On the calculation of the 3-loop invariant and the degree 2 part of the LMO invariant

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1 Introduction

The Kontsevich invariant is a invariant of knots which is universal among all quantum invariants and Vassiliev invariants of knots. The LMO invariant is a invariant of 3-manifolds derived from the Kontsevich invariant, and it is universal among all perturbative invariants and finite type invariants of 3-manifolds. These two invariants are very strong invariants, but their images are presented by infinite sums of some types of graphs, and it is very hard to determine all terms of them. It must be important problems to determine the images of these strong invariants in order to clarify the set of all knots or 3-manifolds.

The Kontsevich invariant has a special expansion, called the "loop expansion", and it can be one approach to investigate the image of the Kontsevich invariant [5, 7]. In [17], the author define the 3-loop invariant (or, the 3-loop polynomial) which present the 3-loop part of the loop expansion of the Kontsevich invariant, and perform some calculations for the 3-loop polynomial. Also, in [6], Garoufalidis and Kricker found a formula for the LMO invariant of cyclic branched covers of knots by using the loop expansion of the Kontsevich invariant. By using this formula and the result in [17], the author calculate the degree 2 part of the LMO invariant of cyclic branched covers of knots in [18].

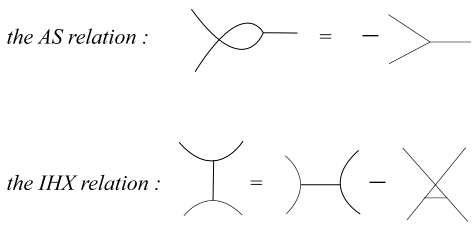
This report is a rough explanation of these results.

2 Preliminaries

2.1 The Kontsevich invariant and its loop expansion

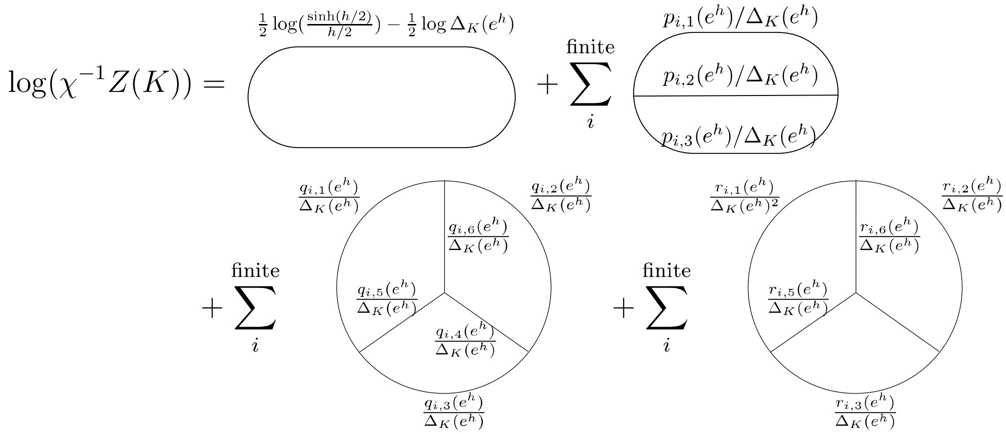
In this section, we review the Kontsevich invariant and its loop expansion, and we define the 3-loop invariant. For details, see [5, 7, 11, 17].

An *open Jacobi diagram* is an uni-trivalent graphs such that a cyclic order of the three edges around each trivalent vertex is fixed, in other words, each trivalent vertex is *vertex-oriented*. When we draw a Jacobi diagram on $\emptyset$, each trivalent vertex is vertex-oriented in the counterclockwise order. Furthermore, we define the *degree* of a Jacobi diagram to be half the number of all vertices of the graph of the Jacobi diagram. We define $\mathcal{B}$ to be the quotient vector space spanned by Jacobi diagrams subject to the AS, IHX relations.



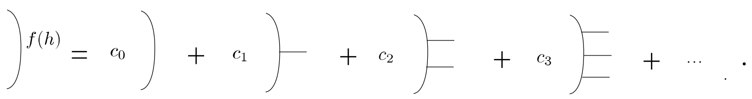
$\mathcal{B}$ forms a commutative algebra whose product is given by disjoint union.

