On an algebraic system similar to logic using trice

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Our aim is to challenge the principle of bivalence and dualism. In mathematical logic, there is true and false. However, looking at the world in general, it is not always possible to consider absolute true or false. What is believed in one country may not be believed in another, or may have completely opposite values. We want to constitute a means of thinking about this kind of world. Łukasiewicz logic and Kleene logic are many-valued logics, but there are ‘true’ and ‘false’ values in them, and they introduce intermediate values orundetermined . This is not what we are aiming for. What we have created does not contain 'false'. We do not believe that there is only one implication. We dare to think of a way to analyze the world that embraces the antinomy. Hence, we assume that what we are aiming for is what we are told we cannot call logic. We do research on special concept 'trice'. It play an important role with regard to this attempt. Therefore, the title is "On an algebraic system similar to logic using trice". We introduce the "stealth absorption law," an even weaker form of the roundabout absorption law. It's a modest concept that is likely to be overlooked by ordinary logic. However, it can be interpreted as a property that is meant to invalidate the premise.

1 Preliminaries

This section reviews the definitions and examples of the algebras used.
A semilattice $(S,\ *)$ is a set S with a single binary, idempotent, commutative and associative operation $*$.

$$a * a = a \qquad (idempotent) \tag{1}$$
$$a * b = b * a \qquad (commutative) \tag{2}$$
$$a * (b * c) = (a * b) * c \qquad (associative) \tag{3}$$

Under the relation defined by $a \leq_* b \iff a * b = b$, any semilattice $(S,\ *)$ is a partially ordered set $(S,\ \leq_*)$. For A a nonempty set and n a positive integer, let $(A,\ *_1, *_2, ..., *_n)$ be an algebra with n binary operations, and $(A,\ *_i)$ be a semilattice for every $i \in \{1, 2, ..., n\}$. Then, $(A,\ *_1, *_2, ..., *_n)$ is called a **n-semilattice**. We denote each order on A by $a \leq_i b \iff a *_i b = b$, respectively.

Definition 1 Let $(A,\ *_1, *_2, ..., *_n)$ be an algebra with n binary operations, and let S_n be the symmetric group on $\{1, 2, ...n\}$. An algebra $(A, *_1, *_2, ..., *_n)$ has the **n-roundabout-absorption law** if it satisfies the following $n!$ identities:

$$((((a *_{\sigma(1)} b) *_{\sigma(2)} b) *_{\sigma(3)} b) ... *_{\sigma(n)} b) = b. \tag{4}$$

for all $a, b \in A$ and for all $\sigma \in S_n$.

The 2-semilattice $(A, *_1, *_2)$ which satisfies the 2-roundabout-absorption law is a lattice. The operations $*_1$ and $*_2$ are denoted by $\vee$ and $\wedge$. In this case, $(A, \leq_\vee)$ and $(A, \leq_\wedge)$ are exactly opposite ordered sets. Hence, only one of orders is to be considered.
Let T be a set. If there is a semilattice, the order is derived, and if the order can be shown, the semilattice can be identified. We introduce three orders into T, However, condition, two elements of T have a least upper bound for each order, is required. Then, we can construct the set into a triple-semilattice. A concrete way to show the trice is to draw a Hasse diagram of the three orders. Note that only those with isomorphisms of three orders were considered here.

The operations $*_1$, $*_2$ and $*_3$ will be often denoted by $\vee_1$, $\vee_2$ and $\vee_3$. We have used the operational symbols with arrows in previous papers, but refrain from doing so this time to avoid confusion (see [3] [4] [5] [6] [7]). This is because we use arrows for symbols of implication.

Definition 2 A triple semilattice $(T, *_1, *_2, *_3)$ which satisfies the 3-roundabout-absorption law is said to be a **trice**.

Example 1 Let T be a set which consists of three points. We introduce three orders of Fig. 1. Then, T is a trice. This trice is called Lambda because of its shape.

