Multipliers and weak multipliers of algebras *

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1 Introduction

Multipliers of algebras, in particular, multipliers of Banach algebras, have been discussed in analysis.

Let B be a Banach algebra. A mapping $T: B \to B$ is called a multiplier of B, if it satisfies the condition (I) $xT(y) = T(xy) = T(x)y$ $(x, y \in B)$. Let $M(B)$ denote the collection of all multipliers of B, and let $B(B)$ be the collection of all bounded linear operators on B. Then $M(B)$ forms an algebra and $B(B)$ forms a Banach algebra. B is called *without order* if it has no nonzero left (or right) annihilator. If B is without order, then $M(B)$ forms a commutative closed subalgebra of $B(B)$ (see [2], Proposition 1.4.11). In 1952, Wendel [6] proved an important result that the multiplier algebra of $L^1(G)$ on a locally compact group G is isometrically isomorphic to the measure algebra on G. The general theory of multipliers of Banach algebras has been developed by Johnson [1].

When B is without order, T is a multiplier if it satisfies the condition (II) $xT(y) = T(x)y$ $(x, y \in B)$. Many researchers had been unaware of difference between conditions (I) and (II) until Zivari-Kazempour [7] (see also [8]) recently clearly stated the difference. We call a mapping T satisfying (II) a weak multiplier and denote the set of weak multipliers of B by $M'(B)$. Then, $M(B)$ is in general a proper subset of $M'(B)$. Furthermore, (weak) multipliers can be defined for an algebra A not necessarily associative, and they are not linear mappings in general. We denote the spaces of linear multipliers and linear weak multipliers of A by $LM(A)$ and $LM'(A)$ respectively. $M(A)$ and $LM(A)$ are subalgebras of the algebra A^A consisting of all mappings from A to itself. Meanwhile, $M'(A)$ and $LM'(A)$ are closed under the operation $\circ$ defined by $T \circ S = TS + ST$, and they form a Jordan algebra.

In this note we will discuss the (weak) multipliers (not necessarily associative) algebras in a purely algebraic manner. The complete classification of associative algebras of dimension 3 over an algebraically closed field of characteristic not equal to 2 were given in Kobayashi et al, [3]. Here, we choose a typical algebra from them and calculate its (weak) multipliers.

*This is a digest version of [4].

2 Multipliers and weak multipliers

Let K be a field and A be a (not necessarily associative) algebra over K. The set A^A of all mappings from A to A forms an associative algebra over K in the usual manner. Let $L(A)$ denotes the subalgebra of A^A of all linear mappings from A to A.

A mapping $T\colon A \to A$ is a *weak multiplier* of A, if

$$xT(y) = T(x)y$$

holds for any $x, y \in A$, and T is a *multiplier*, if

$$xT(y) = T(xy) = T(x)y$$

for any $x, y \in A$. Let $M(A)$ (resp. $M'(A)$) denote the set of all multipliers (resp. weak multipliers) of A. Define

$$LM(A) = M(A) \cap L(A) \text{ and } LM'(A) = M'(A) \cap L(A).$$

Proposition 2.1. *$M(A)$ (resp. $LM(A)$) is a unital subalgebra of A^A (resp. $L(A)$), and $M'(A)$ (resp. $LM'(A)$) is a Jordan subalgebra of A^A (resp. $L(A)$).*

Let $\mathrm{Ann}_l(A)$ (resp. $\mathrm{Ann}_r(A)$) be the left (resp. right) annihilator of A and let A_0 be their intersection, that is,

$$\mathrm{Ann}_l(A) = \{a \in A \mid ax = 0 \text{ for all } x \in A\},$$

$$\mathrm{Ann}_r(A) = \{a \in A \mid xa = 0 \text{ for all } x \in A\}$$

and

$$A_0 = \mathrm{Ann}_l(A) \cap \mathrm{Ann}_r(A).$$

For a subset X of A, $\langle X \rangle$ denotes the subspace of A generated by X.