The *Kontsevich invariant* $\chi^{-1}Z(K)$ of a knot K is defined to be in $\mathcal{B}$ (Strictly speaking, it is defined to be in the completion of $\mathcal{B}$ with respect to the degree). Note that $\chi^{-1}Z(K)$ is group-like, which means that it is exponential of series of connected diagrams. The loop expansion of the Kontsevich invariant of a 0-framed knot K is a presentation of the following form [5, 7],

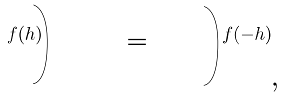


+ (terms of (> 3)-loop part),

where $\Delta_K(t)$ denotes the Alexander polynomial, and $p_{i,j}(e^h), q_{i,j}(e^h), r_{i,j}(e^h)$ are polynomials in $e^{\pm h}$. Here, a labeling of $f(h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \cdots$ implies that ,



Note that



by the AS relation. Then, we define the *3-loop invariant* of K by

$$\begin{aligned}
&\Lambda_K(t_1,t_2,t_3,t_4)\\
&= \sum_{\substack{i\\ \tau\in\mathfrak{S}_4}} \frac{q_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,4}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(3)}^{-\mathrm{sgn}\tau})q_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})q_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})}\\
&+ \sum_{\substack{i\\ \tau\in\mathfrak{S}_4}} \frac{r_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})r_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_{\tau(1)}t_{\tau(4)}^{-1})^2\Delta_K(t_{\tau(2)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(1)}^{-1})\Delta_K(t_{\tau(1)}t_{\tau(2)}^{-1})}\\
&\in \frac{1}{\hat{\Delta}^2}\cdot \mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1),
\end{aligned}$$

where we put

$$\hat{\Delta} = \Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1}).$$

In particular, if $\Delta_K(t)=1$, then $\Lambda_K(t_1,t_2,t_3,t_4)$ is a polynomial, so in this case, we call it the *3-loop polynomial*. For details about the 3-loop invariant, see [17]. The 3-loop invariant is a rational form presenting the 3-loop part of the Kontsevich invariant of knots.

Remark 1. The 2-loop part of the Kontsevich invariant of knots is presented by the 2-loop polynomial. The 2-loop polynomial $\Theta_K(t_1,t_2,t_3)$ is defined by

$$\Theta_K(t_1,t_2,t_3) = \sum_{\substack{i\\ \epsilon=\pm1\\ \sigma\in\mathfrak{S}_3}} p_{i,1}(t_{\sigma(1)}^{\epsilon})p_{i,2}(t_{\sigma(2)}^{\epsilon})p_{i,3}(t_{\sigma(3)}^{\epsilon}) \in \mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1}]/(\mathfrak{S}_3\times\mathbb{Z}/2\mathbb{Z}, t_1t_2t_3=1).$$

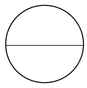
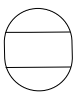
There are examples of calculations of the 2-loop polynomial, for example, see [4, 12, 13].

2.2 The LMO invariant and the formula of Garoufalidis and Kricker

In this section, we review the LMO invariant.

A *Jacobi diagram* on $\emptyset$ is a trivalent graph such that each trivalent vertex is *vertex-oriented*. We define $\mathcal{A}(\emptyset)$ to be the quotient vector space spanned by Jacobi diagrams on $\emptyset$ subject to *the AS, IHX relations*.

The *LMO invariant* $Z^{LMO}(M)$ of a closed 3-manifold M is defined to be in $\mathcal{A}(\emptyset)$ (Strictly speaking, it is defined to be in the completion of $\mathcal{A}(\emptyset)$ with respect to the degree). The LMO invariant is presented by

$$Z^{LMO}(M) = \exp\Bigg(c_1(M)$$  $$+\, c_2(M)$$  $$+ \text{(terms of connected diagrams of degree} > 2)\Bigg),$$

where $c_i(M)$ is a scalar invariant of M. Note that $c_1(M)$ is equal to $(-1)^{b_1(M)}\lambda(M)/2$, where $\lambda(M)$ is the Casson-Walker-Lescop invariant and $b_1(M)$ is the first Betti number of M. For details, see for example [8, 10].