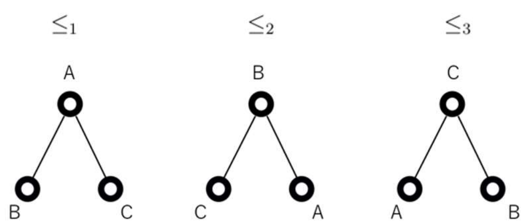

Figure 1: Lambda trice

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Next, we deal with the concept of complement. Let $(S, *)$ be a semilattice with maximum, denote it by 1, that is, $x \leq_* 1$ $(x * 1 = 1)$ for all $x \in S$. An element a' is a **pseudocompliment** of $a \in S$ iff $a * a' = 1$ and $a * x = 1$ implies that $a' \leq_* x$. Usually, this definition is written in terms of a dual concept with the order reversed (see [1] [2]). Please attention should be paid to the concept pseudocompliment of in

Heyting algebra. We deal with the case where there is exactly one pseudo-complement of x. We would like to assume from the outset that $'$ is a one-to-one onto function from S to S such that $x'' = x$ and x' is a pseudocompliment of x.
Let $(T, \vee_A, \vee_B, \vee_C)$ be a triple semilattice. The order derived from each of $\vee_A$, $\vee_B$ and $\vee_C$ is written as $\leq_A$, $\leq_B$, $\leq_C$. Suppose $(T, \leq_A)$, $(T, \leq_2)$ and $(T, \leq_3)$ have maximum A, B and C respectively. Then, $(T, \vee_A, \vee_B, \vee_C)$ is called bounded.

Definition 3 If there is a one-to-one onto function $'$ from T to T so that for all $x \in T$, $x \vee_A x' = A$, $x \vee_B x' = B$, $x \vee_C x' = C$ and $x'' = x$ then, we will call T is **common pseudocomplimented triple-semilattice (CPTS)** and x' is **c-p-compliment** of x.

Some examples of CPTS consisting of six points are shown below.

Example 2 We introduce three orders of Fig. 2. We will call this Pantograph.

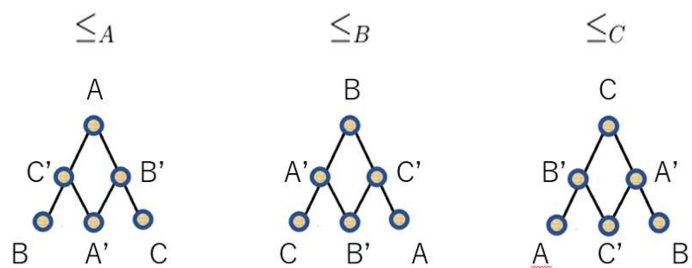

Figure 2: Pantograph trice

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Example 3 We introduce three orders of Fig. 3. We will call this Hexagon.

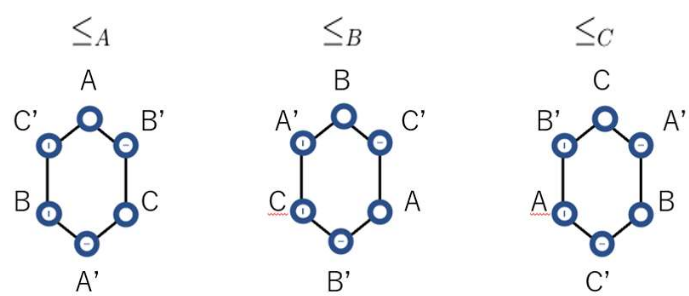

Figure 3: Hexagon trice

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Pantograph of Example 2 and Hexagon of Example 3 are trices. Next examples are not trices.

Example 4 We introduce three orders of Fig. 4. This is not a trice but a CPTS. We will call this Umbrella.

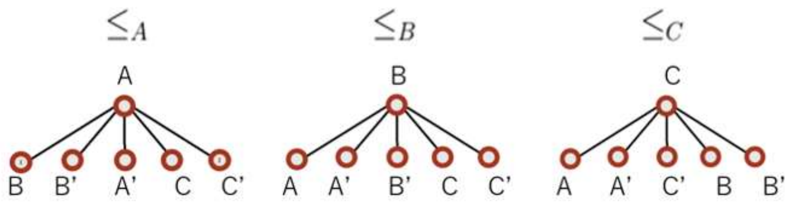

Figure 4: Umbrella triple-semilattice

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Example 5 We introduce three orders of Fig. 5. This is not a trice but a CPTS. We will call this Octahedron.

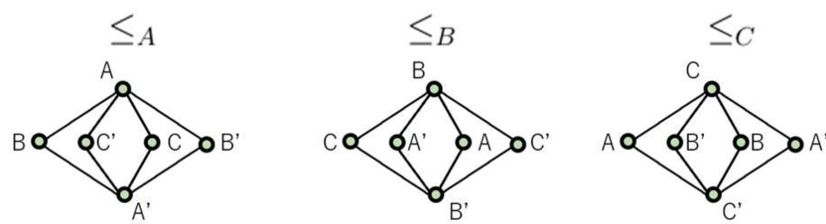

Figure 5: Octahedron triple-semilattice

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Also shown is an example of a six-piece set that does not satisfy the property of CPTS requirements.

