Forced rapidly dissipative Navier–Stokes flows

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1 Introduction

This is a note of survey based on the joint work with Lorenzo Brandolese.

We consider the incompressible Navier-Stokes equations on the whole space in $\mathbb{R}^n$, $n \geq 2$.

$$\text{(NS)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = \nabla \cdot f & \text{in } \mathbb{R}^n \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot, 0) = a & \text{in } \mathbb{R}^n, \end{cases}$$

where, $u = u(x,t) = \big(u_1(x,t), \dots, u_n(x,t)\big)$ and $\pi = \pi(x,t)$ denote the unknown velocity vector and the pressure of the fluid at $(x,t) \in \mathbb{R}^n \times (0,\infty)$, respectively. $f = \big(f_{k\ell}(x,t)\big)_{k,\ell=1,\dots,n}$ denotes the external force. While, $a = a(x) = \big(a_1(x), \dots, a_n(x)\big)$ is the given initial data.

In this note, we consider the decay rate of a rapidly dissipative Navier-Stokes flow compared with the well known optimal rate. For the energy decay problem of the Navier-Stokes flows, there are many results, for examples, Masuda [13], Schonbek [16], Wiegner [17], Kajikiya and Miyakawa [9]. So, nowadays, it is well known that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n+2}{4}}, \quad t > 0, \tag{1.1}$$

especially, if $a \in L^2(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} (1+|x|)|a(x)|\,dx < \infty$. The rate as in (1.1) is optimal in general, describing the rate of the leading order terms of the nonlinear terms. Indeed, Fujigaki and Miyakawa [8] gave a precise asymptotic expansions for the linear part and the nonlinear part, respectively, such as

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \bigg\| u_j(t) + \sum_{k=1}^n \partial_k E_t(\cdot) \int_{\mathbb{R}^n} y_k a_j(y)\,dy + \sum_{k,\ell=1}^n F_{j,k\ell}(\cdot,t) \int_0^\infty \int_{\mathbb{R}^n} u_k u_\ell(y,s)\,dyds \bigg\|_{L^q(\mathbb{R}^n)} = 0 \tag{1.2}$$

for all $1 \le q \le \infty$ and for all $j = 1, \dots, n$, if $a \in L^n(\mathbb{R}^n)$ is small enough and satisfies $\int_{\mathbb{R}^n}(1+|x|)|a(x)|\,dx < \infty$. Here, $E_t(x) = (4\pi t)^{-\frac{n}{2}} \exp\bigl(-\frac{|x|^2}{4t}\bigr)$ is the usual heat kernel and $F_{j,k\ell}(x,t)$ denotes the kernel function which represents the operator '$e^{t\Delta}\mathbb{P}\nabla\cdot$', defined by

$$F_{j,k\ell}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_t^\infty \partial_j\partial_k\partial_\ell E_s(x)\,ds \quad \text{for } j,k,\ell = 1,\dots,n.$$

Since the principal terms is explicitly obtained, if we make the flow rapidly decaying, it suffices to investigate the coefficients of the principal terms in the asymptotic expansions (1.2). Indeed, Miyakawa and Schonbek [15] clarified that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = o\bigl(t^{-\frac{n+2}{4}}\bigr) \quad \text{as } t \to \infty,$$

if and only if

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} y \otimes a(y)\,dy = 0, \\ \displaystyle\int_0^\infty \int_{\mathbb{R}^n} u_k u_\ell(y,s)\,dyds = c\delta_{k\ell} \quad \cdots\text{(A)}. \end{cases}$$

for some $c \in \mathbb{R}$. However, in general, it is difficult to confirm the condition (A), since we need the whole information of the unknown velocity both on the space and time region.

For the condition (A), Brandolese [1] introduce a spatial symmetry, so-called cyclic symmetry, as follows:

(a) u_j is odd function with respect to x_j and is even function with the others,

(b) $u_1(x_1, x_2, \dots, x_n) = u_2(x_n, x_1, \dots, x_{n-1}) = u_n(x_2, \dots, x_n, x_1)$.

On the other hand, as a viewpoint of a control problem, it is natural to consider that for any initial data whether we are able to make the flow faster decaying by another approach.

