On unique solvability of the time-periodic problem for the Navier-Stokes equation

Tomoyuki Nakatsuka

Department of Economics,
Matsuyama University

1 Introduction

This is a summary of the paper [17] and thus the details are left to the original paper [17].

Let $n \geq 3$. We consider the time-periodic motion of a viscous incompressible fluid governed by the Navier-Stokes equation:

$$(1.1) \qquad \begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + \nabla q = \operatorname{div} F & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ v(\cdot, x) \to 0 & \text{as } |x| \to \infty, \\ v(t, \cdot) = v(t+T, \cdot) & \text{for all } t \in \mathbb{R}. \end{cases}$$

Here $v = (v_1(t,x), \cdots, v_n(t,x))$ and $q = q(t,x)$ denote, respectively, the unknown velocity and pressure of the fluid, while $F = (F_{ij}(t,x))_{i,j=1}^n$ is a given periodic tensor with $\operatorname{div} F = (\sum_{i=1}^n \partial_{x_i} F_{ij}(t,x))_{j=1}^n$ denoting the periodic external force. Furthermore, $T > 0$ denotes a fixed period.

The existence and uniqueness of (1.1) are studied in many manuscripts such as [13, 8, 19, 4, 18, 2]. The time-periodic problem is traditionally investigated via the initial value problem, however, a new method to analyze the time-periodic problem without discussing the initial value problem was invented by Kyed [10]. He introduced the reformulation of the time-periodic problem on a group $G := \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n$. The advantage of the reformulation is the availability of the Fourier transform on G. The time-periodic Navier-Stokes equation is studied by using this reformulation method in, for instance, [10, 12, 6, 7, 5]. In these papers, the time-periodic Navier-Stokes equation in which $v(\cdot, x)$ goes to nonzero vector at spatial infinity, instead of $(1.1)_3$, is considered. This is, as is well-known, a crucial difference from our problem (1.1). As far as the author knows, the time-periodic Navier-Stokes equation with $(1.1)_3$ is investigated without discussing the initial value problem only in [15, 16, 1]. In [15, 16], the author established the existence of solutions v to (1.1) with the pointwise decay properties such as $|v(t,x)| = O(|x|^{1-n})$ and $|\nabla v(t,x)| = O(|x|^{-n})$ as $|x| \to \infty$ uniformly in time and, furthermore, it was shown that the decay rates of this solution are optimal. It was proved in [1] that the time-periodic Navier-Stokes equation in three dimensional exterior domains admits a strong solution v such that $\|v(\cdot,x)\|_{L^p(0,T)}$ with $2<p<\infty$ decays like $|x|^{-1}$.

We state our strategy to study (1.1) in advance of the main results. We employ the reformulation introduced in [10] to study the problem (1.1) and thus our argument does not depend on the initial value problem. We also consider the decomposition of time-periodic functions introduced by Kyed [10]. For a T-periodic function v, we define its steady part $\mathcal{P}v$ and purely periodic part $\mathcal{P}_\perp v$ by

$$\mathcal{P}v(x) := \frac{1}{T}\int_0^T v(s,x)\,ds, \quad \mathcal{P}_\perp v(t,x) := v(t,x) - \mathcal{P}v(x).$$

In this paper, $\mathcal{P}v$ and $\mathcal{P}_\perp v$ are often abbreviated as v_s and v_p respectively. Using this decomposition, we decompose (1.1) into two equations:

$$\begin{cases} -\Delta v_s + v_s\cdot\nabla v_s + \nabla q_s = \mathrm{div}\ F_s - \mathcal{P}(v_p\cdot\nabla v_p) & \text{in } \mathbb{R}^n, \\ \mathrm{div}\ v_s = 0 & \text{in } \mathbb{R}^n, \\ v_s(x)\to 0 & \text{as } |x|\to\infty, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \partial_t v_p - \Delta v_p + \mathcal{P}_\perp(v_p\cdot\nabla v_p) + \nabla q_p = \mathrm{div}\ F_p - v_s\cdot\nabla v_p - v_p\cdot\nabla v_s & \text{in } \mathbb{R}\times\mathbb{R}^n, \\ \mathrm{div}\ v_p = 0 & \text{in } \mathbb{R}\times\mathbb{R}^n, \\ v_p(\cdot,x)\to 0 & \text{as } |x|\to\infty, \\ v_p(t,\cdot) = v_p(t+T,\cdot) & \text{for all } t\in\mathbb{R}. \end{cases} \tag{1.3}$$

The equation (1.2) can be obtained by substituting $v = v_s + v_p$ in (1.1) and then formally multiplying (1.1) by $\mathcal{P}$. Notice that $\mathcal{P}(\mathcal{P}_\perp v) = 0$ for all T-periodic functions v. Subtracting (1.2) from (1.1) yields the equation (1.3). We can verify that $v = v_s + v_p$ is a solution of (1.1) if and only if v_s and v_p are solutions of (1.2) and (1.3) respectively. We study the problem (1.1) through the analysis of (1.2) and (1.3). The equation (1.2) is the stationary Navier-Stokes equation with the external force $\mathrm{div}(F_s - \mathcal{P}(v_p\otimes v_p))$ and we analyze it using the standard theory for the steady Stokes equation. In the analysis of the equation (1.3), it is important that all the terms in (1.3) are purely periodic, that is, the steady part of each term in (1.3) is 0. The time-periodic Stokes equation with purely periodic data is studied by using the Fourier transform mentioned above and we see that purely periodic solutions of the Stokes equation can have some additional regularity in time. This additional regularity plays an important role in constructing a solution of (1.3).

The aim of the paper [17] is to establish the existence and uniqueness of solutions to (1.1) in the sense of distributions. It is shown that if F is sufficiently small in an appropriate sense, then there exists a solution v of (1.1) with information on the classes of $\mathcal{P}v$ and $\mathcal{P}_\perp v$. The steady part $\mathcal{P}v$ and the purely periodic part $\mathcal{P}_\perp v$ belong to suitable L^p spaces corresponding to the classes of $\mathcal{P}F$ and $\mathcal{P}_\perp F$ respectively. In addition, slightly more regularity of $\mathcal{P}_\perp v$ in time is inherited from that of the purely periodic solutions to the Stokes equation. It should be noted that this additional property of $\mathcal{P}_\perp v$ is obtained thanks to the decomposition of (1.1) into (1.2) and (1.3). Furthermore, we will see that if the solution v is sufficiently small in a suitable sense, then there exists no other solution of (1.1) in the same class as v. We emphasize that this assertion is not a consequence of simple uniqueness theorems which assert the coincidence of two small solutions. In our uniqueness theorem, we assume only the smallness of one solution and no additional condition on the other solution.

This summary is organized as follows. In Section 2, we state the main results. We study the existence of solutions to (1.1) in Section 3. The theories for steady and time-periodic Stokes equations are investigated and we will see the additional regularity in time of purely periodic solutions to the Stokes equation by analyzing their representation via the Fourier transform. Based on the theory of the Stokes equations, solutions v_s and v_p to the nonlinear problems (1.2) and (1.3) are constructed and we will obtain a solution $v = v_s + v_p$ of (1.1). Section 4 is devoted to the study of uniqueness of solutions constructed in Section 3. We will consider the equations which the difference w of two given solutions, its steady part w_s and purely periodic part w_p should obey. Applying the regularity theory for the Stokes equations, we will establish the L^2 property of w_s and w_p. Furthermore, in order to get information on the class of $\partial_t w_p$, we will also show that w_p is indeed a strong solution. Combining the properties of w_s and w_p, we obtain a suitable L^2 property of w and we take w as a test function in the weak form of its equation to derive the uniqueness.