Example 6 We introduce three orders of Fig. 6. In this figure, x' is not c-p-compliment of x (For example, $B \vee_A B' = B' \neq A$) . However, we assume $x'' = x$. This is not a CPTS but a trice. We will call this Tie.

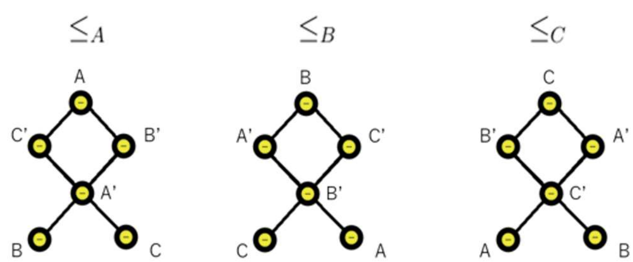

Figure 6: Tie trice

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2 Configuration of something like logic

It is natural to think of logical operations such as $\vee$ and $\wedge$. in mathematical logic as algebraically organised. Boolean algebra was born from this. It has then been studied in connection with more generalised lattice and universal algebras. In mathematical logic, there are T (true) and F (false), and we have to decide between them. This is the principle of bivalence. If intermediate states are not taken into account, this can be seen as dualism. Multi-valued logic allows intermediate states, but true and false exist.

We started with the three A B C options instead of true or false. We could consider more options, but it is reasonable to start with three. We have attempted to extend the binary operation on the two-point set to the three-point set, To do so, we algebraically apply A and B to the two terms T and F. However, we have not decided that A is true and B is false. We then attempted to make the treatment of B and C equivalent. Equivalent (or should we say symmetry) here means that B and C become isomorphic if they are interchanged. This setting is quite strict and limits the extension of operations. Note that A is given special treatment and is not required to be equivalent to B or C.

We have extended the operations of disjunction $\vee$ and implication $\rightarrow$. The operation of disjunction must be semi-lattice. On implication operation, we assume the element A must be right zero and left unitary. And we want to consider $(X \rightarrow Y) \rightarrow Y$ as $X \vee Y$. Since A is special, we denote $\vee_A$ and $\rightarrow_A$. The resulting results are shown in the following table.

$\vee_A$	A	B	C
A	A	A	A
B	A	B	A
C	A	A	C

$\rightarrow_A$	A	B	C
A	A	B	C
B	A	A	C
C	A	B	A

Figure 7: disjunction $\vee_A$ and implication $\rightarrow_A$

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Making B and C equivalent is a severe constraint. Hence, we cannot constitute an extension of conjunction $\wedge$. Other operations are also difficult to extend meaningfully. By using this operation to create $B \vee_A C = C \vee_A B = A$, we expect to be able to express a phenomenon in which something is created out of nothing. Neither B nor C is an error or contradiction, but it would be mysterious to those who believe in A. (From nothing comes something? we would be happy to be the beginning consideration of the universe.)

Similarly, we can create operations that treat A and C equivalently and treat B specially. We denote $\vee_B$ and $\rightarrow_B$. See the following table (Fig. 8). In the same way, $\vee_C$ and $\rightarrow_C$ are also created (Fig. 9).

$\vee_B$	A	B	C
A	A	B	B
B	B	B	B
C	B	B	C

$\to_B$	A	B	C
A	B	B	C
B	A	B	C
C	A	B	B

Figure 8: disjunction $\vee_B$ and implication $\to_B$

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$\vee_C$	A	B	C
A	A	C	C
B	C	B	C
C	C	C	C

$\to_C$	A	B	C
A	C	B	C
B	A	C	C
C	A	B	C

Figure 9: disjunction $\vee_C$ and implication $\to_C$

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Let Ω be the set $\{A, B, C\}$. Then, $(\Omega, \vee_A, \vee_B, \vee_C)$ is Lambda trice of Example 1. In a world where only A and B appear, it goes without saying that $\vee_A$ is $\vee$ for those who hold A to be true. And $\vee_B$ plays the role of $\wedge$. This is the same as in ordinary binary logic. Symmetrically, for those who hold B to be true, $\vee_B$ is the $\vee$ and $\vee_A$ is the $\wedge$. Logic and algebra are closely related, and even weak logic is often related to lattice in structure. What we have created is not a lattice. Nor is it ordinary logic. We can create an interesting situation by including another option C.