So, the aim is to control the flow with the aid of external force. For this purpose, we introduce the forcing term with a divergence form $\nabla \cdot f$. In this situation, we see that

$$\int_0^t [e^{(t-s)\Delta}\mathbb{P}\nabla \cdot f]_j(s)\,ds \sim \sum_{k,\ell=1}^n F_{j,k\ell}(t) \int_0^\infty \int_{\mathbb{R}^n} f_{k\ell}(y,s)\,dyds$$

as $t \to \infty$. Hence, the principal terms of the forcing term are given by the same kernel function $F_{j,k\ell}$ as the nonlinear term, we generalize the condition (A) in case with the external force.

Proposition 1.1 (Brandolese-O. [3]).

$$\|u(t) - e^{t\Delta}a\|_{L^2(\mathbb{R}^n)} = o(t^{-\frac{n+2}{4}}) \quad \textit{as } t \to \infty$$

if and only if

$$\text{(B)} \qquad \int_0^\infty \int_{\mathbb{R}^n} \big(f_{k\ell}(y,s) - u_k u_\ell(y,s)\big)\, dyds = c\delta_{k\ell}, \quad \textit{for } k, \ell = 1, \ldots, n,$$

with some $c \in \mathbb{R}$.

Here, we note that the condition (B) is easy to be obtained as an analogy of Miyakawa and Schonbek [15]. However, we regard the condition (B) as a quantitative condition. Indeed, in the previous result [3], we introduce the following external forces. Let $\Phi \in C_0^\infty\big(\mathbb{R}^n \times [0,\infty)\big)$ with $\int_0^\infty \int_{\mathbb{R}^n} \Phi(x,t)\, dxdt = 1$. Then each component of the forcing term is defined by

$$f_{k\ell}(x,t) = \lambda_{k\ell}\Phi(x,t), \quad k, \ell = 1, \ldots, n, \tag{1.3}$$

for some constants $\lambda_{k\ell} \in \mathbb{R}$. Substituting the external force $f_{k\ell}$ as in (1.3) into the condition (B), we see that

$$\lambda_{k\ell} = \int_0^\infty \int_{\mathbb{R}^n} u_k u_\ell(y,s)\, dyds, \quad k \neq \ell. \tag{1.4}$$

Therefore, our problem is reduced to find the coefficients $\lambda_{k\ell}$ such that (1.4) holds true. Finally, we note that the method that how we construct such $\lambda_{k\ell}$ is the main matter. Moreover, the fewer restrictions of the choice of Φ are the better, as a viewpoint of controlling problem.

2 Main results

In this section, we state recent progress of our problem. To state theorems, we introduce some function spaces.

Let $1 \leq r < \infty$.

$$X_r = \big\{v \in L^\infty_{\mathrm{loc}}\big(0,\infty; L^r(\mathbb{R}^n)\big); \|v\|_{X_r} = \operatorname*{ess\,sup}_{t>0} t^{\frac{1}{2}-\frac{n}{2r}}\|v(t)\|_{L^r(\mathbb{R}^n)} < \infty\big\},$$
$$Y_r = \big\{f \in L^\infty_{\mathrm{loc}}\big(0,\infty; L^r(\mathbb{R}^n)\big); \|f\|_{Y_r} = \operatorname*{ess\,sup}_{t>0} t^{1-\frac{n}{2r}}\|f(t)\|_{L^r(\mathbb{R}^n)} < \infty\big\}.$$

Notice that X_r is usual Kato's space for the velocity field and such space is left invariant by the natural scaling of (NS), $u \mapsto u_\lambda$, with $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$. The space Y_r is the corresponding natural space for the forcing term, and it is left invariant by the natural scaling, i.e., $\|f_\lambda\|_{Y_r} = \|f\|_{Y_r}$, where $f_\lambda(x,t) = \lambda^2 f(\lambda x, \lambda^2 t)$.

We use homogeneous Besov of the form $\dot{B}^s_{r,q}(\mathbb{R}^n)$, with $1 \leq r, q \leq \infty$ and negative regularity s. These are normed by $f \mapsto \big\|2^{js}\|\Delta_j f\|_r\big\|_{\ell^q(\mathbb{Z})}$, where $(\Delta_j f)_{j\in\mathbb{Z}}$ is the Littlewood–Paley decomposition of the tempered distribution $f \in \mathcal{S}'_h$.