2 Main Results

Before stating our results, we introduce some function spaces. In what follows, we adopt the same symbols for vector and scalar function spaces as long as there is no confusion. For $1 \le r \le \infty$, the usual Lebesgue and Sobolev spaces are denoted, respectively, by $L^r(\mathbb{R}^n)$ and $W^{1,r}(\mathbb{R}^n)$ with norms $\|\cdot\|_{L^r(\mathbb{R}^n)}$ and $\|\cdot\|_{W^{1,r}(\mathbb{R}^n)}$. Furthermore, for $1 < r < \infty$, we define the homogeneous Sobolev space $\dot{H}^1_r(\mathbb{R}^n)$ by the completion of $C_0^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support in $\mathbb{R}^n$, in the norm $\|\nabla\cdot\|_{L^r(\mathbb{R}^n)}$.

We need the spaces of T-periodic functions. Set $C^\infty_{0,per}(\mathbb{R}\times\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}\times\mathbb{R}^n); \varphi(t,x) = \varphi(t+T,x) \text{ and } \varphi|_{[0,T]} \in C_0^\infty([0,T]\times\mathbb{R}^n)\}$. For $1 \le r \le \infty$ and a Banach space X with norm $\|\cdot\|_X$, $L^r_{per}(\mathbb{R};X)$ stands for the Banach space of all T-periodic functions $v:\mathbb{R}\to X$ such that the restriction $v|_{[0,T)} \in L^r(0,T;X)$ with norm

$$\|v\|_{L^r_{per}(\mathbb{R};X)} := \left(\frac{1}{T}\int_0^T \|v\|_X^r\,dt\right)^{\frac{1}{r}} \quad (1 \le r < \infty), \qquad \|v\|_{L^\infty_{per}(\mathbb{R};X)} := \operatorname*{ess\,sup}_{0\le t<T} \|v\|_X.$$

Notice that if $r_0 < r_1$, then the embedding $L^{r_1}_{per}(\mathbb{R};X) \subset L^{r_0}_{per}(\mathbb{R};X)$ is continuous. In the case $X = L^r(\mathbb{R}^n)$, we simply write $L^r_{per}(\mathbb{R}\times\mathbb{R}^n)$ with norm $\|\cdot\|_r := \|\cdot\|_{L^r_{per}(\mathbb{R};L^r(\mathbb{R}^n))}$. If $X = L^s(\mathbb{R}^n)$ for $s \ne r$, we denote its norm by $\|\cdot\|_{r,s} := \|\cdot\|_{L^r_{per}(\mathbb{R};L^s(\mathbb{R}^n))}$. Note that $C^\infty_{0,per}(\mathbb{R}\times\mathbb{R}^n)$ is dense in $L^r_{per}(\mathbb{R}\times\mathbb{R}^n)$ for $1 \le r < \infty$. The space $W^{1,2,r}_{per}(\mathbb{R}\times\mathbb{R}^n)$ is defined, for $1 \le r < \infty$, by the completion of $C^\infty_{0,per}(\mathbb{R}\times\mathbb{R}^n)$ in the norm $\|\cdot\|_{1,2,r} := (\|\partial_t\cdot\|_r^r + \sum_{|\alpha|\le 2}\|\partial_x^\alpha\cdot\|_r^r)^{1/r}$. Furthermore, subspaces of these function spaces, consisting of purely periodic functions, are denoted by

$$\begin{aligned} C^\infty_{0,per,\perp}(\mathbb{R}\times\mathbb{R}^n) &:= \{\varphi\in C^\infty_{0,per}(\mathbb{R}\times\mathbb{R}^n); \mathcal{P}\varphi=0\},\\ L^r_{per,\perp}(\mathbb{R};X) &:= \{v\in L^r_{per}(\mathbb{R};X); \mathcal{P}v=0\},\\ W^{1,2,r}_{per,\perp}(\mathbb{R}\times\mathbb{R}^n) &:= \{v\in W^{1,2,r}_{per}(\mathbb{R}\times\mathbb{R}^n); \mathcal{P}v=0\}. \end{aligned}$$

It is easy to see that $C^\infty_{0,per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ is dense in $L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ for $1\le r<\infty$.

We also introduce some exponents used in this paper. For $1<r<n$, we define the exponent r^* by $1/r^* := 1/r-1/n$. For a parameter $0\le\lambda\le 1$ and given $1<r<n$, we define the exponents $\alpha_{\lambda,r}$ and $\beta_{\lambda,r}$ by

$$\begin{cases} \alpha_{\lambda,r}=\dfrac{2r}{2-\lambda r} & \text{if} \quad \lambda r<2,\\ \alpha_{\lambda,r}<\infty & \text{if} \quad \lambda r=2,\\ \alpha_{\lambda,r}=\infty & \text{if} \quad \lambda r>2, \end{cases} \tag{2.1}$$

and

$$\beta_{\lambda,r} := \frac{nr}{n-(1-\lambda)r}. \tag{2.2}$$

The exponents $\alpha_{\lambda,r}$ and $\beta_{\lambda,r}$ are used in this section and Section 3. By $C=C(\cdot,\cdots,\cdot)$ we denote various constants depending only on the quantities in parentheses.

Now we state the main results. The first result is on the existence of solutions to (1.1).

Theorem 2.1. *Let $n\ge 3$ and $n/2+1\le r<n$. Suppose that F satisfies $\mathcal{P}F\in L^{n/2}(\mathbb{R}^n)$ and $\mathcal{P}_\perp F\in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$. There exists a constant $\delta=\delta(n,r,T)>0$ such that if $\|\mathcal{P}F\|_{L^{n/2}(\mathbb{R}^n)}+\|\mathcal{P}_\perp F\|_r\le\delta$, then (1.1) admits a solution $\{v,q\}$ satisfying*

$$\mathcal{P}v\in \dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n), \quad \mathcal{P}q\in L^{\frac{n}{2}}(\mathbb{R}^n) \tag{2.3}$$

and

$$\mathcal{P}_\perp v\in L^r_{per,\perp}(\mathbb{R};W^{1,r}(\mathbb{R}^n))\cap L^{\alpha_{\lambda,r}}_{per,\perp}(\mathbb{R};L^{\beta_{\lambda,r}}(\mathbb{R}^n)), \quad \mathcal{P}_\perp q\in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n) \tag{2.4}$$

for all $0\le\lambda\le 1$. Furthermore, if $\mathcal{P}_\perp F\in L^s_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ for some $1<s<r$, then the solution $\{v,q\}$ satisfies

$$\mathcal{P}_\perp v\in L^s_{per,\perp}(\mathbb{R};W^{1,s}(\mathbb{R}^n))\cap L^{\alpha_{\lambda,s}}_{per,\perp}(\mathbb{R};L^{\beta_{\lambda,s}}(\mathbb{R}^n)), \quad \mathcal{P}_\perp q\in L^s_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$$

for all $0\le\lambda\le 1$.

Remark 2.1. The properties (2.3) and (2.4) of the solution v in Theorem 2.1 imply $\mathcal{P}v\in L^n(\mathbb{R}^n)$ and $\mathcal{P}_\perp v\in L^p_{per,\perp}(\mathbb{R};L^n(\mathbb{R}^n))$ for $p<\infty$ if $r=n/2+1$ and $p=\infty$ if $n/2+1<r<n$. Hence, we have $v\in L^n_{per}(\mathbb{R}\times\mathbb{R}^n)$ if $r=n/2+1$ and $v\in L^\infty_{per}(\mathbb{R};L^n(\mathbb{R}^n))$ if $n/2+1<r<n$.

The next theorem is concerned with the uniqueness of solutions constructed in Theorem 2.1. We emphasize that only the smallness of one solution is assumed and no additional assumption is imposed on the other solution in the next theorem.

