Anisotropically weighted L^q-L^r estimates of the Oseen semigroup in exterior domains, with applications to the Navier-Stokes flow past a rigid body

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1 Introduction

We consider a viscous incompressible flow past a rigid body $\mathscr{O} \subset \mathbb{R}^3$. We suppose that $\mathscr{O}$ is translating with a velocity $-\psi(t)ae_1$, where $a > 0$, $e_1 = (1, 0, \cdots, 0)^\top$. Then by taking frame attached to the body, the fluid motion which occupies the outside of $\mathscr{O}$ obeys

$$\begin{cases} \partial_t u + u \cdot \nabla u = \Delta u - a\partial_{x_1} u - \nabla p, & x \in D,\ t > 0, \\ \nabla \cdot u = 0, & x \in D,\ t \geq 0, \\ u|_{\partial D} = -ae_1, & t > 0, \\ u \to 0 & \text{as } |x| \to \infty, \\ u(x, 0) = u_0, & x \in D \end{cases} \tag{1.1}$$

where $D = \mathbb{R}^3 \setminus \mathscr{O}$ is the exterior domain with C^2 smooth boundary ∂D and the origin of coordinate is assumed to be contained in the interior of $\mathscr{O}$. The functions $u = (u_1(x, t), u_2(x, t), u_3(x, t))^\top$ and $p = p(x, t)$ denote unknown velocity and pressure of the fluid, respectively, while u_0 is a given initial velocity. The large time behavior of solutions to (1.1) is related to the stationary problem

$$\begin{cases} u_s \cdot \nabla u_s = \Delta u_s - a\,\partial_{x_1} u_s - \nabla p_s, & x \in D, \\ \nabla \cdot u_s = 0, & x \in D, \\ u_s|_{\partial D} = -ae_1, & \\ u_s \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.2}$$

The pioneer work due to Leray [27] provided the existence theorem for weak solution to (1.2), what is called D-solution, having finite Dirichlet integral without smallness of data, however, his solution had little information about the anisotropic decay structure caused by the translation. Later on, Finn [10, 12–14] succeeded in constructing a stationary solution, termed by him physically reasonable solution, exhibiting a paraboloidal wake region behind the body, that is,

$$u_s(x) = O((1+|x|)^{-1}(1+|x|-x_1)^{-1}) \tag{1.3}$$

if a is small enough. The L^2 stability of u_s, that is the problem (1.1) with $u_0 = u_s + b$ ($b \in L^2(D)$: perturbation), was first proved by Heywood [17, 18]. On the other hand, by the decay structure (1.3), we have $u_s \in L^q(D)$ for $q > 2$, but $u_s \notin L^2(D)$ in general due to Finn [11], see also Galdi [16], thus it is reasonable to seek a solution to (1.1) in the L^q framework. The L^q stability was proved by Shibata [30] (see also Enomoto and Shibata [7]), in which the key is the L^q-L^r estimate of the Oseen semigroup developed by Kobayashi and Shibata [25] (see also Enomoto and Shibata [6, 7]).

As in the stationary problem (1.2), we expect that nonstationary solutions to (1.1) exhibit the paraboloidal wake region, but the literature for concerning this issue is Knightly [22], Mizumachi [29] and Deuring [4, 5] only. Deuring [4] used a representation formula for the solution to the Oseen system to deduce

$$\nabla^i u(x,t) = O((1+|x|)^{-1-\frac{i}{2}}(1+|x|-x_1)^{-1-\frac{i}{2}})$$

for $i = 0, 1$ uniformly in t under some assumptions on the initial perturbation from the stationary solution and on the solution u. In [5], by employing another integral representation, he also established the estimate

$$\nabla^i\big(u(x,t)-u_s\big) = O((1+|x|)^{-\frac{5}{4}-\frac{i}{2}}(1+|x|-x_1)^{-\frac{5}{4}-\frac{i}{2}})$$

for $t > 0$ without the boundary condition except the zero-flux condition.