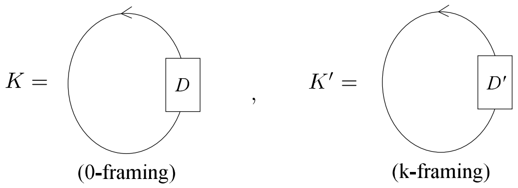
Let K be a knot in S^3, and let Σ_K^p is the *p-fold cyclic branched covers* of K. We call a knot K *p-regular* if Σ_K^p is a rational homology sphere, and we call a knot K *regular* if it is p-regular for all p. It is known that a knot K is p-regular if and only if its Alexander polynomial $\Delta_K(t)$ has no complex pth root of unity. In [6], Garoufalidis and Kricker found the formula which present $Z^{LMO}(\Sigma_K^p)$ by using the loop expansion of the Kontsevich invariant of K.

3 Results about the 3-loop invariant

In this section, we state the results about the 3-loop invariant obtained in [17].

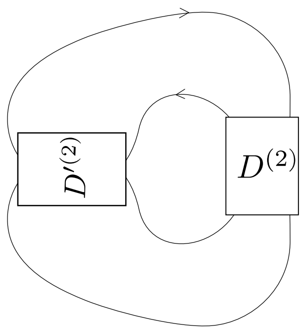
3.1 The 3-loop polynomial of $D(K, K')$

Let K be a 0-framed knot, and let K' be a k-framed knot ($k \in \mathbb{Z}$). Let D, D' be 1-tangles whose closures are K, K', respectively, noting that isotopy classes of D and D' are uniquely determined by K and K'.



(1)

We define $D(K, K')$ to be the following knot,



where $D^{(2)}$ and $D'^{(2)}$ are the doubles of D and D', respectively. We can obtain $D(K, K')$ by *plumbing* of the doubles of K and K', noting that $D(K, K')$ is a genus 1 knot with trivial Alexander polynomial.

For a knot K, we denote the low degree Vassiliev invariants as follows. Let c_n be the coefficient of the Conway polynomial $\nabla_K(z) = \sum c_n z^n$, and let j_n be the coefficient of the Jones polynomial $J_K(e^t) = \sum j_n t^n$. Note that the Conway polynomial is defined by $\nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_K(t)$. Then, we denote

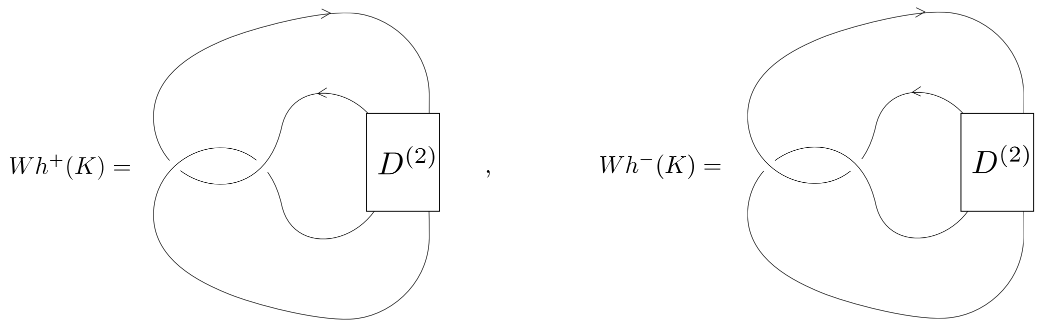
$$a_2 = -\frac{1}{2}c_2, \quad a_3 = -\frac{1}{24}j_3, \quad a_4 = \frac{1}{24}(-12c_4 + 6c_2^2 - c_2) \qquad (2)$$

We put $u_{m,n} = t_m t_n^{-1} + t_m^{-1} t_n - 2$ and $v_{m,n} = t_m t_n^{-1} - t_m^{-1} t_n \qquad (m, n \in \{1, 2, 3, 4\})$.

Theorem 2. *The 3-loop polynomial of $D(K, K')$ is presented by*

$$
\begin{aligned}
&\Lambda_{D(K,K')}(t_1, t_2, t_3, t_4) \\
&= (-16a_2a_2' - k^2a_2 - 8ka_3)(u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\
&\quad + (-\frac{k^2a_2}{12} + 4k^2a_4)(u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\
&\qquad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1}) \\
&\quad + 24k^2a_4(u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3}) \\
&\quad + 8k^2a_2^2(u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\
&\quad - \frac{k^2a_2}{4}(v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\
&\qquad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1}) \\
&\quad \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1).
\end{aligned}
$$

In particular, we can get the 3-loop polynomial of untwisted Whitehead double of K. We denote it by $Wh^{\pm}(K)$.