Theorem 2.2. *Let $n = 3$ and $5/2 < r < 3$. Suppose that the pairs $\{u, p\}$ and $\{v, q\}$ are solutions of* (1.1) *having the properties* (2.3) *and* (2.4). *There exists an absolute constant $\tilde{\delta} > 0$ such that if*

$$\|\mathcal{P}u\|_{L^3(\mathbb{R}^3)} + \|\mathcal{P}_\perp u\|_{\infty,3} \le \tilde{\delta},$$

then $\{u, p\} = \{v, q\}$.

Remark 2.2. The set of solutions satisfying the smallness condition in Theorem 2.2 is not empty. Indeed, we can easily verify that the solution v of (1.1) constructed in Theorem 2.1 is subject to the estimate

$$\|\mathcal{P}v\|_{L^3(\mathbb{R}^3)} + \|\mathcal{P}_\perp v\|_{\infty,3} \le \tilde{C}\left(\|\mathcal{P}F\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|\mathcal{P}_\perp F\|_r\right)$$

for some constant $\tilde{C} = \tilde{C}(r, T)$. Thus, v satisfies the smallness condition in Theorem 2.2 provided that $\|\mathcal{P}F\|_{L^{3/2}(\mathbb{R}^3)} + \|\mathcal{P}_\perp F\|_r \le \tilde{C}^{-1}\tilde{\delta}$. We can deduce from this observation and Theorem 2.1 that if $\|\mathcal{P}F\|_{L^{3/2}(\mathbb{R}^3)} + \|\mathcal{P}_\perp F\|_r \le \min\{\delta, \tilde{C}^{-1}\tilde{\delta}\}$, then (1.1) admits no solution satisfying (2.3) and (2.4) except the one constructed in Theorem 2.1. Here δ is the constant in Theorem 2.1 with $n = 3$.

Remark 2.3. The case $r = 5/2$ is excluded due to a simple observation that (2.4) does not imply $\mathcal{P}_\perp v \in L^\infty_{per,\perp}(\mathbb{R}; L^3(\mathbb{R}^3))$ if $r = 5/2$, see Remark 2.1 above. The uniqueness holds even for $r = 5/2$ if we assume $\mathcal{P}_\perp u, \mathcal{P}_\perp v \in L^\infty_{per,\perp}(\mathbb{R}; L^3(\mathbb{R}^3))$ in addition to the assumptions in Theorem 2.2.

3 Existence

This section is devoted to the proof of Theorem 2.1. We intend to construct solutions $\{v_s, q_s\}$ and $\{v_p, q_p\}$ of (1.2) and (1.3) respectively, and then we obtain a solution $\{v, q\} = \{v_s + v_p, q_s + q_p\}$ of (1.1). In order to study the equations (1.2) and (1.3), we begin with the analysis of the Stokes equations. We recall the unique solvability of the steady Stokes equation:

$$(3.1) \qquad \begin{cases} -\Delta v + \nabla q = \operatorname{div} F & \text{in } \mathbb{R}^n, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^n, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Lemma 3.1 ([3, 9]). *Let $1 < r < \infty$. For every $F \in L^r(\mathbb{R}^n)$, there exists a unique solution $\{v, q\} \in \dot{H}^1_r(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ of* (3.1) *such that*

$$\|\nabla v\|_{L^r(\mathbb{R}^n)} + \|q\|_{L^r(\mathbb{R}^n)} \le C\|F\|_{L^r(\mathbb{R}^n)}$$

with C depending only on n and r.

Next, we consider the time-periodic Stokes equation:

$$
\begin{cases} \partial_t v - \Delta v + \nabla q = \operatorname{div} F & \text{in } \mathbb{R}\times\mathbb{R}^n, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}\times\mathbb{R}^n, \\ v(\cdot,x) \to 0 & \text{as } |x|\to\infty, \\ v(t,\cdot) = v(t+T,\cdot) & \text{for all } t\in\mathbb{R}. \end{cases} \tag{3.2}
$$

The next lemma and proposition on the equation (3.2) are essentially based on the reformulation mentioned in Introduction. We briefly review the theory. Set $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$ and $G := \mathbb{T}\times\mathbb{R}^n$. We define the map $\pi : \mathbb{R}\times\mathbb{R}^n \to G$ by $\pi(t,x) := ([t],x)$ and let $\Pi := \pi|_{[0,T)\times\mathbb{R}^n}$. The restriction Π is a bijection from $[0,T)\times\mathbb{R}^n$ to G, and via Π we identify G with $[0,T)\times\mathbb{R}^n$. Hence, via the compositions with π and Π^{-1}, there is a natural correspondence between T-periodic functions in $\mathbb{R}\times\mathbb{R}^n$ and functions on G. The Haar measure dg on the locally compact abelian group G, unique up to a normalization factor, is chosen as the product of the Lebesgue measures on $\mathbb{R}^n$ and $[0,T)$, and we have

$$
\int_G u(g)\,dg := \frac{1}{T}\int_0^T\int_{\mathbb{R}^n} (u\circ\Pi)(s,y)\,dyds.
$$

The Lebesgue spaces on G are denoted by $L^r(G)$, and $L^r(G)$ is homeomorphism with $L^r_{per}(\mathbb{R}\times\mathbb{R}^n)$. The advantage of the reformulation is the availability of the Fourier transform on G. The Fourier transform and inverse Fourier transform are defined by

$$
\mathscr{F}_G[v] = \frac{1}{T}\int_0^T\int_{\mathbb{R}^n} v(s,y)e^{-ix\cdot\xi - i\frac{2\pi}{T}kt}\,dyds, \quad \mathscr{F}_G^{-1}[v] = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n} v(k,\xi)e^{ix\cdot\xi + i\frac{2\pi}{T}kt}\,d\xi
$$

respectively. Notice that $\mathscr{F}_G = \mathscr{F}_\mathbb{T}\circ\mathscr{F}_{\mathbb{R}^n}$. For more details on the analysis on G, see [10, 11].

The next lemma on the unique existence of a purely periodic solution to (3.2) was essentially proved in [5, Theorem 9] via the so-called transference principle and the Marcinkiewicz multiplier theorem. For later convenience, only the proof of uniqueness is given here.

Lemma 3.2. *Let $1 < r < \infty$. For every $F \in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$, there exists a unique solution $\{v,q\} \in L^r_{per,\perp}(\mathbb{R};W^{1,r}(\mathbb{R}^n)) \times L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ of (3.2) such that*

$$
\|v\|_{L^r_{per}(\mathbb{R};W^{1,r}(\mathbb{R}^n))} + \|q\|_r \le C\|F\|_r
$$

for some constant $C = C(n,r,T)$. Furthermore, if $F \in L^s_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ for some $1 < s < \infty$, then the solution $\{v,q\}$ satisfies

$$
v \in L^s_{per,\perp}(\mathbb{R};W^{1,s}(\mathbb{R}^n)), \quad q \in L^s_{per,\perp}(\mathbb{R}\times\mathbb{R}^n).
$$