As mentioned above, the wake structure uniform with respect to time has been investigated by [4, 5, 22, 29], while the purpose of the present paper is to derive temporal decay rate with the wake structure captured. To accomplish our purpose, we develop the theory of the Oseen semigroup in L^q spaces with the weight $(1+|x|)^\alpha(1+|x|-x_1)^\beta$, in particular, derive the anisotropically weighted L^q-L^r estimates of the Oseen semigroup. We then apply those estimates to construct a nonstationary solution to

$$\begin{cases} \partial_t v = \Delta v - a\partial_{x_1} v - v\cdot\nabla v - v\cdot\nabla u_s - u_s\cdot\nabla v - \nabla\phi, & x\in D,\ t>0,\\ \nabla\cdot v = 0, \quad x\in D,\ t\geq 0, \\ v|_{\partial D} = 0, \quad t>0, \\ v\to 0 \quad \text{as } |x|\to\infty, \\ v(x,0) = b := u_0 - u_s, \quad x\in D \end{cases} \tag{1.4}$$

in the anisotropically weighted L^q framework, where (v,ϕ) is defined by

$$u(x,t) = v(x,t) + u_s, \quad p(x,t) = \phi(x,t) + p_s.$$

Here, we note that the condition

$$-1 < \beta < q-1, \quad -3 < \alpha+\beta < 3(q-1) \tag{1.5}$$

is the necessary and sufficient condition on α, β so that $(1+|x|)^\alpha(1+|x|-x_1)^\beta$ belongs to the Muckenhoupt class $\mathscr{A}_q(\mathbb{R}^3)$, which ensures the weighted L^q boundedness of singular integral operators, see, for instance, García-Cuerva and Rubio de Francia [15, Chapter IV], Torchinsky [32, Chapter IX] and Stein [31, Chapter V]. The proof is accomplished by checking the definition of $\mathscr{A}_q(\mathbb{R}^3)$. This fact with $q=2$ was already known from Farwig [8] and the sufficiency of (1.5) was proved by Kračmar, Novotný and Pokorný [26]. We then employ such weights to apply the weighted L^q theory for the Stokes resolvent problem and the Helmholtz decomposition in weighted L^q spaces developed by Farwig and Sohr [9].

To establish the anisotropically weighted L^q-L^r estimates of the Oseen semigroup e^{-tA_a} in the exterior domain D, it is important to derive the estimates in $\mathbb{R}^3$. Given $q \le r \le \infty$ $(q \neq \infty)$ and $\alpha, \beta > 0$ satisfying $\beta < 1-1/q, \alpha+\beta < 3(1-1/q)$ (which ensures $(1+|x|)^{\alpha q}(1+|x|-x_1)^{\beta q} \in \mathscr{A}_q(\mathbb{R}^3)$), it follows that

$$\begin{aligned}
&\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \nabla^j S_a(t)f\|_{L^r(\mathbb{R}^3)} \\
&\quad \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{j}{2}+\frac{\alpha}{4}+\max\{\frac{\alpha}{4},\frac{\beta}{2}\}+\varepsilon}\|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{L^q(\mathbb{R}^3)}
\end{aligned} \tag{1.6}$$

for $t \ge 1, j = 0,1$, where $S_a(t)$ $(a > 0)$ is the Oseen semigroup in $\mathbb{R}^3$ and $\varepsilon > 0$ is a given positive constant. But, it seems difficult to apply (1.6) to construct a solution in the nonlinear problems. Therefore, in Proposition 3.1 we derive the other estimate

$$\begin{aligned}
&\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \nabla^j S_a(t)f\|_{L^r(\mathbb{R}^3)} \\
&\le Ct^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{r})-\frac{j}{2}}\|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{L^{q_1}(\mathbb{R}^3)} + Ct^{-\frac{3}{2}(\frac{1}{q_2}-\frac{1}{r})-\frac{j}{2}+\alpha}\|(1+|x|-x_1)^\beta f\|_{L^{q_2}(\mathbb{R}^3)} \\
&\quad + Ct^{-\frac{3}{2}(\frac{1}{q_3}-\frac{1}{r})-\frac{j}{2}+\frac{\beta}{2}}\|(1+|x|)^\alpha f\|_{L^{q_3}(\mathbb{R}^3)} + Ct^{-\frac{3}{2}(\frac{1}{q_4}-\frac{1}{r})-\frac{j}{2}+\alpha+\frac{\beta}{2}}\|f\|_{L^{q_4}(\mathbb{R}^3)}
\end{aligned} \tag{1.7}$$

