On keen weakly reducible bridge splittings of links

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1 Introduction

For a link L in a closed orientable 3-manifold M, we say that $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is a (g, b)-*splitting* for the pair (M, L) if $F \cap L = P$ and F separates (M, L) into two components (V_1, t_1) and (V_2, t_2), where

- V_1 and V_2 are handlebodies of genus g,
- $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = F$, and
- $t_i = L \cap V_i$ is a union of b arcs properly embedded in V_i which is parallel to ∂V_i $(i = 1, 2)$.

A (g, b)-splitting $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is said to be *reducible* if there is a pair of essential disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ such that $\partial D_1 = \partial D_2$, and is said to be *irreducible* otherwise. A (g, b)-splitting $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is said to be *weakly reducible* if there is a pair of essential disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ such that $\partial D_1 \cap \partial D_2 = \emptyset$, and is said to be *strongly irreducible* otherwise. A weakly reducible (g, b)-splitting $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is said to be *keen* if the pair of essential disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ such that $\partial D_1 \cap \partial D_2 = \emptyset$ is unique up to isotopy.

In the talk at ILDT (May 26, 2023), the second author presented two theorems (Thm 1 and Thm 2 in [2]) and gave some comments on the proof of a part of the conclusions of Thm 1, that is, the case when "$g = 0$" and "$n = 1$". In this article, we show a bit of detailed arguments of the proof, that is, the outline of the proof of Theorem 3.4 of this article. As mentioned in the talk, the proof of Thm 1 consists of several parts, and each has different flavor. We hope this article would be helpful for readers who are interested in this kind of research subject.

2 Preliminaries

Throughout this paper, for a submanifold Y of a manifold X, $N_X(Y)$ denotes a regular neighborhood of Y in X. When X is clear from the context, we denote $N_X(Y)$ by $N(Y)$ in brief. We denote $\mathrm{cl}_X(Y)$ (or $\mathrm{cl}(Y)$ in brief) the closure of Y in X.

2.1 Curve complexes

Let S be a genus-g orientable surface with e boundary components and p punctures. A simple closed curve in S is *essential* if it does not bound a disk or a once-punctured disk in S and is not parallel to a component of ∂S. We say that S is *non-simple* if there exists an essential simple closed curve in S, and S is *simple* otherwise. By an *arc properly embedded* in S, we mean an arc intersecting ∂S only in its endpoints. An arc properly embedded in S is *essential* if it does not co-bound a disk with no puncture in S together with an arc on ∂S. Two simple closed curves or two arcs in S are *isotopic* if there is an ambient isotopy of S which sends one to the other. We say that S is *sporadic* if either "$g = 0$ and $e + p \leq 4$" or "$g = 1$ and $e + p \leq 1$".

For a non-sporadic surface S, the *curve complex* $\mathcal{C}(S)$ is defined as follows: Each vertex of $\mathcal{C}(S)$ is the isotopy class of an essential simple closed curve in S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in S. For sporadic surfaces, we need to modify the definition of the curve complex slightly. We assume that either $g = 1$ and $e + p \leq 1$ or $g = 0$ and $e + p = 4$ since, otherwise, S is simple. When $g = 1$ and $e + p \leq 1$ (resp. $g = 0$ and $e + p = 4$), a collection of $k + 1$ vertices forms a k-simplex of $\mathcal{C}(S)$ if they can be realized by curves in S which mutually intersect transversely exactly once (resp. twice). The *arc-and-curve complex* $\mathcal{AC}(S)$ is defined similarly: Each vertex of $\mathcal{AC}(S)$ is the isotopy class of an essential properly embedded arc or an essential simple closed curve in S, and a collection of $k+1$ vertices forms a k-simplex of $\mathcal{AC}(S)$ if they can be realized by disjoint arcs or simple closed curves in S. The symbols $\mathcal{C}^0(S)$ and $\mathcal{AC}^0(S)$ denote the 0-skeletons of the curve complexes $\mathcal{C}(S)$ and $\mathcal{AC}(S)$, respectively. Throughout this paper, for a vertex $x \in \mathcal{C}^0(S)$ or $x \in \mathcal{AC}^0(S)$ we often abuse notation and use x to represent (the isotopy class of) a geometric representative of x.