Here, D is a 1-tangle whose closure is K as shown in (1).

Corollary 3. *The 3-loop polynomial of $Wh^{\pm}(K)$ is presented by*

$$
\begin{aligned}
&\Lambda_{Wh^{\pm}(K)}(t_1, t_2, t_3, t_4) \\
&= (-a_2 \pm 8a_3)(u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\
&\quad + (-\frac{a_2}{12} + 4a_4)(u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\
&\qquad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1}) \\
&\quad + 24a_4(u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3}) \\
&\quad + 8a_2^2(u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\
&\quad - \frac{a_2}{4}(v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\
&\qquad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1}) \\
&\quad \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1).
\end{aligned}
$$

Remark 4. The 2-loop polynomial of $Wh^{\pm}(K)$ is presented by [4, 13]

$$\Theta_{Wh^{\pm}(K)}(t_1,t_2,t_3) = \pm 4a_2(t_1+t_1^{-1}+t_2+t_2^{-1}+t_3+t_3^{-1}-6).$$

Moreover, the 2-loop polynomial of genus 1 knots are calculated in [13].

We can prove Theorem 2 by using the rational version of the Aarhus integral. For the Aarhus integral, see [1, 2, 3]. By considering the doubles of knots one of which is 0 framing, the 3-loop polynomial of $D(K,K')$ is relatively easy to calculate, see [17]. This and Remark 4 are some of reasons why we consider $D(K,K')$.

3.2 A connected sum formula for the 3-loop invariant

Let K_1, K_2 be 0-framing knots, and let $K_1\#K_2$ be their connected sum.

Proposition 5. *We get the 3-loop invariant of $K_1\#K_2$ as follows,*

$$\begin{aligned}&\Lambda_{K_1\#K_2}(t_1,t_2,t_3,t_4) = \Lambda_{K_1}(t_1,t_2,t_3,t_4)+\Lambda_{K_2}(t_1,t_2,t_3,t_4)\\ &+\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\frac{t_{\tau(1)}t_{\tau(2)}t_{\tau(4)}^{-2}\Delta'_{K_1}(t_{\tau(1)}t_{\tau(4)}^{-1})\Delta'_{K_1}(t_{\tau(2)}t_{\tau(4)}^{-1})}{24\Delta_{K_1}(t_{\tau(1)}t_{\tau(4)}^{-1})\Delta_{K_1}(t_{\tau(2)}t_{\tau(4)}^{-1})}+\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\frac{t_{\tau(1)}t_{\tau(2)}^{-1}t_{\tau(3)}t_{\tau(4)}^{-1}\Delta'_{K_1}(t_{\tau(3)}t_{\tau(4)}^{-1})\Delta'_{K_1}(t_{\tau(1)}t_{\tau(2)}^{-1})}{24\Delta_{K_1}(t_{\tau(3)}t_{\tau(4)}^{-1})\Delta_{K_1}(t_{\tau(1)}t_{\tau(2)}^{-1})}\\ &\quad\in\frac{1}{\hat{\Delta}^2}\cdot\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4,t_1t_2t_3t_4=1),\end{aligned}$$

In particular, if $\Delta_{K_1}(t)=\Delta_{K_2}(t)=1$, then

$$\Lambda_{K_1\#K_2}(t_1,t_2,t_3,t_4) = \Lambda_{K_1}(t_1,t_2,t_3,t_4)+\Lambda_{K_2}(t_1,t_2,t_3,t_4)\in\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4,t_1t_2t_3t_4=1).$$

Remark 6. The 1-loop part of the Kontsevich invariant (that is, $\log\Delta_K(t)$) of $K_1\#K_2$ and the 2-loop polynomial of $K_1\#K_2$ are presented by

$$\begin{aligned}\log\Delta_{K_1\#K_2}(t) &= \log\Delta_{K_1}(t)+\log\Delta_{K_2}(t),\\ \Theta_{K_1\#K_2}(t_1,t_2,t_3) &= \Theta_{K_1}(t_1,t_2,t_3)+\Theta_{K_2}(t_1,t_2,t_3).\end{aligned}$$

This shows that up to 2-loop parts behave additively for the connected sum of knots. However, more than 2-loop parts do not behave additively, and the 3-loop part behaves as Proposition 5.

