Relationship between Rota's Logic of Information Based on Partitions and Logic of Information Defined in a Closure Space

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1. Introduction

The term "logic" in the title of this paper may be confusing as it is a product of the far-going generalization of the original understanding of logic as a discipline investigating human reasoning and its validity in the search for the truth. Mathematical logic and metamathematics developed a wide range of theoretical tools for this discipline including mathematical structures serving as models for the object of study. Among these structures, the most important was the concept of Boolean algebra.

The traditional logical inquiry has been expanded to include alternative forms of reasoning such as intuitionistic logic, modal logics, probability theory, etc. Finally, structures similar to logical structures were identified in contexts unrelated (at least directly) to reasoning, for instance in quantum theory. The expansion required the gradual generalization of the fundamental structure of a Boolean algebra to a range of different types of lattices understood either as order structures or as algebras. They may be diverse, but also they have very clearly defined and extensively investigated common features justifying the use of the term logic. In this paper, this type of fundamental structure is investigated in the context of information. Information logics considered as mathematical structures are authentic generalizations of the original linguistic logic of a Boolean algebra which in turn can be understood as a special instance of a model for information logic. Indeed, a theory of information that does not include its linguistic aspects does not have any explanatory value.

The subject of this paper is a comparative study of two concepts of information logic. In both the central role is played by lattices. One was proposed by Gian-Carlo Rota in his famous Fubini Lectures (Torino, 3-5 June 1998) *Twelve Problems in Probability No One Likes to Bring Up* [1]. The other was introduced by the author in a series of publications commenced in 2011 [2]. Both concepts were introduced as attempts to develop methods for structural analysis of information, however, in Rota's lectures, the primary subject was probability and the concept of information was secondary and was invoked as a candidate for setting foundations for probability in a reversal of their historical roles.

This paper demonstrates that some of Rota's proposals for future research were already implemented in the studies published more than a half-century earlier but later forgotten. Thus, there are ready answers to some of his questions for use in the study of information. Also, the paper shows that Rota's understanding of the logic of information as a structure of partitions parallel to the Boolean algebraic logic of events in probability is faulty. Moreover, it is demonstrated that Rota's logic of information is related to the logic of information studied by the author, but in an indirect way. Each partition leads to a logic of information and the structure that Rota proposes as a logic of information is a structure of a variety of such logics which with some abuse of language could be called a logic of logics. Surprisingly, this clarification of mutual relationships allows us to justify the claim of the impossibility to introduce hidden variables in physics that would transform the present formalism of quantum mechanics into an incomplete partial theory that could be completed by the addition of these variables to become a form of reestablished classical mechanics.

2. The Original Sin of Information Theory

The advances in the development of information technologies, generate widely spread fear of the loss of control over the products of such design. The only way to retain control is to gain sufficient knowledge and understanding of the phenomena involved in these technologies, in particular understanding of information. Information remains an elusive concept without a fully developed theory.

My call for more study of information may be countered by reference to the immense body of literature on the so-called information theory initiated by Claude Shannon. There is no point to repeat all well-known but mostly ignored arguments that Shannon's work on information transmission does not tell us much about information itself if anything at all. Whether someone is satisfied with the present status of the study of information or not, there are some clear gaps in understanding of information.

Shannon's declaration of his disinterest in the semantics of information makes it clear, that he is not interested in the concept of information at all but in the very different engineering problem of its transmission, actually the transmission of messages [3]. This aspect of Shannon's work met with strong criticism soon after its publication. Probably the earliest, openly negative view was presented by Yehoshua Bar-Hillel and Rudolf Carnap who completely rejected Shannon's "theory of signal transmission" as anything about information and proposed their logical theory of information [4,5]. However, neither their criticism nor their attempt to formulate a semantic theory of information, nor other attempts carried out in the following decades brought a definitive solution. At least there is no general theory of information matching the depth and extension of Shannon's theory of information transmission.