We can define the *distance* between two vertices in the curve complex $\mathcal{C}(S)$ to be the minimal number of 1-simplices of a simplicial path in $\mathcal{C}(S)$ joining the two vertices. We denote by $d_S(a, b)$ the distance in $\mathcal{C}(S)$ between the vertices a and b. For subsets A and B of the vertices of $\mathcal{C}(S)$, we define $\mathrm{diam}_S(A, B) = \mathrm{diam}_S(A \cup B)$. Similarly, we can define the distance $d_{\mathcal{AC}(S)}(a, b)$ and $\mathrm{diam}_{\mathcal{AC}(S)}(A, B)$.

2.2 Subsurface projections

Throughout this paper, $\mathcal{P}(Y)$ denotes the power set of a set Y. Let S be a genus-g orientable surface with e boundary components and p punctures. We say that a subsurface $X(\subset S)$ is *essential* if each component of ∂X is an essential simple closed curve in S. Suppose that X is a non-simple essential subsurface of S. We call the composition $\pi_X := \pi_0 \circ \pi_{AC}$ of maps $\pi_{AC} : \mathcal{C}^0(S) \to \mathcal{P}(\mathcal{AC}^0(X))$ and $\pi_0 : \mathcal{P}(\mathcal{AC}^0(X)) \to \mathcal{P}(\mathcal{C}^0(X))$ a *subsurface projection*, where π_{AC} and π_0 are defined as follows: For a vertex α, take a representative α so that $|\alpha \cap X|$ is minimal, where $|\cdot|$ is the number of connected components. Then

- $\pi_{AC}(\alpha)$ is the set of all isotopy classes of the components of $\alpha \cap X$,
- $\pi_0(\{\alpha_1, \ldots, \alpha_n\})$ is the union for all $i = 1, \ldots, n$ of the set of all isotopy classes of

the components of $\partial N_X(\alpha_i \cup \partial X)$ which are essential in X.

Let Y, Z be non-simple surfaces. Suppose that there exists an embedding $\varphi : Y \to Z$ such that $\varphi(Y)$ is an essential subsurface of Z. Note that φ naturally induces maps $\mathcal{C}^0(Y) \to \mathcal{C}^0(Z)$ and $\mathcal{P}(\mathcal{C}^0(Y)) \to \mathcal{P}(\mathcal{C}^0(Z))$. Throughout this paper, under this setting, we abuse notation and use φ to denote these maps.

3 Results

First we give a description for the pair of essential disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ such that $\partial D_1 \cap \partial D_2 = \emptyset$ for a keen weakly reducible (g, b)-splitting $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$.

The following proposition is due to [5, Theorems 2.2 and 2.3].

Proposition 3.1 (cf. [5, Theorems 2.2 and 2.3]). *Let $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ be a $(1, 1)$-splitting for (M, L). Then the following hold.*

(1) *$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is reducible if and only if L is a trivial knot in M.*

(2) *$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is weakly reducible and irreducible if and only if M is $S^2 \times S^1$ and L is the core knot in M.*

Moreover, by the proofs of [5, Theorems 2.2 and 2.3], we can see the following for a weakly reducible $(1, 1)$-splitting $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ for (M, L).

- $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is keen reducible if and only if L is a trivial knot in M and M is not homeomorphic to $S^2 \times S^1$. In this case, the disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ with $\partial D_1 = \partial D_2$ must be inessential in V_1 and V_2, respectively, and hence, each of ∂D_1 and ∂D_2 cuts off a twice-punctured disk from $F \setminus P$. (See [5, Lemma 4.1 and its proof, and Lemma 4.2].)
- If $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is weakly reducible and irreducible, then it is keen. In this case, for the disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ with $\partial D_1 \cap \partial D_2 = \emptyset$, each of ∂D_1 and ∂D_2 is non-separating on F and $\partial D_1 \cup \partial D_2$ is separating on F. (See the proof of [5, Theorem 2.3] and also [1, Proposition 10.1].)