Proof. We prove only the uniqueness. The uniqueness of purely periodic solutions can be proved in the same way as [11, Theorem 4.8]. Suppose $\{v,q\}, \{\tilde{v},\tilde{q}\} \in L^r_{per,\perp}(\mathbb{R};W^{1,r}(\mathbb{R}^n)) \times L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ are solutions of (3.2). The pair $\{v-\tilde{v}, q-\tilde{q}\}$ is a solution of the Stokes equation (3.2) with $F = 0$. We reformulate the equation on the group G and multiply it by the Helmholtz projection P_H on G, see [11, Lemma 4.3], to eliminate $\nabla(q-\tilde{q})$. Then we apply the Fourier transform to get

$$\left(|\xi|^2 + i\frac{2\pi}{T}k\right)\mathscr{F}_G[v-\tilde{v}] = 0.$$

Since $|\xi|^2 + i\frac{2\pi}{T}k = 0$ if and only if $\{\xi, k\} = \{0, 0\}$, it suffices to show that $\mathscr{F}_G[v-\tilde{v}](0,0) = 0$. We can easily see that $\mathcal{P}_\perp(v-\tilde{v}) = v-\tilde{v}$ holds and the Fourier transform of this relation is given by $(1-\delta_{\mathbb{Z}}(k))\mathscr{F}_G[v-\tilde{v}] = \mathscr{F}_G[v-\tilde{v}]$ by [11, Lemma 4.7]. This implies $\mathscr{F}_G[v-\tilde{v}](\xi, 0) = 0$ for all $\xi \in \mathbb{R}^n$. In particular, we have $\mathscr{F}_G[v-\tilde{v}](0,0) = 0$. Consequently, we derive $v = \tilde{v}$ on G and thus $v = \tilde{v}$ in $\mathbb{R}\times\mathbb{R}^n$. By the equation $(3.2)_1$ with $F = 0$, we get $\nabla(q-\tilde{q}) = 0$, and thus $q - \tilde{q} = h(t)$ for some purely periodic function h. Since $q-\tilde{q} \in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$, we deduce $h(t) = 0$ and thus $q = \tilde{q}$ in $\mathbb{R}\times\mathbb{R}^n$. Therefore, we derive $\{v, q\} = \{\tilde{v}, \tilde{q}\}$. □

Remark 3.1. As we can see in the proof above, the uniqueness of purely periodic solutions to the Stokes equation (3.2) holds even in larger class of solutions. In particular, we can verify the coincidence of $\{v, q\}$ and $\{\tilde{v}, \tilde{q}\}$ even if the latter is a purely periodic strong solution. Lemma 4.5 below is based on this observation.

The L^r theory of the Stokes equation is established in Lemma 3.2, however, this is not sufficient to construct a solution of (1.3). Indeed, Lemma 3.2 does not yield a good estimate of the nonlinear term $\mathcal{P}_\perp(v_p \otimes v_p)$ such as $\|v_p\otimes v_p\|_r \le C\|v_p\|_r^2$. To overcome this difficulty, we need the following proposition. Recall the definitions of the exponents $\alpha_{\lambda,r}$ and $\beta_{\lambda,r}$ in (2.1) and (2.2). This proposition is essentially proved by Galdi-Kyed [7]. The proof is based on the detailed analysis of the representation of the unique solution v in Lemma 3.2 via the Fourier transform:

$$v = \mathscr{F}_G^{-1}\left[\frac{i\xi(1-\delta_{\mathbb{Z}}(k))}{|\xi|^2 + i\frac{2\pi}{T}k}\left(I - \frac{\xi\otimes\xi}{|\xi|^2}\right)\mathscr{F}_G[F]\right].$$

Proposition 3.1. *Let $1 < r < n$ and $0 \le \lambda \le 1$. If $v \in L^r_{per,\perp}(\mathbb{R}; W^{1,r}(\mathbb{R}^n))$ is a solution of the Stokes equation* (3.2) *with $F \in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$, then we have*

$$v \in L^{\alpha_{\lambda,r}}_{per,\perp}(\mathbb{R}; L^{\beta_{\lambda,r}}(\mathbb{R}^n))$$

with the estimate

$$\|v\|_{\alpha_{\lambda,r},\beta_{\lambda,r}} \le C\|F\|_r$$

for some constant $C = C(n, r, \lambda, T)$.

Combining Lemma 3.2 and Proposition 3.1, we get the following.

Corollary 1. *Let $1 < r < n$. For every $F \in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$, there exists a unique solution $\{v, q\} \in L^r_{per,\perp}(\mathbb{R}; W^{1,r}(\mathbb{R}^n)) \times L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ of* (3.2) *such that $v \in L^{\alpha_{\lambda,r}}_{per,\perp}(\mathbb{R}; L^{\beta_{\lambda,r}}(\mathbb{R}^n))$ and*

$$\|v\|_{L^r_{per}(\mathbb{R};W^{1,r}(\mathbb{R}^n))} + \|v\|_{\alpha_{\lambda,r},\beta_{\lambda,r}} + \|q\|_r \le C\|F\|_r$$

with $C = C(n, r, T, \lambda)$.

Based on the theory of the Stokes equations, the equations (1.2) and (1.3) are investigated. Recall that for given T-periodic function v we denote its steady part $\mathcal{P}v$ and purely periodic part $\mathcal{P}_\perp v$ by v_s and v_p respectively:

$$v_s := \mathcal{P}_\perp v, \quad v_p := \mathcal{P}_\perp v.$$

Functions q_s, q_p, F_s and F_p are defined in the same way. For $n/2+1 \le r < n$, we define the space X_r by, if $r = n/2+1$,

$$X_{\frac{n}{2}+1} := L^{\frac{n}{2}+1}_{per,\perp}(\mathbb{R}; W^{1,\frac{n}{2}+1}(\mathbb{R}^n)) \cap L^{n+2}_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$$

with the norm $\|\cdot\|_{X_{\frac{n}{2}+1}} := \max\{\|\cdot\|_{L^{n/2+1}_{per}(\mathbb{R}; W^{1,n/2+1}(\mathbb{R}^n))}, \|\cdot\|_{n+2}\}$ and, if $n/2+1 < r < n$,

$$X_r := L^r_{per,\perp}(\mathbb{R}; W^{1,r}(\mathbb{R}^n)) \cap L^\infty_{per,\perp}(\mathbb{R}; L^n(\mathbb{R}^n))$$

with the norm $\|\cdot\|_{X_r} := \max\{\|\cdot\|_{L^r_{per}(\mathbb{R}; W^{1,r}(\mathbb{R}^n))}, \|\cdot\|_{\infty,n}\}$. Let K_s be a solution operator defined by Lemma 3.1 with $r = n/2$:

$$K_s : F_s \in L^{\frac{n}{2}}(\mathbb{R}^n) \mapsto v_s \in \dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)$$

and K_p defined by Corollary 1:

$$K_p : F_p \in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n) \mapsto v_p \in X_r.$$

Notice that K_p is indeed an operator from $L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$ to X_r thanks to Corollary 1. Furthermore, given $F_s \in L^{n/2}(\mathbb{R}^n)$ and $F_p \in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$, we define the operator K on $\dot{H}^1_{n/2}(\mathbb{R}^n) \times X_r$ by

$$K(v_s, v_p) := \{K_s(F_s - v_s\otimes v_s - \mathcal{P}(v_p\otimes v_p)), K_p(F_p - v_s\otimes v_p - v_p\otimes v_s - \mathcal{P}_\perp(v_p\otimes v_p))\},$$

where $u\otimes v := (u_i v_j)_{i,j=1}^n$. We note that $\mathcal{P}$ commutes with differential operators and thus $\mathcal{P}\mathrm{div}\,(u\otimes v) = \mathrm{div}\,\mathcal{P}(u\otimes v)$. The same is true for $\mathcal{P}_\perp$. The operator K is well-defined as we see in the next lemma.

Lemma 3.3. *Let $n/2+1 \le r < n$. Suppose $F_s \in L^{n/2}(\mathbb{R}^n)$ and $F_p \in L^r_{per,\perp}(\mathbb{R}\times\mathbb{R}^n)$. The operator K maps $\dot{H}^1_{n/2}(\mathbb{R}^n)\times X_r$ to itself with the estimate*

$$\|K(v_s,v_p)\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} \le C\left(\|F_s\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} + \|F_p\|_r + \|\{v_s,v_p\}\|^2_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r}\right)$$

for some constant $C = C(n,r,T)$.