3.3 The 3-loop part of the colored Jones polynomial

The colored Jones polynomial $J_n(K;t)$ is the polynomial invariant of knots, which is obtained by

$$J_n(K;t) = \frac{V_n(K;t)}{V_n(\text{the unknot};t)} = \frac{t^{1/2}-t^{-1/2}}{t^{n/2}-t^{-n/2}} \cdot V_n(K;t),$$

where $V_n(K;t)$ is obtained by $V_n(K;e^{-h}) = W_{\mathfrak{sl}_2,V_n}\big(Z(K)\big)$, and $W_{\mathfrak{sl}_2,V_n}$ denotes the weight system derived from the Lie algebra $\mathfrak{sl}_2$ and its irreducible representation V_n. For details, see [11, 10, 9]. It is known, see Conjecture 1.2 of [15], Theorem 1.2 of [16], Proposition 3.1 of [14], that $J_n(K;t)$ can be presented in the following form,

$$J_n(K;e^h) = \sum_{l\geq 0} h^l \sum_{k\geq 0} d_{l,k}(nh)^k = \sum_{l\geq 0} h^l \frac{P_l(e^{nh})}{\Delta_K(e^{nh})^{2l+1}}$$

for some $P_l(t) \in \mathbb{Q}[t^{\pm 1}]$. This is called the loop expansion of the colored Jones polynomial. The 3-loop part of the colored Jones polynomial is given by $\dfrac{P_2(e^{nh})}{\Delta_K(e^{nh})^5}$.

For a knot K, $\Lambda_K(t^{\frac{1}{2}},t^{\frac{1}{2}},t^{-\frac{1}{2}},t^{-\frac{1}{2}})$ is a symmetric rational form in $t^{\pm 1}$ divisible by $t-1$ (since $\Lambda_K(1,1,1,1)=0$) and, hence, divisible by $(t-1)^2$. We define the *reduced 3-loop invariant* by

$$\hat{\Lambda}_K(t) = \frac{\Lambda_K(t^{\frac{1}{2}},t^{\frac{1}{2}},t^{-\frac{1}{2}},t^{-\frac{1}{2}})}{(t^{1/2}-t^{-1/2})^2} \in \frac{1}{\Delta_K(t)^4} \cdot \mathbb{Q}[t^{\pm 1}],$$

which is symmetricl in $t^{\pm 1}$. If $\Delta_K(t)=1$, then this is a polynomial, so we call it the *reduced 3-loop polynomial.* We denote the reduced 2-loop polynomial by $\hat{\Theta}_K(t) = \dfrac{\Theta_K(t,t^{-1},1)}{(t^{1/2}-t^{-1/2})^2}$ defined in [12].

Proposition 7. *The 3-loop part of the colored Jones polynomial* $\dfrac{P_2(e^{nh})}{\Delta_K(e^{nh})^5}$ *is presented by*

$$\frac{P_2(t)}{\Delta_K(t)^5} = (t^{1/2}-t^{-1/2})^2 \frac{\hat{\Lambda}_K(t)}{\Delta_K(t)} + \frac{(t^{1/2}-t^{-1/2})^4}{2\Delta_K(t)^5}\hat{\Theta}_K(t)^2 + \frac{\Delta_K'(t)t^2}{3(t-1)\Delta_K(t)^2} + \frac{\Delta_K''(t)t^2}{6\Delta_K(t)^2} - \frac{\Delta_K'(t)^2t^2}{3\Delta_K(t)^3}.$$

Remark 8. The 1-loop part of the colored Jones polynomial $\dfrac{P_0(e^{nh})}{\Delta_K(e^{nh})}$ and the 2-loop part of the colored Jones polynomial $\dfrac{P_1(e^{nh})}{\Delta_K(e^{nh})^3}$ are presented by

$$\frac{P_0(e^{nh})}{\Delta_K(e^{nh})} = \frac{1}{\Delta_K(t)}, \quad \frac{P_1(e^{nh})}{\Delta_K(e^{nh})^3} = -\frac{(t^{1/2}-t^{-1/2})^2}{\big(\Delta_K(t)^3\big)}\hat{\Theta}_K(t).$$

For details, see [12].