Proposition 3.2. *Let $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ be a (g, b)-splitting which is keen weakly reducible and irreducible, and let $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ be essential disks such that $\partial D_1 \cap \partial D_2 = \emptyset$. Then either of the following holds.*

(1) *Each of ∂D_1 and ∂D_2 is non-separating on F, and $\partial D_1 \cup \partial D_2$ is separating on F,*

(2) *each of ∂D_1 and ∂D_2 cuts off a twice-punctured disk from $F \setminus P$.*

Proof. Case 1. Each of ∂D_1 and ∂D_2 is non-separating on F.

If $\partial D_1 \cup \partial D_2$ is non-separating on F, then there is an essential simple closed curve γ on $F \setminus P$ intersecting ∂D_1 transversely in one point and $\gamma \cap \partial D_2 = \emptyset$. We may assume that $N_{F \setminus P}(\partial D_1) \cap \gamma$ consists of a single arc γ_1. Let γ_2 be the closure of $\gamma \setminus \gamma_1$, and let D_1' be the disk in $V_1 \setminus t_1$ obtained from two copies of D_1 (whose boundary is $\partial N_{F \setminus P}(\partial D_1)$) by performing a band operation along γ_2. Then D_1' is an essential disk in $V_1 \setminus t_1$ which is not isotopic to D_1 and is disjoint from D_2, which is a contradiction to the uniqueness of the pair (D_1, D_2). Thus, $\partial D_1 \cup \partial D_2$ must be separating on F in this case.

Case 2. Either one of ∂D_1 and ∂D_2, say ∂D_1, is separating on F.

Assume that ∂D_2 is non-separating on F. Then it is easy to see that the arguments for Case 1 above work in this case to lead to a contradiction (together with the fact that $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is irreducible). Hence, each of ∂D_1 and ∂D_2 is separating on F.

Assume that ∂D_1 does not cut off a twice-punctured disk from $F \setminus P$. Let (V_1^1, t_1^1) and (V_1^2, t_1^2) be the closures of the two components obtained by cutting (V_1, t_1) along D_1. If ∂D_2 is contained in ∂V_1^i for $i = 1$ or 2, then we can find an essential disk D_1' in $V_1 \setminus t_1$ such that $\partial D_1' \subset \partial V_1^j$ where $j (\neq i) \in \{1, 2\}$ (and hence, $\partial D_1' \cap \partial D_2 = \emptyset$), and D_1' is not isotopic to D_1. This is a contradiction to the uniqueness of the pair (D_1, D_2). Thus, ∂D_1 must cut off a twice-punctured disk from $F \setminus P$. Similarly, ∂D_2 must cut off a twice-punctured disk from $F \setminus P$ as well. □

We also have the following for reducible (g, b)-splittings.

Proposition 3.3 (cf. [1, Theorem 13.1]). *A (g, b)-splitting $(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ for (M, L) is reducible if and only if one of the following holds.*

(1) *$E(L) = \mathrm{cl}(M \setminus N(L))$ is reducible.*

(2) *$(V_1, t_1) \cup_{(F,P)} (V_2, t_2)$ is stabilized, that is, there are essential disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ such that ∂D_1 and ∂D_2 intersect transversely in one point.*

For the existence of (g, b)-splittings which are keen weakly reducible and irreducible, we have the following.

Theorem 3.4 ([1, Theorems 1.1 and 1.3]). *For any integers g and b with $g \geq 0$ and $b \geq 1$ except for $(g, b) = (0, 1), (0, 3)$, there exists a (g, b)-splitting which is keen weakly reducible and irreducible.*

Theorem 3.5 ([1, Theorem 1.4]). *There does not exist a $(0, 3)$-splitting which is keen weakly reducible and irreducible.*

By Proposition 3.2, we see that, to prove Theorem 3.4, it may be reasonable to use pair of disks $D_1 \subset V_1 \setminus t_1$ and $D_2 \subset V_2 \setminus t_2$ satisfying the following (, and this intuition is shown to be correct in our paper [1]):

- each of ∂D_1 and ∂D_2 is non-separating on F and $\partial D_1 \cup \partial D_2$ is separating on F, if $g \geq 2$ (see the arguments in [1, Section 9]), and
- each of ∂D_1 and ∂D_2 cuts off a twice-punctured disk from $F \setminus P$, if $b \geq 2$ and $(g, b) \neq (0, 2), (0, 3)$ (see the arguments in [1, Sections 10 and 11]).