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1. For small constants $\delta > 0$ and $\mu > 0$ to be determined later, we assume that $\|F_s\|_{L^{n/2}(\mathbb{R}^n)} + \|F_p\|_r \le \delta$ and let $\overline{B_\mu} \subset \dot{H}^1_{n/2}(\mathbb{R}^n) \times X_r$ be a closed ball in $\dot{H}^1_{n/2}(\mathbb{R}^n) \times X_r$ centered at 0 with radius μ. We show that K is a contraction map from $\overline{B_\mu}$ to itself.

For $\{v_s, v_p\} \in \overline{B_\mu}$, we have by Lemma 3.3

$$\begin{aligned}\|K(v_s, v_p)\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} &\le C_1 \left(\|F_s\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} + \|F_p\|_r + \|\{v_s, v_p\}\|^2_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} \right) \\ &\le C_1(\delta + \mu^2),\end{aligned}$$

where C_1 is the constant in Lemma 3.3.

Let $\{v_{1s}, v_{1p}\}, \{v_{2s}, v_{2p}\} \in \overline{B_\mu}$. We put $w_s := v_{1s} - v_{2s}$ and $w_p := v_{1p} - v_{2p}$. Standard calculation yields

$$\begin{aligned}&\|K_s(F_s - v_{1s}\otimes v_{1s} - \mathcal{P}(v_{1p}\otimes v_{1p})) - K_s(F_s - v_{2s}\otimes v_{2s} - \mathcal{P}(v_{2p}\otimes v_{2p}))\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)} \\ &\le C\|v_{1s}\otimes w_s + w_s \otimes v_{2s} + \mathcal{P}(v_{1p}\otimes w_p + w_p\otimes v_{2p})\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \\ &\le C\left(\|\{v_{1s}, v_{1p}\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} + \|\{v_{2s}, v_{2p}\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r}\right) \|\{w_s, w_p\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} \\ &\le 2C\mu \|\{w_s, w_p\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\|K_p(F_p - v_{1s}\otimes v_{1p} - v_{1p}\otimes v_{1s} - \mathcal{P}_\perp(v_{1p}\otimes v_{1p})) \\ &\qquad - K_p(F_p - v_{2s}\otimes v_{2p} - v_{2p}\otimes v_{2s} - \mathcal{P}_\perp(v_{2p}\otimes v_{2p}))\|_{X_r} \\ &\le C\left\| w_s\otimes v_{1p} + v_{2p}\otimes w_s + w_p \otimes v_{1s} + v_{2s}\otimes w_p + \mathcal{P}_\perp(w_p\otimes v_{1p} + v_{2p}\otimes w_p)\right\|_r \\ &\le C\left(\|\{v_{1s}, v_{1p}\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} + \|\{v_{2s}, v_{2p}\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r}\right) \|\{w_s, w_p\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} \\ &\le 2C\mu \|\{w_s, w_p\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r}.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we can verify

$$\|K(v_{1s}, v_{1p}) - K(v_{2s}, v_{2p})\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} \le 2C_1\mu \|\{v_{1s}, v_{1p}\} - \{v_{2s}, v_{2p}\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r}.$$

Now, we set

$$\delta = \frac{2}{9C_1^2}, \quad \mu = \frac{1}{3C_1}, \tag{3.5}$$

so that

$$\begin{aligned}\|K(v_s, v_p)\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} &\le \mu, \\ \|K(v_{1s}, v_{1p}) - K(v_{2s}, v_{2p})\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r} &\le \frac{2}{3}\|\{v_{1s}, v_{1p}\} - \{v_{2s}, v_{2p}\}\|_{\dot{H}^1_{\frac{n}{2}}(\mathbb{R}^n)\times X_r}.\end{aligned}$$

Therefore, K is a contraction map from $\overline{B_\mu} \subset \dot{H}^1_{n/2}(\mathbb{R}^n) \times X_r$ to itself, provided that δ and μ satisfy (3.5). Hence, there exists a unique fixed point $\{v_s, v_p\} \in \overline{B_\mu}$ such that

$$\{v_s, v_p\} = K(v_s, v_p),$$

that is,

$$\begin{aligned} v_s &= K_s(F_s - v_s \otimes v_s - \mathcal{P}(v_p \otimes v_p)), \\ v_p &= K_p(F_p - v_s \otimes v_p - v_p \otimes v_s - \mathcal{P}_\perp(v_p \otimes v_p)). \end{aligned}$$

By the definitions of K_s and K_p, we deduce that $v_s \in \dot{H}^1_{n/2}(\mathbb{R}^n)$ and $v_p \in X_r$ are solutions of (1.2) and (1.3) respectively. The existence of associated pressures $q_s \in L^{n/2}(\mathbb{R}^n)$ and $q_p \in L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$ follows from Lemmas 3.1 and 3.2. Consequently, the pair $\{v, q\} := \{v_s + v_p, q_s + q_p\}$ is a solution of (1.1).

Since $v_p \in L^r_{per,\perp}(\mathbb{R}; W^{1,r}(\mathbb{R}^n))$ is a solution of the Stokes equation (3.2) with F replaced by $F_p - v_s \otimes v_p - v_p \otimes v_s - \mathcal{P}_\perp(v_p \otimes v_p) \in L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$, we see by Proposition 3.1 that

$$v_p \in L^{\alpha_\lambda, r}_{per,\perp}(\mathbb{R}; L^{\beta_\lambda, r}(\mathbb{R}^n))$$

for all $0 \le \lambda \le 1$. The regularity property of v_p follows from the next lemma which can be proved via the bootstrap argument. □

Lemma 3.4. *Let $n/2 + 1 \le r < n$ and $v_s \in L^n(\mathbb{R}^n)$ with div $v_s = 0$ in $\mathbb{R}^n$. Suppose that $\{v_p, q_p\} \in X_r \times L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$ is a solution of (1.3) with $F_p \in L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$. If $F_p \in L^s_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$ for some $1 < s < r$, then the solution $\{v_p, q_p\}$ satisfies*

$$v_p \in L^s_{per,\perp}(\mathbb{R}; W^{1,s}(\mathbb{R}^n)) \cap L^{\alpha_\lambda, s}_{per,\perp}(\mathbb{R}; L^{\beta_\lambda, s}(\mathbb{R}^n)), \quad q_p \in L^s_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$$

for all $0 \le \lambda \le 1$.