The programmatic disregard for semantics was not the only issue. Another was the lack of a clear, uncontested definition of information, the concept known in Shannon's work only by the formula for a quantity, called by him entropy because of its similarity to the physical magnitude of the same name in statistical mechanics, that he declared to be a measure of "information, choice, and uncertainty" [3] expressed in terms of a probability distribution $\{p_i: i\in I\}$ with the finite indexing set I as $H = -\sum p_i \log_2 (p_i)$. However, Shannon also wrote that he would call it "the entropy of the set of probabilities $p_1,..., p_n$". This is followed by a cryptic sentence: "If x is a chance variable we will write H(x) for its entropy; thus x is not an argument of a function but a label for a number, to differentiate it from H(y) say, the entropy of the chance variable y." Of course, he means a random variable x, but what he wants to explain by writing that it is "a label for a number"? There is nowhere in his work a definition of information or a statement identifying information as a particular mathematical object.

It looks like the measure introduced by Shannon is assigned not to probability distribution or random variable as the set of probabilities is not the same as a function that assigns probabilities to the elements of a sample space (or to events), nor the probability distribution of a random variable. In other words, we have here a magnitude programmatically invariant with respect to all permutations of the indexing set I as if information was completely devoid of any structure.

Even more mysterious is how this is related to information. Is information a probability distribution, or rather a random variable for which we want to find a quantitative description? Why do we insist (as Shannon did in his principles) that this magnitude is invariant for all permutations? In probability theory, many measures depend on probability distributions of random variables but not on the values of random variables. However, the reduction of probability distributions to the set of probabilities is unusual.

Once again, as history teaches us, the less clear prophecy, the more successful it is in finding followers. Warren Weaver and other apologists presented many frequently surprisingly absurd explanations of the curious features of Shannon's measure of information. Weaver wrote in the Introductory Note to the book edition of Shannon's work that the exclusion of meaning is perfectly justified in the context of information: "To be sure, this word information in communication theory relates not so much to what you *do* say, as to what you *could* say" [3]. Then he claimed that it is natural that a random sequence of letters carries more information than organized ones. The arguments seem if not convincing then at least acceptable for the study of the transmission of information, in particular, some of its aspects like speed, but not for the study of information. The statement that for the measure of information in the form of entropy the order of characters is irrelevant is equivalent to the statement that this measure does not tell us much about information.

If we do not restrict our interest to the transmission of information but want to understand what it is, what its components are, what its structure is, what its modes of existence are, how we can associate the meaning with information, the knowledge of entropy as a measure of information expressed in terms of probability generates more questions than gives answers.

The most disturbing for generations of researchers attempting to understand information was the fact that in Shannon's study information was conceptualized in terms of not more (as mentioned above actually less) than a probability distribution involved in a process of choice, or simply probability, as probability always refers to a choice or occurrence. This begs the question of whether we need any concept of information. After all, whenever we have a probability distribution or a random variable we can consider associated entropy (possibly infinite in the infinite case). If we believe that there is a need for this concept, then it should be conceptualized independently from probability and only then for specific cases and when it is appropriate probabilistic analysis could be carried out. This view was presented many times in the last decades, but without limited resonance. For instance, René Thom who considered his famous book *Structural Stability and Morphogenesis* a study of information conceptualized in topological and geometric terms openly called for the release of information from the stochastic prison [6].

For the subject of this paper, the most important early revolt against the subjugation of the concept of information to probability was in the paper of Andrey Kolmogorov [7], whose work half a century earlier in the 1930s unified the study of probability into a subdiscipline of mathematics. Kolmogorov called for reversing the relationship between information and probability by the creation of a combinatorial foundation for information and building on this foundation probability theory: "Information theory must precede probability theory, and not to be based on it. By the very essence of this discipline, the foundations of information theory have a finite combinatorial character" [7].

Kolmogorov's implementation of this idea was based on what now is called algorithmic complexity. Thus, it made computing a more fundamental concept than information. Probably for this reason his support for information as a foundation for probability, despite his great authority as a prominent mathematician and the father of modern probability theory, did not radically change the popular view of the relationship between probability and information.