We define $\dot{B}^s_{r,c_0}(\mathbb{R}^n)$ as the closed subspace of $\dot{B}^s_{r,\infty}(\mathbb{R}^n)$ such that

$$\lim_{j\to-\infty} 2^{js}\|\Delta_j f\|_{L^r(\mathbb{R}^n)} = 0.$$

For any $1 \le q < \infty$, we have of course the inclusions $\dot{B}^s_{r,q}(\mathbb{R}^n) \subset \dot{B}^s_{r,c_0}(\mathbb{R}^n) \subset \dot{B}^s_{r,\infty}(\mathbb{R}^n)$.

Now, we state our main theorem.

Theorem 2.1. *Let $2 \le n < r < \infty$. Let*

$$a \in \dot{B}^{-1+\frac{n}{r}}_{r,\infty}(\mathbb{R}^n) \cap \dot{H}^{-1}(\mathbb{R}^n), \qquad \nabla \cdot a = 0.$$

Let $\Phi \in L^\infty_c(\mathbb{R}^n \times \mathbb{R}^+)$, such that $\int_0^\infty \int \Phi = 1$. There exist $\eta_0 > 0$ (only dependent on n and r), and a constant real matrix $(\sigma_{k\ell})$ (dependent on n, r and also on a, Φ), with

$$|\sigma_{k\ell}| \le 1 \qquad (k, \ell = 1, \dots, n),$$

such that if

$$\begin{cases} \|a\|_{\dot{B}^{-1+\frac{n}{r}}_{r,\infty}} + \|a\|^2_{\dot{H}^{-1}} \|\Phi\|_{Y_r} < \eta_0 \\ \|a\|_{\dot{H}^{-1}} \|\Phi\|_{L^{\frac{4+2n}{4+n}}(\mathbb{R}^n \times \mathbb{R}^+)} < \eta_0 \end{cases} \tag{2.1}$$

and if

$$f = (f_{k\ell}), \quad \text{with} \quad f_{k\ell}(x,t) = \sigma_{k\ell}\, \|a\|^2_{\dot{H}^{-1}} \Phi(x,t),$$

then there exists a global solution $u \in X_r \cap L^2(\mathbb{R}^n \times \mathbb{R}^+)$ to (NS) such that

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \|u(t) - e^{t\Delta} a\|_q = 0 \qquad \text{for all } 1 \le q \le \infty. \tag{2.2}$$

The above solution u is rapidly dissipative, i.e., $\|u(t)\|_2^2 = o\big(t^{-\frac{n+2}{2}}\big)$ as $t \to +\infty$, if and only if the initial data a belongs also to $\dot{B}^{-\frac{n+2}{2}}_{2,c_0}(\mathbb{R}^n)$.

Corollary 2.2. *Let $a \in \dot{B}^{-1+\frac{n}{r}}_{r,\infty}(\mathbb{R}^n) \cap \dot{H}^{-1}(\mathbb{R}^n)$, with $n < r < \infty$. Let $R, R' > 0$ and*

$$\Phi(x,t) = R^n R' \phi(Rx)\, \psi(R't),$$

where ϕ is a $L^\infty(\mathbb{R}^n)$-function supported in the unit cube $[0,1]^n$ and ψ is a $L^\infty(\mathbb{R}^+)$-function supported in the interval $[0,1]$, both with integral equal to one. There exists $\eta_0' > 0$ (depending on ϕ, ψ, n and r), such that if

$$\begin{cases} \|a\|_{\dot{B}^{-1+\frac{n}{r}}_{r,\infty}} < \eta_0' \\ \|a\|^2_{\dot{H}^{-1}} R^{n(1-\frac{1}{r})} (R')^{\frac{n}{2r}} < \eta_0' \\ \|a\|_{\dot{H}^{-1}} R^{\frac{n^2}{4+2n}} (R')^{\frac{n}{4+2n}} < \eta_0', \end{cases} \tag{2.3}$$

then the conclusion of Theorem 2.1 applies.