4 Uniqueness

In this section, we consider the uniqueness of solutions to (1.1) constructed in Theorem 2.1. Suppose that the pairs $\{u, p\}$ and $\{v, q\}$ are solutions of (1.1) having the properties (2.3) and (2.4). The difference $w := u - v$ and $\pi := p - q$ obey the equation

$$(4.1) \qquad \begin{cases} \partial_t w - \Delta w + w \cdot \nabla u + v \cdot \nabla w + \nabla \pi = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ \text{div } w = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ w(t, \cdot) = w(t + T, \cdot) & \text{for all } t \in \mathbb{R}. \end{cases}$$

As in the previous section, we analyze this equation by decomposing it into two equations which the steady and purely periodic parts of $\{w, \pi\}$ should satisfy:

$$(4.2) \qquad \begin{cases} -\Delta w_s + w_s \cdot \nabla u_s + v_s \cdot \nabla w_s + \nabla \pi_s = -\mathcal{P}(w_p \cdot \nabla u_p + v_p \cdot \nabla w_p) & \text{in } \mathbb{R}^n, \\ \text{div } w_s = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

and
(4.3)
$$\begin{cases} \partial_t w_p - \Delta w_p + \mathcal{P}_\perp(w_p\cdot\nabla u + v\cdot\nabla w_p) + \nabla\pi_p = -(w_s\cdot\nabla u_p + v_p\cdot\nabla w_s) & \text{in } \mathbb{R}\times\mathbb{R}^n,\\ \text{div } w_p = 0 & \text{in } \mathbb{R}\times\mathbb{R}^n,\\ w_p(t,\cdot) = w_p(t+T,\cdot) & \text{for all } t\in\mathbb{R}. \end{cases}$$

Here w_s, u_s, v_s, π_s and w_p, u_p, v_p, π_p denote the steady and purely periodic parts of w, u, v, π respectively. Note that $\mathcal{P}_\perp(w_p\cdot\nabla u + v\cdot\nabla w_p) = w_p\cdot\nabla u_s + v_s\cdot\nabla w_p + \mathcal{P}_\perp(w_p\cdot\nabla u_p + v_p\cdot\nabla w_p)$. We investigate the equations (4.2) and (4.3) to get some useful information on w_s and w_p. This yields the information on w and we take w as a test function in the weak form of (4.1):

$$\frac{1}{T}\int_0^T -(w,\partial_t\varphi) + (\nabla w,\nabla\varphi) - (w\otimes u,\nabla\varphi) - (v\otimes w,\nabla\varphi) - (\pi,\text{div }\varphi)\,dt = 0 \tag{4.4}$$

for all $\varphi\in C^\infty_{0,per}(\mathbb{R}\times\mathbb{R}^n)$. Here, $(\cdot,\cdot)$ denotes the duality pairing on $\mathbb{R}^n$.

We begin with the analysis of (4.2). Our purpose in the analysis of (4.2) is to establish the L^2 property of the solution $\{w_s,\pi_s\}$. In order to show the regularity property of solutions to (4.2), we employ the idea introduced in [14]. We decompose $v_s\in L^n(\mathbb{R}^n)$ into a small part $\theta_{1,\epsilon}$ and a regular part $\theta_{2,\epsilon}$. This decomposition is indeed possible. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^n(\mathbb{R}^n)$, for each $\epsilon>0$ there exists $\psi_\epsilon\in C_0^\infty(\mathbb{R}^n)$ such that $\|v_s-\psi_\epsilon\|_{L^n(\mathbb{R}^n)}\le\epsilon$. Setting $\theta_{1,\epsilon} := v_s-\psi_\epsilon$ and $\theta_{2,\epsilon} := \psi_\epsilon$, we get

$$v_s = \theta_{1,\epsilon}+\theta_{2,\epsilon},\qquad \|\theta_{1,\epsilon}\|_{L^n(\mathbb{R}^n)}\le\epsilon,\qquad \theta_{2,\epsilon}\in C_0^\infty(\mathbb{R}^n). \tag{4.5}$$

Note furthermore that $v_s\cdot\nabla w_s = \text{div }(\theta_{1,\epsilon}\otimes w_s) + \text{div }(\theta_{2,\epsilon}\otimes w_s)$. With this decomposition of v_s in hand, we consider the regularity property of the perturbed Stokes equation

$$\begin{cases} -\Delta w + w\cdot\nabla u + \text{div }(\theta_{1,\epsilon}\otimes w) + \nabla\pi = \text{div } F & \text{in } \mathbb{R}^n,\\ \text{div } w = 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{4.6}$$

For the proof of the next lemma, see [14, Lemma 4.2].

Lemma 4.1. *Let $1<r_0,r_1<n$ and let $u,\theta_{1,\epsilon}\in L^n(\mathbb{R}^n)$ with (4.5). Suppose that $\{w,\pi\}\in \dot H^1_{r_0}(\mathbb{R}^n)\times L^{r_0}(\mathbb{R}^n)$ is a solution of (4.6) with $F\in L^{r_0}(\mathbb{R}^n)\cap L^{r_1}(\mathbb{R}^n)$. There exist constants $\delta=\delta(n,r_0,r_1)>0$ and $\tilde\delta=\tilde\delta(n,r_0,r_1)>0$ such that if*

$$\|u\|_{L^n(\mathbb{R}^n)}\le\delta,\qquad \epsilon\le\tilde\delta,$$

then

$$w\in\dot H^1_{r_0}(\mathbb{R}^n)\cap\dot H^1_{r_1}(\mathbb{R}^n),\qquad \pi\in L^{r_0}(\mathbb{R}^n)\cap L^{r_1}(\mathbb{R}^n)$$

with the estimate

$$\|\nabla w\|_{L^{r_0}(\mathbb{R}^n)\cap L^{r_1}(\mathbb{R}^n)} + \|\pi\|_{L^{r_0}(\mathbb{R}^n)\cap L^{r_1}(\mathbb{R}^n)} \le C\|F\|_{L^{r_0}(\mathbb{R}^n)\cap L^{r_1}(\mathbb{R}^n)}$$

where $C=C(n,r_0,r_1)$.

Using Lemma 4.1, we get the desired L^2 property of the solution $\{w_s, \pi_s\}$ to (4.2).

Lemma 4.2. *Let $n = 3$. Let $u_s, v_s \in L^3(\mathbb{R}^3)$ and, for $5/2 < r < 3$, let $w_p, u_p, v_p \in L^r_{per,\perp}(\mathbb{R}; W^{1,r}(\mathbb{R}^3))$. Also, assume that div $v_s = 0$ in $\mathbb{R}^3$ and div $w_p =$ div $v_p = 0$ in $\mathbb{R} \times \mathbb{R}^3$. Suppose that $\{w_s, \pi_s\} \in \dot{H}^1_{3/2}(\mathbb{R}^3) \times L^{3/2}(\mathbb{R}^3)$ is a solution of (4.2). There exists an absolute constant $\delta > 0$ such that if*

$$\|u_s\|_{L^3(\mathbb{R}^3)} \le \delta,$$

then

$$w_s \in \dot{H}^1_2(\mathbb{R}^3), \quad \pi_s \in L^2(\mathbb{R}^3).$$

Next, we study the equation (4.3). As in the case of steady equation (4.2), we establish the L^2 property of the solution $\{w_p, \pi_p\}$ to (4.3). We use the notation X_r for $n/2+1 < r < n$ appearing in the previous section again:

$$X_r = L^r_{per,\perp}(\mathbb{R}; W^{1,r}(\mathbb{R}^n)) \cap L^\infty_{per,\perp}(\mathbb{R}; L^n(\mathbb{R}^n)).$$

Lemma 4.3. *Let $n = 3, 4$ and $n/2+1 < r < n$. Let $w_s \in L^n(\mathbb{R}^n)$ and $u, v \in L^\infty_{per}(\mathbb{R}; L^n(\mathbb{R}^n))$ with $u_p, v_p \in L^r_{per,\perp}(\mathbb{R}; W^{1,r}(\mathbb{R}^n))$. Also, assume that div $w_s = 0$ in $\mathbb{R}^3$ and div $v = 0$ in $\mathbb{R} \times \mathbb{R}^3$. If $\{w_p, \pi_p\} \in X_r \times L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$ is a solution of (4.3), then*

$$w_p \in L^q_{per,\perp}(\mathbb{R}; W^{1,q}(\mathbb{R}^n)), \quad \pi_p \in L^q_{per,\perp}(\mathbb{R} \times \mathbb{R}^n) \quad \text{for all } \frac{nr}{n+r} \le q \le r.$$

In particular, we have

$$w_p \in L^2_{per,\perp}(\mathbb{R}; W^{1,2}(\mathbb{R}^n)), \quad \pi_p \in L^2_{per,\perp}(\mathbb{R} \times \mathbb{R}^n).$$

The weak form (4.4) contains the term $\partial_t \varphi$ and thus we need information on the class of $\partial_t w$ to take w as a test function. However, we thus far have no information on the class of $\partial_t w$ $(= \partial_t w_p)$. We can overcome this difficulty thanks to the good uniqueness property of purely periodic solutions to the Stokes equation (3.2), see Remark 3.1. In order to obtain the information on the class of $\partial_t w_p$, we show that the solution $\{w_p, \pi_p\} \in X_r \times L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$ of (4.3) is indeed a strong solution.