Probably the most influential in this long way towards establishing information as a fundamental concept preceding not only probability but also computing (or at least without any reference to computing) were Gian-Carlo Rota's Fubini Lectures (Torino, 3-5 June 1998) *Twelve Problems in Probability No One Likes to Bring Up* [1] mentioned in the introduction to this paper. It took a half-century ride on the "bandwagon" (Shannon's own critical expression for the enthusiastic, but frequently false interpretations of his work) to the wider recognition of the need for re-examination of the mathematical foundations for the study of information. Unfortunately, Rota's description of the relationship between probability and information involves some questionable statements (discussed below).

3. The Logic of Information: Rota's Way

Rota's Fubini Lectures *Twelve Problems in Probability No One Likes to Bring Up* have the form of a presentation of twelve issues that he identified in probability theory. Since this paper has very different objectives and these objectives are present in presentations of several problems but not necessarily as main topics, Rota's claims, and opinions will be paraphrased here to emphasize what for our objectives is of primary importance.

The relationship between the fundamental concepts of probability in terms of sample spaces and random variables is of special importance for us in our search for what we can identify as information in its traditional studies in terms of Shannon's entropy. In the past, we could only speculate that information is associated either with the choice of the σ-algebra $\mathcal{E}$ determined by the choice of a random variable or the probability measure. Rota provides a powerful tool for the study in the form of the lattice theory.

Rota's Lectures were addressed to mathematicians with expertise in the subject of his talks, so he did not present all details of his argumentation and avoided explanations of more basic concepts or lengthy poofs. Even more frustrating may be the lack of references to all except a few most famous sources. More detailed explanations of some of his shortcuts can be found in the book *Combinatorics:The Rota Way* written with his coworkers. In this book, in the context of combinatorics where the restriction to the finite case is well justified, the succinct comparison of the logic of information and logic of probability is made: "[The] lattice of partitions plays for information the role that Boolean algebra of subsets plays for size of probability" [8].

In the Lectures, Rota maintains a more general level of consideration without the restriction to finite sets: "The lattice $\sum$ of σ-subalgebras plays for information the role that the lattice of events plays for probability. To get an inkling of this role, let us first consider a σ-subalgebra π_X that is σ-generated by a countable set of atoms of positive probability, and that is therefore associated with a partition of the sample space. Recall that, if π_X is the information provided by a random variable X, then, as we have seen, we can view the random variable X as a question whose answer will tell us which of the blocks of the partition π_X an unknown sample point ω_0 lies in" [1].

Earlier, Rota explains "Every random variable X determines a unique σ-subalgebra π_X of the σ-algebra of events Π, namely, the minimal σ-subalgebra π_X of Π relative to which X is measurable. Such a σ-subalgebra π_X expresses the ability of the random variable X to distinguish points of the sample space. This ability of distinguishing among points is relevant in the interpretation of random variables as the result of a search, in information theory" [1]. From this, we have a clear identification of instances of information with partitions. These partitions into subsets of the sample space (Rota uses the expression subsets of points in sample space) are uniquely generated by random variables, while at the same time, the unions of the blocks (elements of the same value of the variable) of each countable partition form a σ-subalgebra. Thus we have three alternative key concepts of a random variable, its associated partition, and σ-subalgebra, but the informational aspect is in the way partitions are like nets for catching sample points in their search.

The key point for our comparison of the methods in this paper is that the logic of information is identified with a variety of choices of partitions so that the logic of information is the lattice of partitions. It may be confusing that Rota more frequently refers to the lattice of σ-subalgebras, but they are isomorphic to the lattice of partitions in the case of countable partitions.

Random variables X come with their probability distributions defined on corresponding σ-subalgebra π_X of Π, which in turn allows the definition of entropy. This way we can reintroduce the key concept of Shannonian study of information. Rota defines entropy as a function of appropriate σ-subalgebra, but it is clearly equivalent to defining it on the lattice of partitions.