Let us now consider the implications. What we have created is a world with three kinds of implications. Using $\to_A$, $\to_B$ and $\to_C$ above, $(\Omega, \to_A, \to_B, \to_C)$ is an algebra. This will be called **L-I** (Lambda-Implications). You may find it odd that there are multiple implications. But we can envisage a case where the three countries have different values and different implications. With this approach, two kinds of implications can be considered in binary logic. Let Ω be the set $\{A, B\}$, and restrict operations $\vee_A$ and $\vee_B$ to the range of Ω, $(\Omega, \vee_A, \vee_B)$ can be regarded as lattice. This is isomorphic to $(\Omega, \vee, \wedge)$, with Ω as $\{T, F\}$. In the case of dualism, it might be interesting to say that there are 'honest implication' $\to_T$ $(\to_A)$ and 'underworld implication' $\to_F$ $(\to_B)$.

$\to_T$	T	F
T	T	F
F	T	T

$\to_F$	T	F
T	F	F
F	T	F

Figure 10: dualism-implications

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The number of binary logical operations was $2^{2\times 2}$. With $\to_T$ and $\to_F$, all binary logical operations can be constructed. It is clear from $Y \to_T (X \to_F Y) = X \uparrow Y$.

Let us return to a non-dualistic world, that is, the case of $\Omega = \{A, B, C\}$. The number of binary operations on Ω is $3^{3\times3}$. Let $(\Omega, \rightarrow_A, \rightarrow_B, \rightarrow_C)$ be **L-I**. With $\rightarrow_A$, $\rightarrow_B$ and $\rightarrow_C$, all binary operations on Ω can be constructed. It is functionally complete with respect to $\rightarrow_A$, $\rightarrow_B$ and $\rightarrow_C$. Details of proof are omitted. The basic operations (replacement, rotation) are constructed, and specific configurations can be made by combining them. (The rotaion is Post's negation functor).

3 Stealth-absorption law

Consider a concept that weakens definition 1.

Definition 4 Let $(A, *_1, *_2, ..., *_n)$ be an algebra with n binary operations, and let S_n be the symmetric group on $\{1, 2, ...n\}$. An algebra $(A, *_1, *_2, ..., *_n)$ has the **n-stealth-absorption law** if there exists a function f from $S_n \times A$ to A such that

$$((((a *_{\sigma(1)} b) *_{\sigma(2)} b) *_{\sigma(3)} b)... *_{\sigma(n)} b) = f(\sigma, b) \tag{5}$$

for all $a, b \in A$ and for all $\sigma \in S_n$.

The final result is not determined to be b and depends on the order of operations. But it means that the effect of the first variable a has disappeared.

In case that $(L, \vee, \wedge)$ is a lattice, $(X \vee Y) \wedge Y = Y$ holds. That is, if $(\Omega, \vee_A, \vee_B)$ is a lattice, then $(X \vee_A Y) \vee_B Y = Y$. When $X \vee_A Y = (X \rightarrow_A Y) \rightarrow_A Y$ and $X \vee_B Y = (X \rightarrow_B Y) \rightarrow_B Y$, then $((((X \rightarrow_A Y) \rightarrow_A Y) \rightarrow_B Y) \rightarrow_B Y) = Y$. From this expression, $((X \rightarrow_A Y) \rightarrow_B Y) = Y$ might be expected. Unfortunately, it does not hold. However, the following proposition holds.

Proposition 1 In the case of binary logic, let Ω be $\{T, F\}$. If $\rightarrow_T$ and $\rightarrow_F$ are dualism-implications in Figure 10, then $(\Omega, \rightarrow_T, \rightarrow_F)$ has 2-stealth-absorption law.

Consider the case on a 3-point set.
Let Ω be the three points $\{T, F, I\}$. And determine $\rightarrow_T$ and $\rightarrow_F$ as in the following figure. This $\rightarrow_T$ is the implication of Łukasiewicz logic. And $\rightarrow_F$ is the flip side of the T and F roles in $\rightarrow_T$. Note that the disjunction $X \vee Y$ of Łukasiewicz logic corresponds to $(X \rightarrow_T Y) \rightarrow_T Y$. Then $(\Omega, \rightarrow_T, \rightarrow_F)$ has 2-stealth-absorption low.

