Construction of a Latin Hexahedron

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Abstract

We consider Latin hexahedra satisfying a certain property similar to Latin squares. We review basic properties of Latin hexahedra and study constructions of such objects. In particular, constructions of separable Latin hexahedron are discussed.

1 Latin Hexahedra and related combinatorial designs

The concept of a *Latin hexahedron* is introduced by Yamamura [5] in which Latin hexahedra is shown to have close connection to combinatorial designs that are related to one-factorization of the complete tripartite graph $K(2k,2k,2k)$ and the complete quadripartite graph $K(n,n,n,n)$. We discuss properties of such combinatorial objects and various constructions. A reader is referred to [5] for terminologies and results on Latin hexahedra.

Let us recall the definitions and basic properties of a Latin hexahedron and related concepts. Let A be an $n \times m$ matrix filled with integers in $\{1,2,3,\ldots,k\}$, where $k = max(n,m)$. If no integer appears more than once in any row or column, then A is called a *Latin rectangle*. A *Latin square of order n* is an $n \times n$ Latin rectangle. A reader is referred to [1] for Latin squares.

A regular hexahedron of order n is a polyhedron consisting of six faces, each of which forms an $n \times n$ matrix filled with integers in $\{1,2,3,\ldots,4n\}$. A *net* of a hexahedron is an arrangement of a non-overlapping edge-joined polygon which can be folded along edges to become faces of the hexahedron. A *circuit* of a regular hexahedron of order n is a $1 \times 4n$ subarray in one of its nets. A circuit of a regular hexahedron of order 2 is shown in Figure 1. We note that there exist precisely $3n$ circuits on a regular hexahedron of order n. A regular hexahedron of order n is called *Latin* if every integer in $\{1,2,3,\ldots,4n\}$ appears exactly once in every circuit. A Latin regular hexahedron of order 2 and its net are shown in Figure 2.

A *Sudoku Latin square* is a 9×9 matrix filled with integers in $\{1,2,3,4,5,6,7,8,9\}$ such that each column, each row, and each of the nine 3×3 sub-matrices contain all of the integers from 1 to 9. It appears in the number-placement puzzle. We introduce a similar property into Latin regular hexahedra. Let L be a Latin regular hexahedron of order n. We say L is a *Latin sudoku regular hexahedron* if every integer

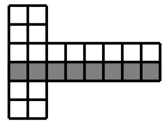

Figure 1: Circuit of a regular hexahedron of order 2

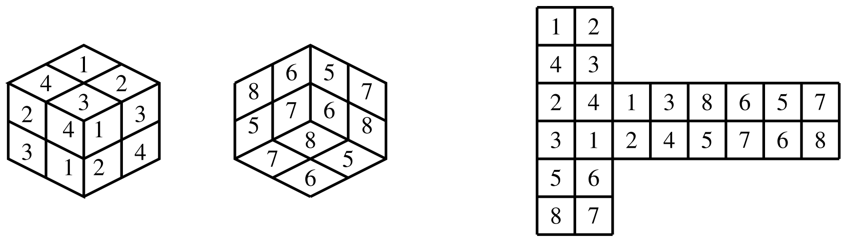

Figure 2: Latin regular hexahedron of order 2 and its net

in $\{1,2,3,\ldots,4n\}$ appears exactly once on each face. For example, a net in Figure 3 gives a Latin sudoku regular hexahedron of order 4. We also say that L is a *Latin quasi-sudoku regular hexahedron with multiplicity m* if every integer in $\{1,2,3,\ldots,4n\}$ appears exactly m times on each face. Existence of a Latin regular hexahedron and a Latin quasi-sudoku regular hexahedron is proved in [5].

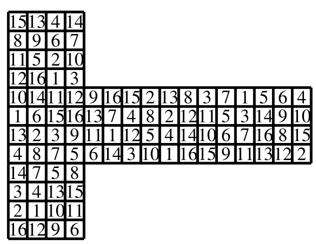

Figure 3: Net of a Latin sudoku regular hexahedron of order 4

Theorem 1.1 ([5]) *Let n be a positive integer.*
(1) A Latin regular hexahedron of order n exists if and only if n is even.
(2) There exists a Latin quasi-sudoku regular hexahedron of order $4n$ *with multiplicity n. In particular, there exists a Latin sudoku regular hexahedron of order 4.*

Two proofs are given in [5]. The first one depends on Hall's marriage theorem, on the other hand, the second one is given by using related combinatorial structures called a Latin three-axis design, which are described next. This proof provides a concrete construction of such combinatorial designs.

