRT invariant

Let us take $H = u_q(\mathfrak{sl}_2^+)$ the small quantum Borel subalgebra with q the n-th primitive root of unity. In this case, for (S^3, f) and $(L(2,1), f')$ in Example 4.2, we have

$$Z(S^3, f; u_q(\mathfrak{sl}_2^+)) = q^{-1},$$

$$Z(L(2,1), f'; u_q(\mathfrak{sl}_2^+)) = 2q^{-1} \frac{1 - q^{-\lfloor \frac{n+1}{2} \rfloor}}{1 - q^{-1}}.$$

When q is a primitive root of unity of odd order N, the above values match, up to multiplication by q, the SO(3) Witten-Reshetikhin-Turaev (WRT) invariant $\tau_N^{\mathrm{SO}(3)}(M)$ [10, 16, 20] times the cardinality $|H_1(M)|$ of the first homology group.

Conjecture 5.4 ([14, Conjecture 5.6])**.** *Let q be a primitive root of unity of odd order N. Then for every closed oriented framed 3-manifold M with $b_1(M) = 0$ there exists an integer k such that*

$$Z(M, f; u_q(\mathfrak{sl}_2^+)) = q^k \cdot |H_1(M)| \cdot \tau_N^{SO(3)}(M).$$

Recall that WRT invariant is an invariant of 2-framed 3-manifold, where one usually chooses canonical 2-framing to compute it [1]. Since framing f induces a 2-framing v_2, we expect that the following holds:

$$Z(M, f; u_q(\mathfrak{sl}_2^+)) = |H_1(M)| \cdot \tau_N^{\mathrm{SO}(3)}(M, v_2).$$

6 Alternative reconstructions and extension of universal invariant based on ideal triangulations

The results in this section will be included in the upcoming paper [18]. We give two alternative reconstructions of the universal invariant for $(1,1)$-tangles. Theorems 6.2 and 6.3 imply that the restrictions of Z provide the universal invariant for regular isotopy classes of $(1,1)$-tangle diagrams. In other words, Z is an extension of the universal invariant. The results in this section are easily generalized to these for string links.

6.1 Reconstruction on crossings

Lemma 6.1 ([18])**.** *Let c be a crossing (as a fundamental tangle). We have*

$$\phi^{\otimes 2} \circ J_R(c, D(A)) = Z(\iota \circ \delta(c), \mathcal{H}(A)) \quad \in \mathcal{H}(A)^{\otimes 4}.$$

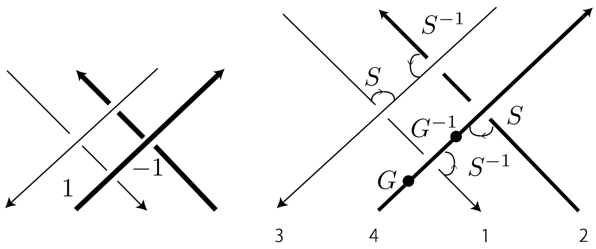
Proof. Let c be a positive crossing where both strands are oriented downwards. Since $J_R(c, D(A)) = R$, it is enough to prove $Z(\iota \circ \delta(c), \mathcal{H}(A)) = \phi^{\otimes 2}(R)$. Recall that $\phi^{\otimes 2}(R) = S''_{14} S_{13} \tilde{S}_{24} S'_{23}$, where the multiplication in the RHS is in $(\mathcal{H}(H) \otimes \mathcal{H}(H)^{\mathrm{op}})^{\otimes 2}$. Thus we should prove

$$\begin{aligned} Z(\iota \circ \delta(c), \mathcal{H}(A)) &= \sum e_i e_j \otimes \tilde{e}_k \tilde{e}_l \otimes e^j e^l \otimes \tilde{e}^i \tilde{e}^k \in (\mathcal{H}(A) \otimes \mathcal{H}(A)^{\mathrm{op}})^{\otimes 2} \\ &= \sum e_i e_j \otimes \tilde{e}_l \tilde{e}_k \otimes e^j e^l \otimes \tilde{e}^k \tilde{e}^i \in \mathcal{H}(A)^{\otimes 4}. \end{aligned}$$