for $t \ge 1, j = 0,1$ and $1 < q_i \le r \le \infty$ $(q_i \neq \infty)$. The estimate (1.6) is not employed in the nonlinear problems, but it is seen that the rate in (1.6) is better than the one in (1.7) with $q_i = q$ $(i = 1,2,3,4)$, that is $t^{-3(1/q-1/r)/2-j/2+\alpha+\beta/2}$. With (1.7) at hand, in Theorem 2.2, we derive the estimate of e^{-tA_a} in D. The proof consists of two steps: one is the decay estimate near the boundary of D; the other is the decay estimate near infinity. This procedure is employed by Iwashita [21], Kobayashi and Kubo [24] for the Stokes semigroup and by Kobayashi and Shibata [25], Enomoto and Shibata [6, 7] and Hishida [20] for the Oseen semigroup. In those papers, they derived the decay rate $t^{-3/2q}$ for given $f \in L^q(D)$ in the first step by carrying out a cut-off procedure based on the L^q-L^r estimates of $S_a(t)$ and on the decay estimate near the boundary of D for initial velocity with compact support, called the local energy decay estimate, see Proposition 3.2. However, since the temporal decay rate should be affected by the spatial decay structure of f, we expect that the decay rate is better than $t^{-3/(2q)}$ if f decays faster than $L^q(D)$, for instance, $(1+|x|)^\alpha(1+|x|-x_1)^\beta f \in L^q(D)$ $(\alpha > 0, \beta \ge 0)$. In fact, we adapt the same procedure as in those papers, but deduce better decay rate $t^{-3/(2q)-\eta}$

if $(1+|x|)^\alpha(1+|x|-x_1)^\beta f \in L^q(D)$ $(\alpha > 0, \beta \geq 0)$, where η is a positive constant dependent on α, β, see Proposition 3.4. The homogeneous estimates of the Stokes semigroup e^{-tA} in isotropically weighted L^q spaces are also deduced by our argument, see the assertion 2 of Theorem 2.2 and the assertion 2 of Theorem 2.3.

The application of Theorem 2.2 and Theorem 2.3 to the problem (1.4) is given by Theorem 2.4, in which the spatial-temporal estimates

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta v(t)\|_{r,D} = o(t^{-\frac{1}{2}+\frac{3}{2r}+\alpha+\frac{\beta}{2}}), \tag{1.8}$$

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \nabla v(t)\|_{3,D} = o(t^{-\frac{1}{2}+\alpha+\frac{\beta}{2}}) \tag{1.9}$$

as $t \to \infty$ for all $r \in [3,\infty]$ are deduced if the velocity a and the L^3 norm of initial perturbation b, which is of class $(1+|x|)^\alpha(1+|x|-x_1)^\beta b \in L^3(D)$, are small enough. The proof of this theorem is accomplished by adapting the argument due to Enomoto and Shibata [7] to analyze four norms appeared in the RHS of (1.7). We note that the smallness of $\|(1+|x|)^\alpha(1+|x|-x_1)^\beta b\|_{L^3(D)}$ is not assumed in this theorem. The rate in (1.8) with $\beta = 0$ is $-1/2+3/(2r) = -3(1/3-1/r)/2$, which is same as the one of the usual L^3-L^r estimate of the Oseen semigroup, and the loss α. This loss is less than the one in Bae and Roh [1].

The next section is devoted to stating the main theorems. In Section 3, we give the outline of the proof of Theorem 2.2 and Theorem 2.3.

2 Main theorems

In this section, we provide our main theorems. Given $1 < q < \infty$ and α, β satisfying

$$-\frac{1}{q} < \beta < 1-\frac{1}{q}, \quad -\frac{3}{q} < \alpha+\beta < 3\left(1-\frac{1}{q}\right), \tag{2.1}$$

we set

$$\rho(x) = (1+|x|)^{\alpha q}(1+|x|-x_1)^{\beta q}. \tag{2.2}$$

By checking the definition of the Muckenhoupt class, we find that the weight ρ belongs to $\mathscr{A}_q(D)$ as well as $\mathscr{A}_q(\mathbb{R}^3)$. Therefore, due to Farwig and Sohr [9], we have the Helmholtz decomposition and the bounded projection operator $P_D : L^q_\rho(D) \to L^q_{\rho,\sigma}(D)$, then define the Oseen operator $A_a : L^q_{\rho,\sigma}(D) \to L^q_{\rho,\sigma}(D)$ $(a \in \mathbb{R})$ by $\mathscr{D}(A_a) = W^{2,q}_\rho(D) \cap W^{1,q}_0(D) \cap L^q_{\rho,\sigma}(D)$, $A_a u = -P_D[\Delta u - a\partial_{x_1} u]$. We simply write the Stokes operator $A = A_0$. We already know from [9] that $-A$ generates an analytic C_0-semigroup (Stokes semigroup) in weighted L^q space whenever the weight belongs to $\mathscr{A}_q(D)$ and the Stokes semigroup is bounded in $L^q_{(1+|x|)^{\alpha q}}(D)$, see [9, Theorem 1.5]. Its L^q-L^r smoothing action near the initial time was derived by Kobayashi and Kubo [24, Theorem 1], see also [23]. We state in the following theorem that $-A_a$ generates an analytic C_0-semigroup in $L^q_{\rho,\sigma}(D)$ possessing the L^q-L^r smoothing action near the initial time.