We consider a combinatorial structure related to Latin hexahedra. It can be applied to construct a Latin hexahedron as we see next. Suppose A, C, D are $n \times n$ squares. We consider a combination obtained by pasting these squares along edges in three dimensional space. Then the triple (A, C, D) is considered as a complex with three axes op, oq and or shown in Figure 4. The triple (A, C, D) is called a *Latin three-axis design of order n*. Each face is coordinated by two of its axes. This implies that any subarray of any net of the complex in Figure 4 is a Latin rectangle where $\rotatebox{90}{$D$}$ represents the array obtained from D by rotating $\frac{\pi}{2}$ counterclockwise. Figure 5 shows a Latin three-axis design of order 2.

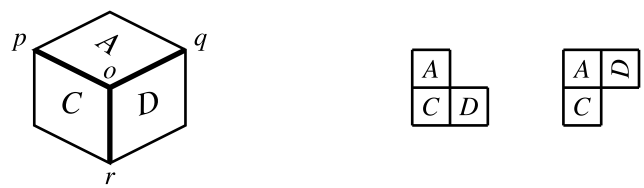

Figure 4: Latin three-axis design and its net

x_1	1	2		
x_2	4	3		
z_1	2	4	1	3
z_2	3	1	2	4
	y_1	y_2	x_2	x_1

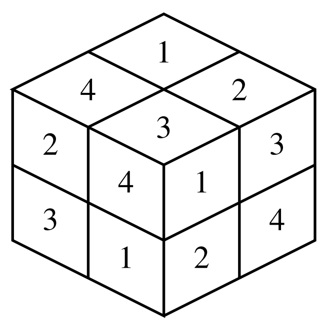

Figure 5: Latin three-axis design of order 2

Next we introduce another combinatorial design. Suppose A, B, C, D, E, F are $n \times n$ squares. The sextuple (A, B, C, D, E, F) is considered as a complex with four axes op, oq, or and os shown in Figure 6. Each face is coordinated by two of its axes. If every array in (1.1) is a Latin rectangle, we say that (A, B, C, D, E, F) is a *Latin four-axis design of order n*.

$$\boxed{A \mid B \mid C} \quad \boxed{\rotatebox{90}{C} \mid \rotatebox{90}{F} \mid E} \quad \boxed{\rotatebox{90}{B} \mid D \mid F} \quad \boxed{\rotatebox{90}{A} \mid \rotatebox{90}{E} \mid \rotatebox{90}{D}} \tag{1.1}$$

Existence of a Latin three-axis design and a Latin four-axis design are proved in [5]. We summarize results on existence of such combinatorial designs as Theorem 1.2.

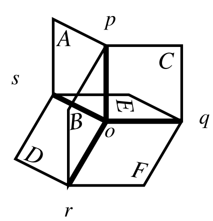

Figure 6: Latin four-axis design

Theorem 1.2 ([5]) *Let n be a positive integer.*
(1) There exists no Latin three-axis design of order $2n-1$.
(2) There exists a Latin three-axis design of order $2n$.
(3) There exists a Latin four-axis design of order n.

Latin three-axis design and a Latin four-axis design are closely related to pone-factorization of the complete tripartite graph $K(2n,2n,2n)$ and the complete quadripartite graph $K(n,n,n,n)$. Let us recall terminologies in graph theory. A subgraph of a graph G is called a *factor* if it includes all of the vertices of G. (see [2, 3, 4]). If every vertex of a factor has degree h, then it is called a *h-factor*. Therefore, a 1-factor of a graph $G=(V,E)$ is a subgraph such that the set of vertices is V and every vertex has exactly one edge incident on it. If E can be partitioned into disjoint subsets so that G decomposes into 1-factors, then G is called *one-factorizable*. It is known that every complete bipartite graph $K(n,n)$ is one-factorizable. Likewise, a Latin three-axis design and a Latin four-axis design have close connection with one-factorization of the complete tripartite graph and the complete quadripartite graph.

Theorem 1.3 ([5]) *Let n be a positive integer.*
(1) The complete tripartite graph $K(2n,2n,2n)$ *is one-factorizable if and only if there exists a Latin three-axis design of side* $2n$.
(2) The complete quadripartite graph $K(n,n,n,n)$ *is one-factorizable if and only if there exists a Latin four-axis design of side n.*

Aa a direct consequence of Theorem 1.2 and 1.3, we can conclude the existence of one-factorization of the complete tripartite graph and the complete quadripartite graph as follows.