The integral normal o-graph $\iota \circ \delta(c)$ has four crossings and one of them has a non-trivial integer on its edges as in Figure 15 (a). We attach copies of $S^{\pm}$ and $G^{\pm}$ as in Figure 15 (b). By using $S^{-1} = \sum_a \gamma(e_a) \otimes e^a = \sum_a e_a \otimes \gamma^*(e^a)$ and $S = \sum_a e_a \otimes e^a = \sum_a \gamma(e_a) \otimes \bar{\gamma}^*(e^a) = \sum_a \tilde{e}_a \otimes \tilde{e}^a$, we have

$$\begin{aligned} Z(\iota \circ \delta(c), \mathcal{H}(A)) &= \sum e_i e_j \otimes \gamma(e_l) e_k \otimes e^j e^l \otimes e^k G^{-1}(\gamma^*(e^i)) G \\ &= \sum e_i e_j \otimes \gamma(e_l) e_k \otimes e^j e^l \otimes e^k (\bar{\gamma}^*)^2 (\gamma^*(e^i)) \\ &= \sum e_i e_j \otimes \tilde{e}_l \tilde{e}_k \otimes e^j e^l \otimes \tilde{e}^k \tilde{e}^i \end{aligned}$$

as desired.



(a) (b)

Figure 15: (a) integral normal o-graoh $\iota \circ \delta(c)$, (b) labels

For other positive crossings c', we arrange c' by planar isotopy to be a composition of c and a maximum and a minimum. If c' is a crossings where both strands are oriented upward, or the upper left strand is oriented upward and the upper right strand is oriented downward, then we have $J_R(c', D(A)) = J_R(c', D(A))$ and $Z(\iota \circ \delta(c'), \mathcal{H}(A)) = Z(\iota \circ \delta(c), \mathcal{H}(A))$, thus we have the assertion. If c' is a crossings where the upper left strand is oriented downward and the upper right strand is oriented upward, then by Lemma 5.1, we have

$$\begin{aligned}\phi^{\otimes 2}(J_R(c', D(A))) &= \phi^{\otimes 2}((u \otimes 1) J_R(c, D(A))(u^{-1} \otimes 1)) \\ &= ((G^{-1})^{\otimes 2} \otimes 1^{\otimes 2}) Z(\iota \circ \delta(c), \mathcal{H}(A))(G^{\otimes 2} \otimes 1^{\otimes 2}) \\ &= Z(\iota \circ \delta(c'), \mathcal{H}(A)).\end{aligned}$$

We can similarly prove the assertion for negative crossings. Thus we have the assertion. □

6.2 Reconstruction of universal invariant version 2

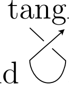
For a $(1,1)$-tangle diagram T, we define a reduced version of the universal invariant as $\bar{J}_R(T, D(A)) = u^{-w(T)} J_R(T, D(A))$, where $w(T)$ is the number of $\smile\!\nearrow$ minus the number of $\frown\!\searrow$ in T. For an integral normal o-tangle Γ, we define a reduced version of the invariant as $\bar{Z}(\gamma, \mathcal{H}(A)) = G^{-I(\gamma)} Z(\gamma, \mathcal{H}(A))$, where we define $I(\Gamma)$ to be the sum of all integer weight on Γ. Observe that $\bar{J}$ is itself an invariant under regular isotopy of tangle diagrams, and $\bar{Z}$ is itself an invariant under $\sim$.

Theorem 6.2 ([18])**.** *We have*

$$\phi \circ \bar{J}_R(T, D(A)) = \bar{Z}((\iota \circ \delta)(T), \mathcal{H}(A)) \quad \in \mathcal{H}(A)^{\otimes 2}$$

Proof. Note that the maxima and minima to which we attach $u^{\pm 1}$ on T correspond to the maxima and minima on the thin (resp. thick) strands of $\iota \circ \delta(T)$ to which we attach $G^{\mp 1}$ (resp. $G^{\pm 1}$). Taking into account that the thick strand goes in reverse compared to T, by Lemma 5.1, the assertion is reduced to the case when T is a crossing, which is already shown in Lemma 6.1. □

6.3 Reconstruction of universal invariant version 3

Let T be a tangle diagram consisting of copies of the fundamental tangles. Recall that $T_{(\leftarrow)}$ is obtained from T by replacing each of $\frown$ and $\smile$ with  and , respectively.