Surprisingly, Rota claims that "The lattice $\sum$ of all Boolean σ-subalgebras has never been properly structurally characterized" [1]. while there is a very extensive 1942 study by Oystein

Ore on this subject [9]. Moreover, Rota in the same paragraph refers to the result of Whitman (although without mentioning his name) whose paper on this subject heavily depends on Ore's paper [10]. There is also a 1951 paper by Usa Sasaki and Shigeru Fujiwra *The Characterization of Partition Lattices* [11]. Both Rota and Ore considered the structure (complete lattice) of partitions on an arbitrary, not necessarily finite set S with the order defined by refinement. Ore identified the relationship (lattice isomorphisms) between this structure and the structures (complete lattices) of (1) fields of subsets of S, (2) σ-fields of subsets of S, (3) complete fields of subsets of S. Rota restricted his interests to σ-fields (due to his interest in probability) while Ore proved isomorphism with lattices of complete fields. Ore provided an extensive study of these structures appended later by Sasaki and Fujiwara.

4. The Logic of Information Defined in a Closure Space

An alternative approach to the logic of information developed in my publications was motivated by the need for a structural analysis of information and an explanation of its semantic aspects. Because I intended to have a theory that describes all types of information, including quantum information, I was guided in the choice of formalism by the studies of logical aspects of quantum theory initiated by Garret Birkhoff and John von Neumann in their 1936 paper *The Logic of Quantum Mechanics* [12] which led to the development of an alternative more general mathematical methodology for quantum theory [13,14]. Typical interpretations of this approach focus on the analogy to linguistic logic (through references to quantum propositions) or to operational concepts (yes-no experiments). My interpretation is that the structures of quantum logics are instances of the informational structures of reality not of the language of its description or experimental procedures. The term logic in its traditional linguistic understanding should be understood as a metalogic for the logics of information that can be identified in all informational structures of reality. In general, structural characteristics of information have to be formulated without restrictions imposed by linguistic, probabilistic, or operational interpretations.

To some extent, my view of information logic is parallel to John Stuart Mill's concept of connotation from his 1843 *System of Logic* [15]. Mill's connotation of a term is the structure of attributes that can be predicated on what the term stands for and by which it is defined in contrast to denotation which consists of objects to which the term applies. However, in my view, the objects of denotation do not have different ontological status. They also can only be identified by usually larger, more complete structures of attributes. For Mill, the meaning is understood as a combination of both denotation and connotation, but in my view, the meaning is simply the relationship between informational structures.

With the status of information logic as a structure present in reality comes the naturalization of information. Information is not necessarily an artifact. Even when information is created by humans, its existence requires a medium that makes it an independent entity. In this, it is consistent with Landauer's *Information is Physical*, although in my opinion the use of the word "physical" is deceptive. I would reformulate it as *Information is Real*. For this reason, the methodology of the study of information should include the most powerful tool of natural sciences, the study of symmetry. The theoretical framework that I propagate makes the logic of information consistent with the study of symmetry [16].

My approach is focusing on filters defined in closure spaces, thus to make this paper self-sufficient an explanation of the most important concepts will follow. More details about mathematical concepts involved in the formulation of my theory of information can be found in expositions of theories of latices or partially ordered sets [17].

Def. A *closure space* $\langle S,f\rangle$ is a set S with a function f: $2^S \to 2^S$ on the power set of S called a *closure operator* that satisfies three conditions: (i) $\forall A \subseteq S$: $A \subseteq f(A)$, (ii) $\forall A,B \subseteq S$: If $A \subseteq B$, then $f(A) \subseteq f(B)$, (iii) $\forall A \subseteq S$: $f(f(A)) = f(A)$.

Every closure space $\langle S,f\rangle$ can be defined in an equivalent (cryptomorphic) way by a Moore family of subsets of S, i.e. family closed with respect to arbitrary intersections and including the set S. Every Moore family $\mathcal{M}$ defines a transitive operator: $f(A) = \cap \{M \subseteq \mathcal{M}: A \subseteq M\}$ and in turn, the family f-Cl = $\{M \subseteq S: f(M) = M\}$ is a Moore family. The family f-Cl is a complete lattice $\mathcal{L}_f$ with respect to the set inclusion $\subseteq$. This lattice is called **the *logic* of $\langle S,f\rangle$**.