Corollary 1.4 ([5]) *Let n be a positive integer.*
(1) The complete tripartite graph $K(2n,2n,2n)$ *is one-factorizable.*
(2) The complete quadripartite graph $K(n,n,n,n)$ *is one-factorizable.*

2 Construction of a Latin regular hexahedron using Latin three-axis designs

The existence of a Latin regular hexahedron of order $2n$ has been already proved in Theorem 1.2 in which Hall's marriage theorem is used. We have another proof that concretely construct a Latin regular hexahedron of order $2n$ using Latin three-axis designs. We review such a construction.

Suppose L_1 and L_2 are Latin three-axis designs such that $|L_1| = \{1,2,3,\ldots,4n\}$ and $|L_2| = \{4n+1, 4n+2, 4n+3, \ldots, 8n\}$, nets of which are given in (2.1), where A, B, C, D, E and F are $2n \times 2n$ arrays, respectively.

(2.1)

We can obtain a Latin regular hexahedron pasting L_1 and L_2 along edges. Two of its nets are shown in Figure 7. As a matter of fact, the Latin regular hexahedron given in Figure 2 is constructed in this fashion.

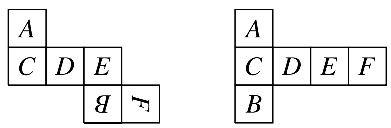

Figure 7: Nets of Latin regular hexahedron obtained by pasting L_1 and L_2

Let us call a Latin box *separable* if it can be constructed by pasting two three-axis designs D_1 and D_2 as above, and *inseparable* otherwise. We shall show every Latin hexahedron is not necessarily separable, that is, there exists an inseparable Latin hexahedron.

On the other hand, we have verified that all Latin boxes of side 2 are separable by a computer experiment, that is, every Latin regular hexahedron of order 2 can be constructed by pasting two disjoint three-axis designs and transposing integers on cells in contrapositions.

Let L be a Latin sudoku box assembled by a development given in Figure 8. Suppose L is obtained from a Latin box L' of side 4 that is obtained from three-axis designs D_1 and D_2 with $|D_1| \cap |D_2| = \emptyset$ by the method given above and transposing integers at contrapositions. We set $P_i = |D_i|$ for $i = 1, 2$, respectively. Then each P_i $(i = 1, 2)$ contains exactly 8 integers, respectively, and $P_1 \cap P_2 = \emptyset$. Let $L(i)$ be the set of integers placed at the contrapositions of the cells on which an integer i is placed on L. Similarly, let $L'(i)$ be the set of integers placed at the contrapositions of the cells on which an integer i is placed on L'. Since L is obtained from L' by transposing integers placed on transpositions, we have $L(i) = L'(i)$ for every i. We may assume that 1 is located on D_1, that is, 1 belongs to P_1. Checking Figure 8, we can verify the integers 2, 4, 8, 12 and 16 are placed at the contrapositions of the cells where 1 is placed on L. Therefore we have $\{2,4,8,12,16\} = L(1) = L'(1) \subset P_2$ because $1 \in P_1$ and the contrapositions of the cells where 1 is placed are located on D_2. Similarly we have $\{1,3,5,7,9,10,11,13,14,15\} = L(2) \cup L(4) \cup L(8) \cup L(12) \cup L(16) = L'(2) \cup L'(4) \cup L'(8) \cup L'(12) \cup L'(16) \subset P_1$. We continue the process and obtain Table 1. It follows that $P_1 = P_2 = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$, which contradicts to that P_1 and P_2 are disjoint. See Table 1. Consequently, L cannot be constructed from two three-axis designs by transposing integers at contrapositions. It follows that not every Latin regular hexahedron can be constructed in the method above.

3	1	4	5												
2	7	6	8												
15	16	11	12												
14	13	10	9												
5	6	13	14	1	2	9	10	7	8	15	16	11	12	3	4
7	8	15	16	3	4	11	12	1	2	9	10	13	14	5	6
9	10	1	2	5	6	13	14	11	12	3	4	15	16	7	8
11	12	3	4	7	8	15	16	13	14	5	6	9	10	1	2
12	11	16	15												
13	14	9	10												
1	4	5	3												
8	2	7	6												

Figure 8: Inseparable Latin box

P_1	1															
P_2	2	4	8	12	16											
P_1	1	3	5	7	9	10	11	13	14	15						
P_2	2	4	6	7	8	9	10	11	12	13	14	15	16			
P_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 1: Partition

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