The case when $(g, b) = (1, 1)$ is done by Proposition 3.1, and the case when $(g, b) = (0, 2)$ can be treated easily since each $V_i \setminus t_i$ $(i = 1, 2)$ admits a unique essential disk up to isotopy and the curve complex of the 4-punctured sphere is well-known (see, for example, [1, Appendix B]).

In the remainder, we give an outline of the proof of Theorem 3.4 for the case where $g = 0$ (and $b \geq 4$).

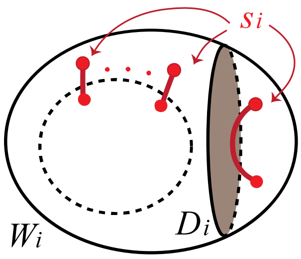

Figure 1: (W_i, s_i) and D_i.

Let F be a 2-sphere and let P be the union of $2b$ points on F. Let α_0 and α_1 be simple closed curves on $F \setminus P$ such that $\alpha_0 \cap \alpha_1 = \emptyset$ and that $\alpha_0 \cup \alpha_1$ cuts off two twice-punctured disks from $F \setminus P$ which are disjoint to each other. For $i = 1, 2$, let $V_i^{*,0}$ be a 3-ball and $t_i^{*,0}$ be the union of b arcs $t_i^1, t_i^2, \ldots, t_i^b$ properly embedded in $V_i^{*,0}$ which is parallel to $\partial V_i^{*,0}$. Let $V_i \,(\subset V_i^{*,0})$ be a 3-ball such that

- $t_i := t_i^{*,0} \cap V_i$ is the union of $(b-1)$ arcs which is parallel to ∂V_i,
- $W_i := \mathrm{cl}(V_i^{*,0} \setminus V_i) \cong \Sigma \times I$, where Σ is a 2-sphere and $I = [0, 1]$, and
- $s_i := t_i^{*,0} \cap W_i$ is the union of $2(b-1)$ I-fibers $(\subset \Sigma \times I)$ and t_i^b.

Let D_i be the disk properly embedded in W_i as in Figure 1. Then $W_i \setminus D_i$ consists of two components W_i^1 and W_i^2 such that $\mathrm{cl}(W_i^1) \cong \Sigma \times I$, where $s_i^1 (:= t_i^{*,0} \cap \mathrm{cl}(W_i^1))$ is the union of $2(b-1)$ I-fibers, $\mathrm{cl}(W_i^2)$ is a 3-ball and $s_i^2 (:= t_i^b)$ is an arc parallel to ∂W_i^2. Let $\partial_- W_i$ be the component of ∂W_i disjoint from D_i, and let $\partial_+ W_i := \partial W_i \setminus \partial_- W_i$. Note that $s_i \cap \partial_- W_i$ consists of $(2b-2)$ points, and $s_i \cap \partial_+ W_i$ consists of $2b$ points. Let F_i be the subsurface $\partial_+ W_i \cap \mathrm{cl}(W_i^1)$ of $\partial_+ W_i$. Let $\pi_{F_i \setminus s_i} : \mathcal{C}^0(\partial_+ W_i \setminus s_i) \to \mathcal{P}(\mathcal{C}^0(F_i \setminus s_i))$ be the subsurface projection, and let $P_i : F_i \setminus s_i \to (F_i \setminus s_i) \cup D_i \to \partial_- W_i \setminus s_i$ be the natural projection. Let $\Phi_i : \mathcal{C}^0(\partial_+ W_i \setminus s_i) \to \mathcal{P}(\mathcal{C}^0(\partial_- W_i \setminus s_i))$ be the composition $P_i \circ \pi_{F_i \setminus s_i}$.