Proposition 2.2. *A weak multiplier T of A such that $\langle T(A) \rangle \cap A_0 = \{0\}$ is a linear mapping over K.*

Cororally 2.3. *If $A_0 = \{0\}$, then any weak multiplier is a linear mapping, that is, $M'(A) = LM'(A)$ and $M(A) = LM(A)$.*

Proposition 2.4. *If T is a weak multiplier, then $T(\mathrm{Ann}_l(A)) \subseteq \mathrm{Ann}_l(A)$, $T(\mathrm{Ann}_r(A)) \subseteq \mathrm{Ann}_r(A)$ and $T(A_0) \subseteq A_0$.*

In this note we denote the subset $\{xy \mid x, y \in A\}$ of A by A^2.

Proposition 2.5. *Any mapping $T\colon A \to A$ such that $T(A) \subseteq A_0$ is a weak multiplier. Such a mapping T is a multiplier if and only if $T(A^2) = \{0\}$. In particular, if A is the zero algebra, every mapping T is a weak multiplier, and it is a multiplier if only if $T(0) = 0$.*

3 Nihil decomposition

Let A_1 be a subspace of A such that

$$A = A_1 \oplus A_0. \tag{1}$$

Here, A_1 is not unique, but choosing an appropriate A_1 will be important. When A_1 is fixed, any mapping $T \in A^A$ is uniquely decomposed as

$$T = T_1 + T_0 \tag{2}$$

with $T_1(A) \subseteq A_1$ and $T_0(A) \subseteq A_0$. We call (1) and (2) *nihil decompositions* of A and T respectively.

Let $M_1(A)$ (resp. $M_0(A)$) denote the set of all multipliers T of A with $T(A) \subseteq A_1$ (resp. $T(A) \subseteq A_0$). Similarly, the sets $M'_1(A)$ and $M'_0(A)$ of weak multipliers of A are defined. Also, set

$$LM_i(A) = M_i(A) \cap L(A) \ \text{ and } \ LM'_i(A) = M'_i(A) \cap L(A)$$

for $i = 0, 1$. By Proposition 2.2 we see

$$M'_1(A) = LM'_1(A) \ \text{ and } \ M_1(A) = LM_1(A),$$

and by Proposition 2.5 we have

$$M'_0(A) = A_0^A \text{ and } M_0(A) = \left\{T \in A_0^A \,|\, T(A^2) = \{0\}\right\}. \tag{3}$$

Theorem 3.1. *Let $A = A_1 \oplus A_0$ and $T = T_1 + T_0$ be nihil decompositions of A and $T \in A^A$ respectively.*

(i) T is a weak multiplier, if and only if T_1 is a weak multiplier. If T is a weak multiplier, T_1 is a linear mapping satisfying $T_1(A_0) = \{0\}$.

(ii) If T_1 is a multiplier and $T_0(A^2) = \{0\}$, then T is a multiplier.

(iii) If A_1 is a subalgebra of A, the converse of (ii) is also true, and $M_1(A)$ (resp. $M'_1(A)$) is isomorphic to $M(A_1)$ (resp. $M'(A_1)$).

Theorem 3.1 implies

$$M'(A) = M'_1(A) \oplus M'_0(A) \text{ and } M_1(A) \oplus M_0(A) \subseteq M(A),$$

where $M'_0(A)$ and $M_0(A)$ are given as (3). Moreover, if A_1 is a subalgebra, we have

$$M'(A) \cong M'(A_1) \oplus (A_0)^A \text{ and } M(A) \cong M(A_1) \oplus \left\{T \in (A_0)^A \,|\, T(A^2) = \{0\}\right\}.$$

Cororally 3.2. *Any weak multiplier T is written as*

$$T = T_1 + R$$

with $T_1 \in LM'_1(A)$ and $R \in (A_0)^A$, and it is a multiplier if and only if

$$R(x_1 y_1) = x_1 T_1(y_1) - T_1(x_1 y_1)$$

for any $x_1, y_1 \in A_1$.

4 Linear multipliers and matrix equation

Let A be an n-dimensional algebra over K with basis $E = \{e_1, e_2, \ldots, e_n\}$.

Lemma 4.1. *A linear mapping $T\colon A \to A$ is a weak multiplier if and only if*

$$e_i T(e_j) = T(e_i) e_j,$$

and it is a multiplier if and only if

$$T(e_i e_j) = e_i T(e_j) = T(e_i) e_j,$$

for all $e_i, e_j \in E$.