Remark 2.1. In the corollary, it suffices to assume that one of R or R' is small enough. Hence, we can control the flow by acting an external force for arbitrarily short time, or acting a force in arbitrarily tiny space region, we can make the flow decaying faster.

3 Construction of the forcing term and approximate solutions

In order to discuss the way to find external forces, we shall recall the condition (B). Since (B) can be regarded as a quantitative condition, the method of inductive approximations seems to be effective. Indeed, the essential idea to find the coefficient $\lambda_{k\ell}$ is the following:

$$\lambda_{k\ell}^{(m+1)} = \int_0^\infty \int_{\mathbb{R}^n} u_k^{(m)} u_\ell^{(m)}(y,s)\,dyds, \quad k \neq \ell,\ m = 0,1,2,\dots.$$

More precisely, we consider the following procedure:

$$\begin{cases} f^{(0)}(x,t) \equiv 0, \\ u^{(m)}(t) = e^{t\Delta}a + \displaystyle\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot f^{(m)}(s)ds \\ \qquad\qquad + \displaystyle\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u^{(m)}\otimes u^{(m)})(s)ds, \quad m = 0,1,2,\dots, \end{cases}$$

where, $f^{(m)}$ is defined by

$$f_{k\ell}^{(m)}(x,t) = \begin{cases} c_{k\ell}^{(m-1)}\Phi(x,t), & k \neq \ell, \\ \left(c_{kk}^{(m-1)} - \dfrac{1}{n}\big(c_{11}^{(m-1)} + \cdots + c_{nn}^{(m-1)}\big)\right)\Phi(x,t), & k = \ell, \end{cases}$$

for $m = 1,2,\dots$, and

$$c_{k\ell}^{(m-1)} = \int_0^\infty \int_{\mathbb{R}^n} u_k^{(m-1)} u_\ell^{(m-1)}(y,s)\,dyds,$$

for $m = 1,2,\dots$.

Here, we give remark on the definition of $\{f_{k\ell}^{(m)}\}_{m\in\mathbb{N}}$. Let us assume that

$$\|u^{(0)}(t)\|_{L^2(\mathbb{R}^n)} = o\big(t^{-\frac{n+2}{4}}\big) \quad \text{as } t\to\infty,$$

where $f^{(0)} = 0$. In this situation, we have nothing to do, i.e., we have no need to consider the nontrivial external force to make the flow decaying faster. Indeed, by the condition (A) in Miyakawa and Schonbek [15], we observe that

$$\begin{cases} c_{k\ell}^{(0)} = \displaystyle\int_0^\infty\int_{\mathbb{R}^n} u_k^{(0)}u_\ell^{(0)}(y,s)\,dyds = 0, & k\neq\ell, \\ c_{kk}^{(0)} = \displaystyle\int_0^\infty\int_{\mathbb{R}^n} u_k^{(0)}(y,s)^2\,dyds = \int_0^\infty\int_{\mathbb{R}^n} u_\ell^{(0)}(y,s)^2\,dyds = c_{\ell\ell}^{(0)}, & k=\ell. \end{cases}$$

Therefore, we have the consistency, i.e.,

$$f_{k\ell}^{(m)}(x,t) \equiv 0 \quad \text{for all } m = 0,1,2,\dots.$$

Due to the definition of the sequence of external forces $\{f_{k\ell}^{(m)}\}_{m\in\mathbb{N}}$, we can observe that for some fixed $n < r < \infty$,

$$\|f^{(m)}\|_{Y_r} \lesssim \|a\|_{\dot{H}^{-1}}^2 \|\Phi\|_{Y_r}, \quad m = 1, 2, \dots,$$

under some additional conditions. So, if we assume

$$\|a\|_{\dot{B}^{-1+\frac{n}{r}}} + \|a\|_{\dot{H}^{-1}}^2 \|\Phi\|_{Y_r} < \eta_0$$

is small enough with some $\eta_0 > 0$ independent of $m \in \mathbb{N}$, then we can expect the global in time existence of $u^{(m)}(t)$ for each $m \in \mathbb{N}$, by the usual Fujita-Kato method.