Identify $(\partial_+ W_1, s_1 \cap \partial_+ W_1)$ and $(\partial_+ W_2, s_2 \cap \partial_+ W_2)$ with (F, P) so that $\partial D_1 = \alpha_0$ and $\partial D_2 = \alpha_1$. Let $h_i : \partial V_i \setminus t_i \to \partial_- W_i \setminus s_i$ be homeomorphisms such that

$$d_{\partial_- W_1 \setminus s_1}(\Phi_1(\alpha_1), h_1(\mathcal{D}^0(V_1 \setminus t_1))) > 3, \tag{1}$$

$$d_{\partial_- W_2 \setminus s_2}(\Phi_2(\alpha_0), h_2(\mathcal{D}^0(V_2 \setminus t_2))) > 3. \tag{2}$$

(The existence of such homeomorphisms is guaranteed by [3, Claim2].) Let $\overline{h}_i : (\partial V_i, \partial t_i) \to (\partial_- W_i, s_i \cap \partial_- W_i)$ be the homeomorphism of the pairs induced from h_i. Let $(V_i^*, t_i^*) := (W_i, s_i) \cup_{\overline{h}_i} (V_i, t_i)$. Then $(V_1^*, t_1^*) \cup_{(F,P)} (V_2^*, t_2^*)$ is a $(0, b)$-splitting of a link. Note that $(V_1^*, t_1^*) \cup_{(F,P)} (V_2^*, t_2^*)$ is weakly reducible since $\partial D_1 \cap \partial D_2 = \emptyset$. Let $\mathcal{D}_i$ be the set of essential disks in $V_i^* \setminus t_i^*$ for $i = 1, 2$. To show that $(V_1^*, t_1^*) \cup_{(F,P)} (V_2^*, t_2^*)$ is keen and irreducible, we prove the following.

Assertion 3.6. *$\partial E_1 \cap \partial E_2 \neq \emptyset$ for any $E_1 \in \mathcal{D}_1$ and $E_2 \in \mathcal{D}_2$ with $(E_1, E_2) \neq (D_1, D_2)$.*

To prove the above assertion, we divide $\mathcal{D}_i$ $(i = 1, 2)$ into three sets $\mathcal{D}_i^1$, $\mathcal{D}_i^2$, $\mathcal{D}_i^3$, where

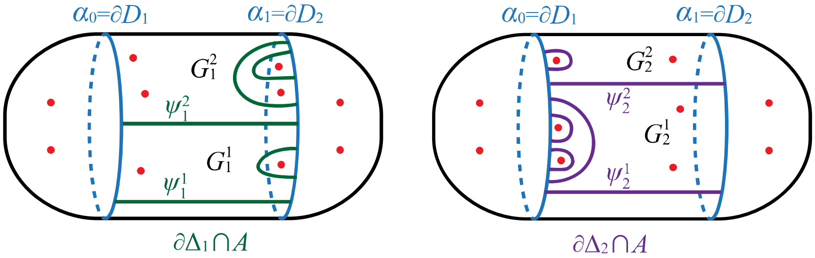

Figure 2: ψ_i^1, ψ_i^2, G_i^1 and G_i^2.

- $\mathcal{D}_i^1$ consists of the single disk D_i,
- $\mathcal{D}_i^2$ consists of disks which are disjoint from D_i, and not isotopic to D_i,
- $\mathcal{D}_i^3$ consists of disks which cannot be isotoped to be disjoint from D_i.

Without loss of generality, we may assume that either one of the following holds.

- $E_1 \in \mathcal{D}_1^1$ and $E_2 \in \mathcal{D}_2^2$,
- $E_1 \in \mathcal{D}_1^1$ and $E_2 \in \mathcal{D}_2^3$,
- $E_1 \in \mathcal{D}_1^2$ and $E_2 \in \mathcal{D}_2^2$,
- $E_1 \in \mathcal{D}_1^2$ and $E_2 \in \mathcal{D}_2^3$,
- $E_1 \in \mathcal{D}_1^3$ and $E_2 \in \mathcal{D}_2^3$.