Let $\boldsymbol{A}$ (we use the bold character for matrix with elements in A) be the multiplication table of A on E. $\boldsymbol{A}$ is a matrix whose elements are from A defined by

$$\boldsymbol{A} = \boldsymbol{E}^t \boldsymbol{E},$$

where $\boldsymbol{E} = (e_1, e_2, \ldots, e_n)$ is the row vector consisting the basis elements. For a linear mapping T on A and a matrix $\boldsymbol{B}$ over A, $T(\boldsymbol{B})$ denotes the broadcasting of $\boldsymbol{B}$ by T, in the sense that the matrix obtained by applying T component-wise, that is, the (i, j)-element of $T(\boldsymbol{B})$ is $T(b_{ij})$ for the (i, j)-element b_{ij} of $\boldsymbol{B}$ (cf. [5]). We use the same character T for the representation matrix of T on E, that is,

$$T(\boldsymbol{E}) = \boldsymbol{E}T.$$

.

Theorem 4.2. *A linear mapping T is a weak multiplier of A if and only if*

$$\boldsymbol{A}T = T^t \boldsymbol{A}, \tag{4}$$

and T is a multiplier if and only if

$$T(\boldsymbol{A}) = \boldsymbol{A}T = T^t \boldsymbol{A}. \tag{5}$$

The multiplication table of the opposite algebra A^{op} of A is the transpose $\boldsymbol{A}^{\mathrm{t}}$ of $\boldsymbol{A}$. The algebras with multiplication tables transposed to each other have the same (weak) multipliers.

5 3-dimensional associative algebras

We have 24 families of 3-dimensional associative algebras, up to isomorphism (see [3]). We can determine the (weak) multipliers of these algebras utilizing the above results. Here, we pick one algebra A defined by the following multiplication table on a basis $\{e, f, g\}$:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & -e & 0 \end{pmatrix}, \qquad \begin{cases} e^2 = ef = fe = f^2 = 0, \\ eg = ge = g^2 = 0, \\ fg = e,\ gf = -e. \end{cases}$$

Then, $A_0 = \mathrm{Ann}_l(A) = \mathrm{Ann}_r(A) = Ke$, and we have a nihil decomposition $A = A_1 \oplus A_0$ with $A_1 = Kf + Kg$. Let T be a weak multiplier of A and let $T = T_1 + T_0$ be the nihil decomposition. By Theorem 3.1, T_1 is a linear mapping such that $T_1(Ke) = \{0\}$. So, it is represented as

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & r \\ 0 & t & u \end{pmatrix}$$

with $q, r, t, u \in K$, T_1 is a weak multiplier by Theorem 4.2, if and only if

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & te & ue \\ 0 & -qe & -re \end{pmatrix} = \boldsymbol{A}T_1 = T_1^{\mathrm{t}}\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -te & qe \\ 0 & -ue & re \end{pmatrix},$$

if and only if $r = t = 0$, $q = u$. Hence, we have

$$M_1'(A) = \{T_q \mid q \in K\},$$

where $T_q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}$. Therefore, we find

$$M'(A) = \{T_q \mid q \in K\} \oplus (Ke)^A$$

and

$$LM'(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \,\Big|\, a, b, c, q \in K \right\}.$$

Next, by Corollary 3.2, any weak multiplier T is expressed as

$$T = T_q + R \in M'(A) \quad \left(R \in (Ke)^A\right),$$

and it is a multiplier, if and only if

$$R((xv - yz)e) = R(\alpha\beta) = \alpha T_q(\beta) - T_q(\alpha\beta) = q(xv - yz)e$$

for any $\alpha = xf + yg, \beta = zf + vg \in A$ $(x, y, z, v \in K)$, if and only if $R(xe) = qxe$ for all $x \in K$. Because $(T - S_q)(Ke) = \{0\}$ with $S_q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}$, we have

$$M(A) = \{S_q \mid q \in K\} \oplus \left\{R \in (Ke)^A \mid R(Ke) = \{0\}\right\},$$

and

$$LM(A) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \,\Big|\, a, b, c \in K \right\}.$$

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