4 Convergence of the forcing terms $\{f^{(m)}\}$ and approximate solutions $\{u^{(m)}\}$

In the previous section, we establish the way of construction for external forces and corresponding Navier-Stokes flows. Then we emphasize that our problem is reduced to the convergence of the coefficients $c_{k\ell}^{(m)}$ as $m \to \infty$, i.e., the convergence of the fores $\{f^{(m)}\}_{m\in\mathbb{N}}$ and simultaneously of the solutions $\{u^{(m)}\}_{m\in\mathbb{N}}$.

We focus on the coefficients $c_{k\ell}^{(m)}$. We see that

$$|c_{k\ell}^{(m)}| \le \int_0^\infty \int_{\mathbb{R}^n} |u^{(m)}(y,s)|^2 \, dyds = \|u^{(m)}\|_{L^2(\mathbb{R}^n \times \mathbb{R}^+)}^2.$$

To derive the convergence of coefficients, we have to deal with $L^2(\mathbb{R}^n \times \mathbb{R}^+)$ estimates of the solutions $u^{(m)}$ and their boundedness in $L^2(\mathbb{R}^n \times \mathbb{R}^+)$. So, in the previous work [3], we derived the above estimate via quantitative energy decay estimate of the solutions $u^{(m)}(t)$. However, to derive quantitative decay estimates which are independent of $m \in \mathbb{N}$, we need many steps and a lot of calculations. To avoid such a complicated scheme, we consider to derive the $L^2(\mathbb{R}^n \times \mathbb{R}^+)$ estimates directly at the moment of construction of the approximate solution $u^{(m)}$.

Let the bilinear form $G[\cdot,\cdot]$ be defined by

$$G[u,v](t) = -\int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (u \otimes v)(s) \, ds.$$

Then we have the following lemma for the bilinear form $G[\cdot,\cdot]$.

Lemma 4.1. *The bilinear operator is continuous:*

i) $G\colon X_r \times X_r \to X_r, \quad n < r < \infty$, *i.e.,*

$$\|G[u,v]\|_{X_r} \lesssim \|u\|_{X_r} \|v\|_{X_r}.$$

ii) $G\colon L^2\big(\mathbb{R}^+; L^2(\mathbb{R}^n)\big) \times X_r \to L^2\big(\mathbb{R}^+; L^2(\mathbb{R}^n)\big)$, *for* $2 \le n < r < \infty$,

$$\|G[u,v]\|_{L^2(\mathbb{R}^n \times \mathbb{R}^+)} \lesssim \|u\|_{L^2(\mathbb{R}^n \times \mathbb{R}^+)} \|v\|_{X_r}.$$

By the virtue of Lemma 4.1, if X_r-norm is small enough then we can obtain $u^{(m)} \in X_r$ and also $u^{(m)} \in L^2(\mathbb{R}^n \times \mathbb{R}^+)$ for every $m \in \mathbb{N}$. Furthermore, we observe the convergence of $u^{(m)}$ in $L^2(\mathbb{R}^n \times \mathbb{R}^+)$ under some additional smallness on a and Φ.

Indeed, we see that

$$\begin{aligned} u^{(m+1)}(t) - u^{(m)}(t) &= \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot [f^{(m+1)} - f^{(m)}](s)\,ds \\ &\quad + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big[(u^{(m+1)} - u^{(m)}) \otimes u^{(m+1)}\big](s)\,ds \\ &\quad + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot \big[u^{(m)} \otimes (u^{(m+1)} - u^{(m)})\big](s)\,ds \\ &=: \mathcal{I}_1^{(m)}(t) + \mathcal{I}_2^{(m)}(t) + \mathcal{I}_3^{(m)}(t). \end{aligned}$$

Applying Lemma 4.1, Item ii), we obtain

$$\begin{aligned} \|\mathcal{I}_2\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)} + \|\mathcal{I}_3\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)} &\lesssim \Big(\|u^{(m)}\|_{X_r} + \|u^{(m+1)}\|_{X_r}\Big)\|u^{(m+1)} - u^{(m)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)} \\ &\lesssim \eta\,\|u^{(m+1)} - u^{(m)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)}. \end{aligned}$$