The proof of Assertion 3.6 is carried out by deriving a contradiction for each of the above cases. The following is an outline of the proof for the last case. Suppose that $|E_i \cap D_i|$ $(i = 1, 2)$ is minimal. Let Δ_i be the closure of $E_i \setminus D_i$ that is outermost in E_i $(i = 1, 2)$. Then by using the inequalities (1) and (2), we can see that $\Delta_1 \cap \partial D_2 \neq \emptyset$ and $\Delta_2 \cap \partial D_1 \neq \emptyset$. Let A be the subsurface of $F \setminus P$ bounded by $\partial D_1 \cup \partial D_2$. Then we can see that, for $i = 1, 2$, $\Delta_i \cap A$ contains exactly two arcs ψ_i^1, ψ_i^2 joining α_0 and α_1, and the other components are disjoint from α_{i-1} (see Figure 2). This shows that there are exactly two components of $A \setminus \Delta_i$ that are adjacent to α_{i-1}. Let G_i^1 and G_i^2 be the closures of the components. Then we can prove that G_i^j contains at most one puncture $(i, j \in \{1, 2\})$, by using the inequality (1) or (2) again (see [1, Claim 11.3]). We can show that this implies that $b = 4$, and each G_i^j contains exactly one puncture. Then there exists a simple closed curve γ in A (and hence in F_1) that bounds a twice-punctured disk, say D_γ, in $A(\subset F_1)$, that intersects $\partial\Delta_1$ twice, and is disjoint from α_1 (see Figure 3). Note that F_1 contains 6 punctures, and hence $F_1 \setminus D_\gamma$ contains 4 punctures. Since $F_1 \setminus (\Delta_1 \cup D_\gamma)$ consists of two components, either of the components must contain at least 2 punctures. Then there exists a simple closed curve δ that bounds a twice-punctured disk in (the interior of) the component. Since $\delta \cap \Delta_1 = \emptyset$, we can find a disk $\overline{\Delta}_1$ in $V_1 \setminus t_1$ disjoint from $\Phi_1(\delta)\,(\neq \emptyset)$ and show that

$$d_{\partial_- W_1 \setminus s_1}(\Phi_1(\delta), h_1(\mathcal{D}^0(V_1 \setminus t_1)) \leq d_{\partial_- W_1 \setminus s_1}(\Phi_1(\delta), h_1(\partial\overline{\Delta}_1)) \leq 1.$$

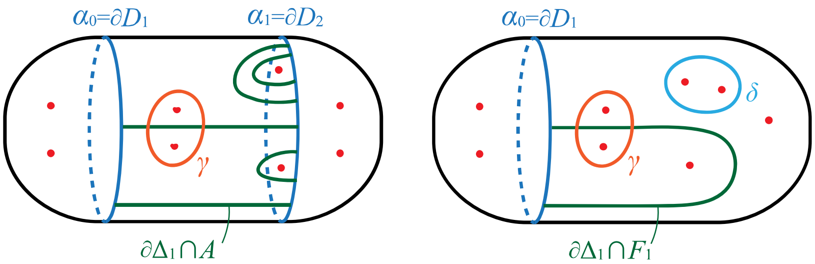

Figure 3: γ and δ.

Note also that $\alpha_1 \cap \gamma = \emptyset$, $\gamma \cap \delta = \emptyset$, and $\Phi_1(\alpha_1) \neq \emptyset$, $\Phi_1(\gamma) \neq \emptyset$. By using these, we can see that

$$\begin{aligned} d_{\partial_- W_1 \setminus s_1}(\Phi_1(\alpha_1), h_1(\mathcal{D}^0(V_1 \setminus t_1))) &\leq d_{\partial_- W_1 \setminus s_1}(\Phi_1(\alpha_1), \Phi_1(\gamma)) \\ &\quad + d_{\partial_- W_1 \setminus s_1}(\Phi_1(\gamma), \Phi_1(\delta)) \\ &\quad + d_{\partial_- W_1 \setminus s_1}(\Phi_1(\delta), h_1(\mathcal{D}^0(V_1 \setminus t_1))) \\ &\leq 1 + 1 + 1 = 3, \end{aligned}$$

a contradiction to the inequality (1).

References

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[2] A. Ido, Y. Jang and T. Kobayashi, *On keen bridge splittings of links*, slide of the talk at Intelligence of Low-dimensional Topology (2023), available at https://www.kurims.kyoto-u.ac.jp/~ildt/slides23/jang23slide.pdf.

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