$\rightarrow_T$	T	I	F
T	T	I	F
I	T	T	I
F	T	T	T

$\rightarrow_F$	T	I	F
T	F	F	F
I	I	F	F
F	T	I	F

Figure 11: Łukasiewicz-implications

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Let Ω be the three points set$\{T, F, U\}$. And determine $\rightarrow_T$ and $\rightarrow_F$ as in the following figure. This $\rightarrow_T$ is the implication of Kleene logic. And $\rightarrow_F$ is the flip side of the T and F roles in $\rightarrow_T$. Then $(\Omega, \rightarrow_T, \rightarrow_F)$ doesn't have 2-stealth-absorption low.

$\rightarrow_T$	T	U	F
T	T	U	F
U	T	U	U
F	T	T	T

$\rightarrow_F$	T	U	F
T	F	F	F
U	U	U	F
F	T	U	F

Figure 12: Kleene-implications

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Proposition 2 Let Ω be the three point set $\{A, B, C\}$ and let $(\Omega, \rightarrow_A, \rightarrow_B, \rightarrow_C)$ be **L-I**. Then, $(\Omega, \rightarrow_A, \rightarrow_B, \rightarrow_C)$ has 3-stealth-absorption law.

Proposition 1 and Proposition 2 are the result obtained by examining all the cases.

4 Extension of truth value

Extending truth values to improve on already existing logic has often been used. Łukasiewicz logic is an extension from two-valued logic to three-valued logic. It was extended to n-valued logic. It was further extended to infinitely many valued. Extensions to infinite continua, such as fuzzy logic, had also been considered. Four-valued and six-valued logic exists in an attempt to generalise Kleene logic, which is a three-valued logic. What is absent from our Lambda up to this point is the concept of negation. To add the concept of negation, we also try to extend truth-values. If we add its negation to three values that are not in a negation relationship with each other, we have six values. It is possible to create something desirable that has six truths. Examples 2, 3, 4, and 5 can appear here. Example 6 is listed for comparison. In the previous section, when considering $(\Omega, \vee_A, \vee_B, \vee_C)$ to $(\Omega, \rightarrow_A, \rightarrow_B, \rightarrow_C)$, we considered $(X \rightarrow Y) \rightarrow Y = X \vee Y$ to be valid. In the 6-valued case, that is not possible. However, since we already have the negation operation, we can use $X \rightarrow Y = X' \vee Y$ to construct it. Let $\Omega = \{A, B, C, A', B', C'\}$. When $(\Omega, \vee_A, \vee_B, \vee_C)$ is Pantograph of Example 2, we will also call $(\Omega, \rightarrow_A, \rightarrow_B, \rightarrow_C)$, which is composed from it, **P-I** (Pantograph-implications) if there is no confusion. Similarly, **H-I** (Hexagon-implications), **U-I** (Umbrella -implications), **O-I** (Octahedron-implications) and **T-I** (Tie-implications) are also constructed.

$\rightarrow_A$	A	B	C	A'	B'	C'
A	A	C'	B'	A'	B'	C'
B	A	A	B'	B'	B'	A
C	A	C'	A	C'	A	C'
A'	A	A	A	A	A	A
B'	A	B	A	C'	A	C'
C'	A	A	C	B'	B'	A

$\rightarrow_B$	A	B	C	A'	B'	C'
A	B	B	A'	A'	A'	B
B	C'	B	A'	A'	B'	C'
C	C'	B	B	B	C'	C'
A'	A	B	B	B	C'	C'
B'	B	B	B	B	B	B
C'	B	B	C	A'	A'	B

$\rightarrow_C$	A	B	C	A'	B'	C'
A	C	A'	C	A'	C	A'
B	B'	C	C	C	B'	B'
C	B'	A'	C	A'	B'	C'
A'	A	C	C	C	B'	B'
B'	C	B	C	A'	C	A'
C'	C	C	C	C	C	C

Figure 13: Pantograph-implications

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$\to_A$	A	B	C	A'	B'	C'
A	A	B	C	A'	B'	C'
B	A	A	B'	B'	B'	A
C	A	C'	A	C'	A	C'
A'	A	A	A	A	A	A
B'	A	B	A	B	A	C'
C'	A	A	C	C	B'	A