The concept of a closure space $\langle S,f\rangle$ and its logic $\mathcal{L}_f$ can be defined on an arbitrary complete lattice $\mathcal{L}$ instead of the power set 2^S by replacing every occurrence of the set inclusion $\subseteq$ with the symbol of the partial order $\leq$ of $\mathcal{L}$.

The family $\Im$ of subsets of S is called a *filter*, if it satisfies two conditions:

- $\forall A,B \subseteq S$: If $A \in \Im$ and $A \subseteq B$, then $B \in \Im$.
- $\forall A,B \subseteq S$: If $A \in \Im$ and $B \in \Im$, then $A \cap B \in \Im$.

A proper filter does not have the empty set as its element. The maximal (proper) filter on S is called an *ultrafilter*. As before, the concept of a filter can be extended to any complete lattice instead of the power set 2^S of set S.

With these mathematical preliminaries, we can introduce the mathematical theory of information. We will consider a closure space $\langle S,f\rangle$ with its corresponding Moore family $\mathcal{M}$ of closed subsets as an **information system**. The specific choice of closure space depends on the choice of the type of information system. For instance, we can consider geometric, topological, logical information, etc.

The family of closed subsets $\mathcal{M}$ = f-Cl is equipped with the structure of a complete lattice $\mathcal{L}_f$ which we can consider to be the **logic of information**. It plays a role in the generalization of traditional logic for information systems, although it does not have to be a Boolean algebra

***Encoding of information* (or instance of information)** is a distinction of a subfamily $\Im$ of $\mathcal{M}$, which is a filter in the lattice $\mathcal{L}_f$.

Our theory of information includes the case of quantum information for which the closure space is defined by the orthomodular lattice $\mathcal{L}(H)$ of closed subspaces in a Hilbert space. Every quantum logic defined by an appropriate set of axioms can always be represented as $\mathcal{L}(H)$. This is in some sense example of information logic at the opposite extreme from the atomic Boolean algebraic logic. While the latter is characterized as a distributive lattice, the former is not distributive, and in many cases not even modular. While atomic Boolean algebras are always reducible (factorizable) to the direct products of simple two-element subalgebras, purely quantum logic cannot be factorized at all.

In this approach, we can resolve one of the outstanding issues in the study of information in the lack of semantics. It is easy to see that the connotation of a concept is a (principal) filter generated by our concept within the logic of our information system. However, in the general case, the logic $\mathscr{L}(S,f)$ is not necessarily a Boolean lattice. So, we have to be careful not to import the properties of filters used in the Stone Theorem. In particular, ultrafilters are not necessarily prime filters and prime filters may not exist at all for instance in $\mathscr{L}$(H).

There are many measure spaces, including quantum logics and Rota's pointless probability spaces, where complete measures (i.e. measures such that the subsets of sets of measure 0 are measurable) define filters $\Im$ by the condition $a \in \Im$ *iff* m(a) = 1. However, we have to be aware of the fact that the concept of a measure requires orthocomplementation defined on the logic of information. This is a quite strong restriction, but at least we can consider within this restriction both Rota's approach and quantum logic of information.

5. Rota's Way vs. Closure Space Information Logic

Now we can compare the two views of the logic of information, Rota's lattice of partitions and my lattice $\mathscr{L}_f$ of closed subsets in a closure space $\langle S,f \rangle$. First, basic facts about partitions.

- Every partition of a set S can be associated with a function α from S to some index set I, (α: S $\rightarrow$ I). We will assume that it is surjective without limiting generality.
- In the case of random variables, it is usually assumed that I is a subset of the set of real numbers $\mathbb{R}$ (α: S $\rightarrow$ I $\subseteq \mathbb{R}$), but there is no compelling reason to do so in the general case.
- The partitions of S can be ordered by the following inverse rectification order:

If α: S $\rightarrow$ I and β: S $\rightarrow$ J, then $\alpha \leq \beta$ *iff* $R_\alpha \leq R_\beta$ *iff* $\forall i \in I\ \exists j \in J$: $A_i \subseteq B_j$, where R_α and R_β are the equivalence relations associated with partitions α, β respectively, and A_i, B_j, are corresponding classes of abstractions or blocks of partitions.