So choosing $\eta > 0$ is small enough, independent of $m \in \mathbb{N}$, we see that the two last term can be absorbed by the first one. Namely,

$$\|u^{(m+1)} - u^{(m)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)} \lesssim \|\mathcal{I}_1^{(m)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)}.$$

Then, we see that

$$\|\mathcal{I}_1^{(m)}\|_{L^2(\mathbb{R}^n\times\mathbb{R})} \lesssim \|a\|_{\dot{H}^{-1}}\|\Phi\|_{L^{\frac{4+2n}{4+n}}(\mathbb{R}^n\times\mathbb{R}^+)}\|u^{(m)} - u^{(m-1)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)}.$$

Therefore, if we assume $\|a\|_{\dot{H}^{-1}}\|\Phi\|_{L^{\frac{4+2n}{4+n}}(\mathbb{R}^n\times\mathbb{R}^+)}$ is small enough, then we obtain that

$$\|u^{(m+1)} - u^{(m)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)} \le \frac{1}{2}\|u^{(m)} - u^{(m-1)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)},$$

which yields that the convergence of the solutions $\{u^{(m)}\}_{m\in\mathbb{N}}$ in $L^2(\mathbb{R}^n \times \mathbb{R}^+)$. Moreover, arguing as before, but using Lemma 4.1, Item i), we see that

$$\begin{aligned} \|u^{(m+1)} - u^{(m)}\|_{X_r} &\lesssim \|\mathcal{I}_1^{(m)}\|_{X_r} \\ &\lesssim \|f^{(m+1)} - f^{(m)}\|_{Y_r} \\ &\lesssim \|a\|_{\dot{H}^{-1}}\|\chi\|_{Y_r}\|u^{(m)} - u^{(m-1)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)}. \end{aligned}$$

Therefore,

$$\|u^{(m+1)} - u^{(m)}\|_{X_r} \lesssim \frac{1}{2}\|u^{(m)} - u^{(m-1)}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^+)}.$$

This proves that $\{u^{(m)}\}_{m\in\mathbb{N}}$ converges also in X_r. A similar argument establishes the convergence of $\{u^{(m)}\}_{m\in\mathbb{N}}$ in X_∞.

We denote by v the limit in $L^2(\mathbb{R}^n \times \mathbb{R}^+)$ and in $X_r \cap X_\infty$ of $u^{(m)}$. As $v \in L^2(\mathbb{R}^n \times \mathbb{R}^+)$, we can define the limit forcing term

$$f_{k\ell}(x,t) = \begin{cases} c_{k\ell}^{(\infty)}\Phi(x,t) & k \neq \ell, \\ (c_{kk}^{(\infty)} - \bar{c}^{(\infty)})\Phi(x,t) & k = \ell, \end{cases}$$

where

$$c_{k\ell}^{(\infty)} = \int_0^\infty \int_{\mathbb{R}^n} v_k v_\ell(y,s)\, dyds \quad (k,\ell = 1,\dots,n),$$

and

$$\bar{c}^{(\infty)} = \frac{1}{n}\big(c_{11}^{(\infty)} + \cdots + c_{nn}^{(\infty)}\big) = \frac{1}{n}\int_0^\infty \int_{\mathbb{R}^n} |v(y,s)|^2\, dyds.$$

By the convergence $u^{(m)} \to v$ in $L^2(\mathbb{R}^n \times \mathbb{R}^+)$, we can see that the convergence $c_{k\ell}^{(m)} \to c_{k\ell}^{(\infty)}$ as $m \to \infty$ for $k, \ell = 1, \dots, n$. Hence, we obtain that

$$f^{(m)} \to f \quad \text{strongly in } Y_r \cap L^{\frac{4+2n}{4+n}}(\mathbb{R}^n \times \mathbb{R}^+),$$

as $m \to \infty$. Finally, we obtain that

$$v(t) = e^{t\Delta}a + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot f(s)\, ds - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (v\otimes v)(y,s)\, dyds.$$

So, v and f is a desired Navier-Stokes flow and an external force.