4 Results about the degree 2 part of the LMO invariant

In this section, we state the results about the degree 2 part of the LMO invariant obtained in [18].

Proposition 9. *For all p and p-regular knot K, we have*

$$
\begin{aligned}
&c_2(\Sigma_K^p) \\
&= \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_K(\omega_1^{\frac{3}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{\frac{3}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{\frac{3}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}) + l_p^K.
\end{aligned}
$$

Here, l_p^K is a scaler invariant of a knot K, which can be calculated by an equivariant linking matrix of a surgery link in $S^3\backslash K$. For details, see [6].

Remark 10. For all p and p-regular knot K, we have

$$
\begin{aligned}
\big(c_0(\Sigma_K^p) = \big)\ |H_1(\Sigma_K^p)| &= \left| \prod_{\omega^p=1} \Delta_K(\omega) \right|, \\
c_1(\Sigma_K^p) &= \frac{1}{12p} \sum_{\omega_1^p=\omega_2^p=1} \frac{\Theta_K(\omega_1, \omega_2, (\omega_1\omega_2)^{-1})}{\Delta_K(\omega_1)\Delta_K(\omega_2)\Delta_K((\omega_1\omega_2)^{-1})} + \frac{1}{16}\sigma_p(K),
\end{aligned}
$$

where $\sigma_p(K)$ is p-signature of K.

Let K be a regular knot. For $i = 0, 1, 2$, we can regard $\{c_i(\Sigma_K^p)\}_{p=1,2,\cdots}$ as families of invariants of K. Proposition 9 and Remark 10 show that i-loop part (i-loop polynomial) of the Kontsevich invariant of K is an universal invariant among $\{c_i(\Sigma_K^p)\}_{p=1,2,\cdots}$.

For $D(K, K')$, we can show that $l_p^{D(K,K')} = 0$. Therefore, we obtain the following theorem and corollary.

Theorem 11. *For all $p > 2$, we have*

$$
\begin{aligned}
&c_2(\Sigma_{D(K,K')}^p) \\
&= \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_{D(K,K')}(\omega_1^{\frac{3}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{\frac{3}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{\frac{3}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}) \\
&= \left(4a_2a_2' + \frac{1}{6}k^2a_2 + 2ka_3 + 10k^2a_4 + 6k^2a_2^2\right)p.
\end{aligned}
$$

Corollary 12. *For all $p > 2$, we have*

$$\begin{aligned} &c_2(\Sigma^p_{Wh^\pm(K)}) \\ &= \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_{Wh^\pm(K)}(\omega_1^{\frac{3}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{\frac{3}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{\frac{3}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}) \\ &= \left(\frac{1}{6}a_2 \mp 2a_3 + 10a_4 + 6a_2^2\right) p. \end{aligned}$$

5 Future directions

Lastly, we consider some problems about the calculation of Λ_K and c_2 for future directions.

For the 2-loop polynomial, some clasper surgery formulas are concretely presented in [13]. Clasper surgery formulas are useful to calculate them for some classes of knots. However, such clasper surgery formulas have not been presented concretely for the 3-loop invariant so far. Thus, it is one problem to present clasper surgery formulas concretely for the 3-loop invariant for clasper surgery along some (simple) claspers.

Further, it is another problem to determine the set of possible values of triple $\big(\Delta_K(t), \Theta_K(t_1,t_2,t_3), \Lambda_K(t_1,t_2,t_3,t_4)\big)$. This problem would be hard to solve in general, so it may be good to consider some further simplified cases, for example, to determine the set of possible values of the 3-loop polynomial of K with $\Delta_K(t) = 1$ and $\Theta_K(t_1,t_2,t_3) = 0$. Moreover it is good to find some useful formula or method to calculate 2-loop polynomial and 3-loop invariant concretely.

For a closed 3-manifold M, the invariant $c_1(M)$ is equivalent to the Casson-Walker-Lescop invariant of M, and it is well-studied. However, not much is known about $c_2(M)$ so far. Thus, it is good to calculate $c_2(M)$ for other examples, or to find some formula or method to calculate it.

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