We review the existence and uniqueness of strong solutions to the Stokes equation

$$\begin{cases} \partial_t v - \Delta v + \nabla q = f & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ \text{div } v = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ v(\cdot, x) \to 0 & \text{as } |x| \to \infty, \\ v(t, \cdot) = v(t+T, \cdot) & \text{for all } t \in \mathbb{R}. \end{cases} \tag{4.7}$$

Lemma 4.4 ([11, 5])**.** *Let $1 < r < n$. For every $f \in L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$, there exists a unique solution $\{v, q\} \in W^{1,2,r}_{per,\perp}(\mathbb{R} \times \mathbb{R}^n) \times L^r_{per,\perp}(\mathbb{R}; \dot{H}^1_r(\mathbb{R}^n))$ of (4.7) such that*

$$\|v\|_{1,2,r} + \|\nabla q\|_r \le C\|f\|_r$$

with $C = C(n, r, T)$.

We regard the equation (4.3) as the Stokes equation (4.7) with $f = -w_s \cdot \nabla u_p - v_p \cdot \nabla w_s - w_p \cdot \nabla u_s - v_s \cdot \nabla w_p - \mathcal{P}_\perp(w_p \cdot \nabla u_p + v_p \cdot \nabla w_p)$ and use the uniqueness of purely periodic solutions to the Stokes equation in order to get the following lemma.

Lemma 4.5. *Let $n = 3, 4$ and $n/2 + 1 < r < n$. Let $u_s, v_s, w_s \in \dot{H}^1_{n/2}(\mathbb{R}^n)$ and $u_p, v_p \in X_r$. Suppose $\{w_p, \pi_p\} \in X_r \times L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^n)$ is a solution of (4.3). Then we have*

$$w_p \in W^{1,2,\frac{nr}{n+r}}_{per,\perp}(\mathbb{R} \times \mathbb{R}^n), \quad \pi_p \in L^{\frac{nr}{n+r}}_{per,\perp}(\mathbb{R}; \dot{H}^1_{\frac{nr}{n+r}}(\mathbb{R}^n)).$$

The next lemmas on the density property can be proved in a standard manner via the mollification.

Lemma 4.6. *Let $1 < r_0, r_1 < \infty$. Suppose $w \in L^{r_0}_{per}(\mathbb{R}; \dot{H}^1_{r_0}(\mathbb{R}^n))$ with $\partial_t w \in L^{r_1}_{per}(\mathbb{R} \times \mathbb{R}^n)$. There exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset C^\infty_{0,per}(\mathbb{R} \times \mathbb{R}^n)$ such that*

$$\begin{aligned} \nabla \varphi_n &\to \nabla w \qquad \text{in } L^{r_0}_{per}(\mathbb{R} \times \mathbb{R}^n), \\ \partial_t \varphi_n &\to \partial_t w \qquad \text{in } L^{r_1}_{per}(\mathbb{R} \times \mathbb{R}^n), \end{aligned}$$

as $n \to \infty$.

Lemma 4.7. *Let $1 < r_0, r_1 < \infty$. Suppose $w \in L^{r_0}_{per}(\mathbb{R} \times \mathbb{R}^n)$ with $\partial_t w \in L^{r_1}_{per}(\mathbb{R} \times \mathbb{R}^n)$. There exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset C^\infty_{0,per}(\mathbb{R} \times \mathbb{R}^n)$ such that*

$$\begin{aligned} \varphi_n &\to w \qquad \text{in } L^{r_0}_{per}(\mathbb{R} \times \mathbb{R}^n), \\ \partial_t \varphi_n &\to \partial_t w \qquad \text{in } L^{r_1}_{per}(\mathbb{R} \times \mathbb{R}^n), \end{aligned}$$

as $n \to \infty$.

We are now in a position to give the proof of Theorem 2.2.

Proof of Theorem 2.2. We put $w := u - v$ and $\pi := p - q$. The pair $\{w, \pi\}$ is a solution of (4.1). We decompose u, v and $\{w, \pi\}$ into steady and purely periodic parts. Then the pairs $\{w_s, \pi_s\} \in \dot{H}^1_{3/2}(\mathbb{R}^3) \times L^{3/2}(\mathbb{R}^3)$ and $\{w_p, \pi_p\} \in X_r \times L^r_{per,\perp}(\mathbb{R} \times \mathbb{R}^3)$ are solutions of (4.2) and (4.3) respectively. We assume that

$$\|u_s\|_{L^3(\mathbb{R}^3)} \le \delta_1,$$

where δ_1 is the absolute constant in Lemma 4.2, and it follows from Lemma 4.2 that

$$w_s \in \dot{H}^1_2(\mathbb{R}^3), \quad \pi_s \in L^2(\mathbb{R}^3). \tag{4.8}$$

Furthermore, Lemmas 4.3 and 4.5 yield

$$w_p \in L^2_{per,\perp}(\mathbb{R}; W^{1,2}(\mathbb{R}^3)), \quad \partial_t w_p \in L^{\frac{3r}{r+3}}_{per,\perp}(\mathbb{R} \times \mathbb{R}^3), \quad \pi_p \in L^2_{per,\perp}(\mathbb{R} \times \mathbb{R}^3). \tag{4.9}$$

Since $W^{1,2}(\mathbb{R}^3) \subset \dot{H}^1_2(\mathbb{R}^3)$, we deduce from (4.8) and (4.9) that $w = w_s + w_p$ and $\pi = \pi_s + \pi_p$ satisfy

$$w \in L^2_{per}(\mathbb{R}; \dot{H}^1_2(\mathbb{R}^3)), \quad \partial_t w \in L^{\frac{3r}{r+3}}_{per,\perp}(\mathbb{R} \times \mathbb{R}^3), \quad \pi \in L^2_{per}(\mathbb{R} \times \mathbb{R}^3).$$

In addition, we have $w_s \in L^{3r/(2r-3)}(\mathbb{R}^3)$ and $w_p \in L^{3r/(2r-3)}_{per,\perp}(\mathbb{R}\times\mathbb{R}^3)$. The former property is a consequence of $\dot{H}^1_{3/2}(\mathbb{R}^3)\cap\dot{H}^1_2(\mathbb{R}^3) \subset \dot{H}^1_{r/(r-1)}(\mathbb{R}^3)$ and the Sobolev inequality $\dot{H}^1_{r/(r-1)}(\mathbb{R}^3) \subset L^{3r/(2r-3)}(\mathbb{R}^3)$. The latter property follows from Proposition 3.1 and the choice $\lambda = 3-6/r < 1$. Hence we have

$$w \in L^{\frac{3r}{2r-3}}_{per}(\mathbb{R}\times\mathbb{R}^3).$$

We also observe that $\nabla w + w\otimes u + v\otimes w \in L^2_{per}(\mathbb{R}\times\mathbb{R}^3)$, since $u, v \in L^\infty_{per}(\mathbb{R}; L^3(\mathbb{R}^3))$ and $w \in L^2_{per}(\mathbb{R}; L^6(\mathbb{R}^3))$.