$\to_B$	A	B	C	A'	B'	C'
A	B	B	A'	A'	A'	B
B	A	B	C	A'	B'	C'
C	C'	B	B	B	C'	C'
A'	A	B	B	B	A	C'
B'	B	B	B	B	B	B
C'	B	B	C	A'	C	B

$\to_C$	A	B	C	A'	B'	C'
A	C	A'	C	A'	C	A'
B	B'	C	C	C	B'	B'
C	A	B	C	A'	B'	C'
A'	A	C	C	C	B'	A
B'	C	B	C	A'	C	B
C'	C	C	C	C	C	C

Figure 14: Hexagon-implications

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$\to_A$	A	B	C	A'	B'	C'
A	A	A	A	A'	A	A
B	A	A	A	A	B'	A
C	A	A	A	A	A	C'
A'	A	A	A	A	A	A
B'	A	B	A	A	A	A
C'	A	A	C	A	A	A

$\to_B$	A	B	C	A'	B'	C'
A	B	B	B	A'	B	B
B	B	B	B	B	B'	B
C	B	B	B	B	B	C'
A'	A	B	B	B	B	B
B'	B	B	B	B	B	B
C'	B	B	C	B	B	B

$\to_C$	A	B	C	A'	B'	C'
A	C	C	C	A'	C	C
B	C	C	C	C	B'	C
C	C	C	C	C	C	C`
A'	A	C	C	C	C	C
B'	C	B	C	C	C	C
C'	C	C	C	C	C	C

Figure 15: Umbrella-implications

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$\to_A$	A	B	C	A'	B'	C'
A	A	B	C	A'	B'	C'
B	A	A	A	B'	B'	A
C	A	A	A	C'	A	C'
A'	A	A	A	A	A	A
B'	A	B	A	B	A	A
C'	A	A	C	C	A	A

$\to_B$	A	B	C	A'	B'	C'
A	B	B	B	A'	A'	B
B	A	B	C	A'	B'	C'
C	B	B	B	B	C'	C'
A'	A	B	B	B	A	B
B'	B	B	B	B	B	B
C'	B	B	C	B	C	B

$\to_C$	A	B	C	A'	B'	C'
A	C	C	C	A'	C	A'
B	C	C	C	C	B'	B'
C	A	B	C	A'	B'	C'
A'	A	C	C	C	C	A
B'	C	B	C	C	C	B
C'	C	C	C	C	C	C

Figure 16: Octahedron-implications

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$\to_A$	A	B	C	A'	B'	C'
A	A	A'	A'	A'	B'	C'
B	A	B'	B'	B'	B'	A
C	A	C'	C'	C'	A	C'
A'	A	A	A	A	A	A
B'	A	B	A'	A'	B'	C'
C'	A	A'	C	A'	B'	C'

$\to_B$	A	B	C	A'	B'	C'
A	A'	B	A'	A'	A'	B
B	B'	B	B'	A'	B'	C'
C	C'	B	C'	B	C'	C'
A'	A	B	B'	A'	B'	C'
B'	B	B	B	B	B	B
C'	B'	B	C	A'	B'	C'

$\to_C$	A	B	C	A'	B'	C'
A	A'	A'	C	A'	C	A'
B	B'	B'	C	C	B'	B'
C	C'	C'	C	A'	B'	C'
A'	A	C'	C	A'	B'	C'
B'	C'	B	C	A'	B'	C'
C'	C	C	C	C	C	C

Figure 17: Tie-implications

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Let $\Omega = \{A, B, C, A', B', C'\}$. The table below shows the status of whether 3-stealth absorption low is satisfied when the algebras $(\Omega, \vee_A, \vee_B, \vee_C)$ are **P-I**, **H-I**, **U-I**, **O-I** and **T-I**. Looking at this table, if trice and CPTS hold, it is expected that 3-stealth absorption low will hold, but the relationship is not clear.

	trice?	CPTS?	3 stealth-absorption law?
P-I	○	○	○
H-I	○	○	○
U-I	×	○	○
O-I	×	○	×
T-I	○	×	×

Figure 18: 3-stealth-absorption?

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We first wrote, " looking at the world in general, it is not always possible to consider absolute true or false". The real issues are complex: Ideological struggles, religious conflicts, transnational conflicts, etc. Sometimes the arguments don't mesh, and strange conclusions are drawn. It can ruin the premise. When there are multiple implications that are different, as in the algebra we have created here, and when the "stealth-absorption law" is in place, this is what sometimes happens. Our attempts may be useful for some analysis in the real world.

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