5 Revisiting Fujigaki and Miyakawa asymptotic profiles

Since we only assume that $a \in \dot{H}^{-1}(\mathbb{R}^n)$, the energy decay is $\|u(t)\|_{L^2(\mathbb{R}^n)} = O\big(t^{-\frac{1}{2}}\big)$ as $t \to \infty$, which is not enough to apply the theory of the Fujigaki and Miyakawa asymptotic expansions. Hence, in this section we investigate the asymptotic expansions of the nonlinear terms.

Here, we notice that the kernel function $F_{j,k\ell}$ has the following scale property;

$$F_{j,k\ell}(x,t) = t^{-\frac{n+1}{2}} F_{j,k\ell}\left(\frac{x}{\sqrt{t}}, 1\right), \quad j,k,\ell = 1,\dots,n.$$

Noting this scale property, we investigate the followings.

Let M be a measurable function on $\mathbb{R}^n \times \mathbb{R}^+$ which satisfies the following scaling properties

$$\text{(5.1a)} \qquad M(x,t) = t^{-\frac{n+1}{2}} M(x/\sqrt{t}, 1), \qquad x \in \mathbb{R}^n,\ t > 0.$$

We also assume that

(5.1b)

$$M(\cdot,1) \in W^{1,\infty}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n) \quad \text{and} \quad x\cdot\nabla M(\cdot,1) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Here, we note that $\partial_t\big(M(x/\sqrt{t},1)\big) = -\frac{1}{2t\sqrt{t}} x \cdot \nabla M(x/\sqrt{t},1)$ and that $M(\cdot,1)$ is uniformly continuous on $\mathbb{R}^n$ by the Morrey inequality.

Then we have the following expansion theorem.

Theorem 5.1. *Let $n \geq 1$, $\mathcal{W} \in L^1(\mathbb{R}^n \times \mathbb{R}^+)$, with $\|\mathcal{W}(t)\|_{L^1(\mathbb{R}^n)} = \mathcal{O}(\frac{1}{t})$ as $t \to \infty$. Let us introduce the constant $\lambda = \int_0^\infty\!\!\int \mathcal{W}(y,s)\,dyds$ and let also*

$$\Psi(x,t) = \int_0^t \int_{\mathbb{R}^n} M(x-y,t-s)\mathcal{W}(y,s)\,dyds.$$

Then, as $t \to \infty$,

$$\Big\|\Psi(t) - \lambda M(\cdot,t)\Big\|_{L^q(\mathbb{R}^n)} = o\big(t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\big), \qquad \text{with} \quad 1 \leq q < \frac{n}{n-1}. \tag{5.2}$$

The above results extend to $\frac{n}{n-1} \leq q \leq \infty$, provided $\mathcal{W}$ satisfies also $\|\mathcal{W}(t)\|_{L^\beta(\mathbb{R}^n)} = \mathcal{O}(t^{-1-\frac{n}{2}(1-\frac{1}{\beta})})$ as $t \to \infty$, for some β such that $\frac{1}{q} \leq \frac{1}{\beta} < \frac{1}{q} + \frac{1}{n}$.

Let us apply Theorem 5.1 to each one of the terms of the summation in the right-hand side, with

$$M = F_{j,k\ell} \qquad \text{and} \qquad \mathcal{W}_{k\ell} = v_k v_\ell - f_{k\ell} \qquad (j,k,\ell = 1,\dots,n).$$

The required conditions on M (5.1) do hold, by the properties of F.

Let us check the needed conditions on $\mathcal{W}$. We have $v \in L^2(\mathbb{R}^n \times \mathbb{R}^+)$ and $\Phi \in L^1(\mathbb{R}^n \times \mathbb{R}^+)$, hence $f \in L^1(\mathbb{R}^n \times \mathbb{R}^+)$ and so $\mathcal{W}$ does belong to $L^1(\mathbb{R}^n \times \mathbb{R}^+)$. Moreover, the conditions that we put on a and Φ insure, In particular, $\|v(t)\|_2^2 = O(t^{-1})$ as $t \to \infty$. We see that $\|\mathcal{W}(t)\|_1 = \mathcal{O}(t^{-1})$. Then Theorem 5.1 applies and we get, at least for $1 \leq q < n/(n-1)$,