According to Lemma 4.6, there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset C^\infty_{0,per}(\mathbb{R}\times\mathbb{R}^n)$ such that $\nabla\varphi_n \to \nabla w$ in $L^2_{per}(\mathbb{R}\times\mathbb{R}^3)$ and $\partial_t\varphi_n \to \partial_t w$ in $L^{3r/(r+3)}_{per}(\mathbb{R}\times\mathbb{R}^3)$ as $n\to\infty$. We take φ_n as a test function in (4.4) and pass to the limit $n\to\infty$ to get

$$\frac{1}{T}\int_0^T -(w,\partial_t w) + \|\nabla w\|^2_{L^2(\mathbb{R}^3)} - (w\otimes u, \nabla w) - (v\otimes w,\nabla w)\,dt = 0. \tag{4.10}$$

In view of Lemma 4.7, we can take a sequence $\{\tilde{\varphi}_n\}_{n=1}^\infty \subset C^\infty_{0,per}(\mathbb{R}\times\mathbb{R}^n)$ such that $\tilde{\varphi}_n \to w$ in $L^{3r/(2r-3)}_{per}(\mathbb{R}\times\mathbb{R}^3)$ and $\partial_t\tilde{\varphi}_n \to \partial_t w$ in $L^{3r/(r+3)}_{per}(\mathbb{R}\times\mathbb{R}^3)$ as $n\to\infty$. The integration by parts yields

$$\int_0^T (w,\partial_t\tilde{\varphi}_n)\,dt = -\int_0^T(\partial_t w,\tilde{\varphi}_n)\,dt + (w(T),\tilde{\varphi}_n(T)) - (w(0),\tilde{\varphi}_n(0)).$$

Since w and $\tilde{\varphi}_n$ are T-periodic, we pass to the limit $n\to\infty$ to deduce

$$\int_0^T (w,\partial_t w)\,dt = -\int_0^T(\partial_t w, w)\,dt.$$

Hence, we get

$$\int_0^T (w,\partial_t w)\,dt = 0. \tag{4.11}$$

Also, the integration by parts yields $(v\otimes w,\nabla w) = 0$ and thus

$$\int_0^T (v\otimes w,\nabla w)\,dt = 0. \tag{4.12}$$

Combining (4.10), (4.11) and (4.12), we deduce

$$\|\nabla w\|_2^2 = \frac{1}{T}\int_0^T (w\otimes u,\nabla w)\,dt.$$

Therefore,

$$\|\nabla w\|_2^2 \le \|w\|_{2,6}\|u\|_{\infty,3}\|\nabla w\|_2 \le C_1\|u\|_{\infty,3}\|\nabla w\|_2^2.$$

Here C_1 is the constant in the Sobolev inequality $\|\cdot\|_{L^6(\mathbb{R}^3)} \le C_1\|\nabla\cdot\|_{L^2(\mathbb{R}^3)}$. If u satisfies

$$\|u\|_{\infty,3} < C_1^{-1},$$

we obtain

$$\|\nabla w\|_2^2 = 0. \tag{4.13}$$

Now, we select the constant $\tilde{\delta}$ so that

$$0 < \tilde{\delta} < \min\{\delta_1, C_1^{-1}\}$$

and assume

$$\|u_s\|_{L^3(\mathbb{R}^3)} + \|u_p\|_{\infty,3} \leq \tilde{\delta}.$$

Then, all the arguments above are justified. We deduce from (4.13) that $w = h(t)$ with T-periodic function h. By the class of w, we derive $w = 0$, that is, $u = v$ in $\mathbb{R} \times \mathbb{R}^3$. This together with $(4.1)_1$ gives $\nabla \pi = 0$ and thus π is a T-periodic function with one variable t. Since $\pi \in L^2_{per}(\mathbb{R} \times \mathbb{R}^3)$, we derive $\pi = 0$, that is, $p = q$ in $\mathbb{R} \times \mathbb{R}^3$. The proof is complete. □

References

[1] T. Eiter, M. Kyed, Y. Shibata, *On periodic solutions for one-phase and two-phase problems of the Navier-Stokes equations*, J. Evol. Equ. **21** (2021), 2955-3014.

[2] R. Farwig, T. Nakatsuka, Y. Taniuchi, *Uniqueness of solutions on the whole time axis to the Navier-Stokes equations in unbounded domains*, Comm. Partial Differential Equations **40** (2015), 1884–1904.

[3] G.P. Galdi, C.G. Simader, *Existence, uniqueness and L^q-estimates for the Stokes problem in exterior domains*, Arch. Ration. Mech. Anal. **112** (1990), 291-318.

[4] G. P. Galdi, H. Sohr, *Existence and uniqueness of time-periodic physically reasonable Navier-Stokes flow past a body*, Arch. Ration. Mech. Anal. **172** (2004), 363-406.

[5] G. P. Galdi, M. Kyed, Time-periodic solutions to the Navier-Stokes equations, Handbook of mathematical analysis in mechanics of viscous fluids, Springer, 2018, 509–578.

[6] G. P. Galdi, M. Kyed, *Time-periodic solutions to the Navier-Stokes equations in the three-dimensional whole-space with a non-zero drift term: asymptotic profile at spatial infinity*, Mathematical analysis in fluid mechanics: selected recent results. American Mathematical Society, Contemporary Mathematics 710, (2018) 121-144.

[7] G. P. Galdi, M. Kyed, *Time-periodic flow of a viscous liquid past a body*, Partial differential equations in fluid mechanics, Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 452, (2018), 20-49.

[8] H. Kozono, M. Nakao, *Periodic solutions of the Navier-Stokes equations in unbounded domains*, Tohoku Math. J., **48** (1996), 33-50.

[9] H. Kozono, H. Sohr, *On a new class of generalized solutions for the Stokes equations in exterior domains*, Ann. Scuola Norm. Sup. Pisa **19** (1992), 155-181.

[10] M. Kyed, *Time-Periodic Solutions to the Navier-Stokes Equations*, Habilitationsschrift, Technische Universität Darmstadt, 2012.

[11] M. Kyed, *Maximal regularity of the time-periodic linearized Navier-Stokes system*, J. Math. Fluid Mech. **16** (2014), 523-538.

[12] M. Kyed, *The existence and regularity of time-periodic solutions to the three-dimensional Navier-Stokes equations in the whole space*, Nonlinearity **27** (2014), 2909-2935.

[13] P. Maremonti, *Existence and stability of time-periodic solutions to the Navier-Stokes equations in the whole space*, Nonlinearity **4** (1991), 503-529.

[14] T. Nakatsuka, *Uniqueness of steady Navier-Stokes flows in exterior domains*, Funkcial. Ekvac. **56** (2013), 323–337.

[15] T. Nakatsuka, *On time-periodic Navier-Stokes flows with fast spatial decay in the whole space*, J. Elliptic Parabol. Equ. **4** (2018), 51-67.

[16] T. Nakatsuka, *On pointwise decay rates of time-periodic solutions to the Navier-Stokes equation*, Math. Nachr. **294** (2021), 98–117.

[17] T. Nakatsuka, *On unique solvability of the time-periodic problem for the Navier-Stokes equation*, submitted.

[18] Y. Taniuchi, *On the uniqueness of time-periodic solutions to the Navier-Stokes equations in unbounded domains*, Math. Z. **261** (2009), 597-615.

[19] M. Yamazaki, *The Navier-Stokes equations in the weak-L^n space with time-dependent external force*, Math. Ann. **317** (2000), 635-675.

Tomoyuki Nakatsuka

Department of Economics
Matsuyama University
790-8578 Matsuyama, Japan
E-mail: t.nakatsuka@g.matsuyama-u.ac.jp