Theorem 6.3 ([18]). *We have*

$$\phi \circ J_R(T, D(A)) = Z((\iota \circ \delta)(T_{(\leftarrow)}), \mathcal{H}(A)) \quad \in \mathcal{H}(A)^{\otimes 2}.$$

Proof. By (2) in the proof of Theorem 3.1 we have $\phi \circ J_R(T, D(A)) = \phi \circ J_R(T_{(\leftarrow)}, D(A))$. Since $T_{(\leftarrow)}$ has no more $\frown$ and $\smile$, the labels for J_R are attached only on crossings of T. Thus we have $J_R(T_{(\leftarrow)}, D(A)) = Z((\iota \circ \delta)(T_{(\leftarrow)}), \mathcal{H}(A))$ by Lemma 6.1, which completes the proof. □

6.4 Comparison of reconstructions

In the reconstruction version 1, J'_R, we attach the canonical element $S^{\pm 1}$ of the Heisenberg double after applying the antipode, depending on the direction of a crossing c which are treated as fundamental tangles of slice diagrams. In this construction, the image of the universal invariant $J_R(c, D(A))$ under the Kashaev embedding can be exactly realized as $J'_R(c, \mathcal{H}(A))$ without any additional structure like framing. However, we cannot extend J'_R to be an invariant of integral normal o-graphs or framed 3-manifolds because it is not invariant under planar isotopy of integral normal o-graphs. It is worth noting that J'_R is planar isotopy invariant for tangles, but we cannot move the strands of the duplicated diagram independently.

In versions 2 and 3 of the reconstruction using Z, we adopt a canonical approach to attach $S^{\pm 1}$ to the crossings of $\delta(c)$. Specifically, we attach S to each positive crossing and S^{-1} to each negative crossing, regardless of the orientation of the crossing. However, attaching $S^{\pm 1}$ alone is not sufficient because the value of $\delta(c)$ does not directly correspond to $\phi^{\otimes}(R)$ and it is not an invariant of 3-manifolds; its value depends on the choice of branched ideal triangulation. To overcome this limitation, we incorporate framing through the map ι to obtain an invariant of 3-manifolds.

We compare Version 2 and Version 3 of the reconstruction. In Version 2, we observe that it focuses on the essential parts of J_R and Z. The original versions, J_R and Z, can be obtained from the reduced versions, $\bar{J}_R$ and $\bar{Z}$, by multiplying invertible elements that can be easily calculated from the diagrams. However, these reduced versions are not the original invariants themselves. In contrast, Version 3 does not require modifying the original invariants. Instead, it involves a transformation where the diagram T is alternated to $T_{(\leftarrow)}$. This transformation corresponds to altering the corresponding branched ideal triangulation and framing of the 3-manifold.

7 Summary

We have presented three types of reconstructions of the universal invariant. The original construction associates the universal R-matrix or its inverse with each crossing of link diagrams, while the reconstructions associate the S-tensor or its inverse with each ideal tetrahedron of 3-manifolds.

The first reconstruction (Theorem 3.1) uses slice diagrams of tangles and provides a topological realization of Kashaev's embedding at each crossing of tangles. However, it cannot be extended to be an invariant of 3-manifolds.

By introducing integral normal o-graphs, we can represent framed 3-manifolds. In particular, for closed framed 3-manifolds with a vanishing first Betti number, we establish a one-to-one correspondence (Theorem 4.1). This allows us to construct an invariant of closed framed 3-manifolds (Theorem 5.3).

Taking the framing structure into account, in the context of the universal invariant, we provide two alternative reconstructions (Theorems 6.2 and 6.3) using integral normal o-graphs. This means that the invariant Z extends the universal invariant in a three-dimensional manner.

We anticipate that our framework will provide a new approach to studying quantum invariants in a three-dimensional context.

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