$$\Big\| v_j(x,t) - e^{t\Delta}a_j(x) + \sum_{\ell,k=1}^n F_{j,k\ell}(\cdot,t) \int_0^\infty \!\! \int \mathcal{W}_{k\ell}(y,s)\,dyds \Big\|_{L^q(\mathbb{R}^n)} = o\big(t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\big), \tag{5.3}$$

as $t \to \infty$. Let us extend the range of the parameter q for the above asymptotic profile. For any $1 \leq q \leq \infty$, we note that

$$\|\mathcal{W}(t)\|_q \leq \|v(t)\|_{2q}^2 + \|f(t)\|_q \lesssim t^{-1-\frac{n}{2}(1-\frac{1}{q})}.$$

By the last assertion of Theorem 5.1, we now deduce that (5.3) holds true for any $1 \leq q \leq \infty$.

However, our constructed solution v and external force f satisfies

$$\int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_{k\ell}(y,s)\,dyds = c\delta_{k\ell} \quad \text{for } k,\ell = 1,\dots,n,$$

with some $c \in \mathbb{R}$. Therefore, we obtain our main theorem on the rapid energy decay for the Navier-Stokes flow with an external force.

References

[1] L. Brandolese, *On the localization of symmetric and asymmetric solutions of the Navier-Stokes equations in* $\mathbb{R}^n$, C. R. Acad. Sci. Paris Sér. I Math., **332** (2001), 125–130.

[2] L. Brandolese, *Space-time decay of Navier-Stokes flows invariant under rotations*, Math. Ann., **329** (2004), 685–706.

[3] L. Brandolese, T. Okabe, *Annihilation of slowly-decaying terms of Navier-Stokes flows by external forcing*, Nonlinearity, **34**, (2021), N.3, 1733–1757.

[4] L. Brandolese, M.E. Schonbek, *Large time behavior of the Navier-Stokes flow*, Handbook of mathematical analysis in mechanics of viscous fluids, Springer, (2018), 579–645.

[5] L. Brandolese, *On a non-solenoidal approximation to the incompressible Navier-Stokes equations*, J. Lond. Math. Soc. (2), **96**, 2017, N.2, 326–344.

[6] H. Bahouri, J.-Y. Chemin, R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **343**, Springer, Heidelberg, 2011.

[7] M. Cannone, *Ondelettes, paraproduits et Navier-Stokes*, Diderot Editeur, Paris, 1995.

[8] Y. Fujigaki and T. Miyakawa, *Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the whole space*, SIAM J. Math. Anal., **33** (2001), 523–544.

[9] R. Kajikiya and T. Miyakawa, *On* L^2 *decay of weak solutions of the Navier-Stokes equations in* $\mathbb{R}^n$, Math. Z., **192** (1986), 135–148.

[10] T. Kato, *Strong* L^p*-solutions of the Navier-Stokes equation in* $\mathbf{R}^m$*, with applications to weak solutions*, Math. Z., **187**(1984), 471–480.

[11] P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Series Chapman & Hall/CRC Research Notes in Mathematics, **431**, Chapman & Hall/CRC, Boca Raton, FL, 2002, ISBN1-58488-220-4.

[12] P.-G. Lemarié-Rieusset, *The Navier-Stokes problem in the 21st century*, CRC Press, Boca Raton, FL, 2016,

[13] K. Masuda, *Weak solutions of Navier-Stokes equations*, Tohoku Math. J. (2), **36** (1984), 623–646.

[14] T. Miyakawa, *Notes on space-time decay properties of nonstationary incompressible Navier-Stokes flows in* $\mathbf{R}^n$, Funkcial. Ekvac., **45** (2002), 271–289.

[15] T. Miyakawa and M. E. Schonbek, *On optimal decay rates for weak solutions to the Navier-Stokes equations in* $\mathbb{R}^n$, in "Proceedings of Partial Differential Equations and Applications (Olomouc, 1999)," Math. Bohem., **126** (2001), 443–455.

[16] M. E. Schonbek, L^2 *decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal., **88** (1985), 209–222.

[17] M. Wiegner, *Decay results for weak solutions of the Navier-Stokes equations on* $\mathbf{R}^n$, J. London Math. Soc. (2), **35** (1987), 303–313.