- If the equivalence relation $R_=$ above is the equality relation =, then all blocks have only one element and there is a bijective correspondence between subsets of S and sets of blocks of $R_=$.
- The order of partitions α: S $\rightarrow$ I and β: S $\rightarrow$ J, defined by $\alpha \leq \beta$ *iff* $\forall i \in I\ \exists j \in J$: $A_i \subseteq B_j$ is a complete lattice which in non-trivial cases is not distributive, but semimodular and atomic.

This lattice of partitions Rota identified as the logic of information. It is isomorphic with the alternative logic defined by σ-fields of subsets of S. In this case, we consider the Moore family of all σ-fields of subsets of S. Then we distinguish the σ-fields of subsets of S generated by partitions (which are families of disjoint subsets of S covering the entire S). Öre showed the correspondence between these two types of logic.

On the other hand, each partition_α: S $\rightarrow$ I defines a closure operator *f* on S defined by:

$\forall A \subseteq S$: $f_\alpha(A) = \cup\{A_i: i \in I \ \& \ A_i \cap A \neq \varnothing\}$. Then $A = f_\alpha(A)$ *iff* $A = \cup\{A_i: i \in J \subseteq I\}$. From this follows directly that $\mathscr{L}(S,f_\alpha)$ is isomorphic to the Boolean lattice of the power set of I, and therefore it is complete, atomic, and distributive. In the special case of the partition by the equality relation $R_=$ we have what Rota calls the logic of probability, i.e. the logic of subsets. Thus, the relationship between the logic of partitions and logic of subsets is very different.

The logics generated by partitions form a special sub-class of logics generated by closure operators on S. They all are Boolean lattices and therefore they are distributive. However, in general logics of information generated by closure operators, for instance quantum logics may be not distributive or even modular.

6. Exclusion of Hidden Variables

The issue of the existence or rather non-existence of hidden variables in quantum mechanics (QM) was discussed for decades and the experimental work on testing theoretical scenarios of the consequences of non-existence always started from the description of a hypothetical physical system described in the formalism of QM and proceeding to the analysis of what happens if there are some hidden (i.e. unknown to us) variables describing the system. Is it possible to recover the complete description of reality in terms of some form of possibly modified classical mechanics CM). All attempts were designed to consider what happens when we add unknown, hidden variables. Here we go the other way. We ask is it possible to hide some variables of a general classical theory to obtain quantum logic (actually more general non-distributive information logic). The answer is in the negative.
When we want to test a lattice for being distributive (completely reducible) we can use as a criterion the presence of two sublattices called M_5 and N_5 (Fig.1).

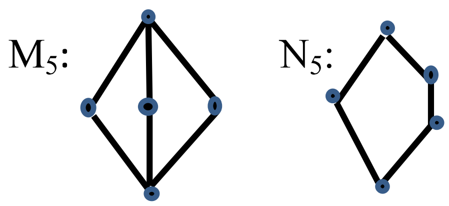

Figure 1. Hasse diagrams of lattices M_5 (left) and N_5 (right).

Lattices that have either M_5 or N_5 as sublattices are not distributive (i.e. they are reducible to products of component lattices. Lattices which have N_5 as a sublattice are not modular (weaker than distributivity but also a fundamental property of lattices). This is a very powerful tool for detecting non-distributive lattices which at the same time tells us that non-distributive lattices cannot be sublattices of distributive ones.

This unusual characterization of distributive lattices by exclusion of sublattices instead of the usual conditions of inclusion of substructures shows that hidden variables cannot complete quantum mechanical to classical theory. If it was possble, quantum logic would have to be a sublattice of classical logic of completed theory in terms of partitions of the phase space by variables. Since a sublattice of a sublattice of any lattice is itself sublattice, either M_5 or N_5 would have been a sublattice of the logic of the completed theory. Thus, the completed theory cannot be distributive, i.e. it cannot be generated by classical variables and their partitions.

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