Theorem 2.1. *Given $a_0 > 0$ arbitrarily, we assume $|a| \leq a_0$.*

1. *Let $1 < q < \infty$ and let α, β satisfy (2.1). Then $-A_a$ generates an analytic C_0-semigroup $\{e^{-tA_a}\}_{t\geq 0}$ in $L^q_{\rho,\sigma}(D)$, where ρ is given by (2.2). If in particular $a = 0$, then the Stokes semigroup $\{e^{-tA}\}_{t\geq 0}$ is a bounded analytic C_0-semigroup in $L^q_{\rho,\sigma}(D)$.*

2. *Let $1 < q \leq r \leq \infty$ ($q \neq \infty$) and let α, β satisfy (2.1). For every multi-index k ($|k| \leq 1$), there exists a constant $C = C(D, a_0, q, r, \alpha, \beta, k)$, independent of a, such that*

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \partial_x^k e^{-tA_a} P_D f\|_{r,D} \leq C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|k|}{2}} \|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{q,D}$$

for all $t \leq 3$, $f \in L^q_\rho(D)$.

In the following theorem, the assertion 1 asserts the large time behavior of the Oseen semigroup. We note that the exponents q_i $(i = 1, 2, 3, 4)$ in the next theorem may not coincide with each other. The assertion 2 yields the homogeneous estimates for the Stokes semigroup in isotropic L^q space.

Theorem 2.2. 1. *Given $a_0 > 0$ arbitrarily, we assume $a \in [0, a_0]$. Let $1 < q_i < \infty$ $(i = 1, 2, 3, 4)$, $1 < r \leq \infty$ and $\alpha, \beta \geq 0$ satisfy*

$$1 < q_4 \leq q_i \leq q_1 \leq r \leq \infty \quad (i = 2, 3), \tag{2.3}$$

$$\alpha < \min\left\{3\left(1-\frac{1}{q_3}\right), 1\right\}, \quad \beta < \min\left\{1-\frac{1}{q_2}, \frac{1}{3}\right\}, \quad \alpha+\beta < \min\left\{3\left(1-\frac{1}{q_1}\right), 1\right\}. \tag{2.4}$$

We set

$$\rho_1(x) = (1+|x|)^{\alpha q_1}(1+|x|-x_1)^{\beta q_1}, \quad \rho_2(x) = (1+|x|-x_1)^{\beta q_2}, \quad \rho_3(x) = (1+|x|)^{\alpha q_3}. \tag{2.5}$$

Then there exists a constant $C(D, a_0, q_1, q_2, q_3, q_4, r, \alpha, \beta)$, independent of a, such that

$$\begin{aligned} &\|(1+|x|)^\alpha(1+|x|-x_1)^\beta e^{-tA_a} P_D f\|_{r,D} \\ &\leq C \sum_{i=1}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})+\eta_i} \|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} f\|_{q_i,D} \end{aligned} \tag{2.6}$$

for all $t \geq 3$, $f \in \bigcap_{i=1}^{3} L^{q_i}_{\rho_i}(D) \cap L^{q_4}(D)$, where $\gamma_i, \delta_i, \eta_i$ are defined by

$$\begin{aligned} &(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (\alpha, 0, \alpha, 0), \quad (\delta_1, \delta_2, \delta_3, \delta_4) = (\beta, \beta, 0, 0), \\ &(\eta_1, \eta_2, \eta_3, \eta_4) = \left(0, \alpha, \frac{\beta}{2}, \alpha + \frac{\beta}{2}\right). \end{aligned} \tag{2.7}$$

2. *Let $a = 0$. Let $1 < q \leq r \leq \infty$ ($q \neq \infty$) and $\alpha \geq 0$ satisfy $0 \leq \alpha < \min\{3(1-1/q), 1\}$. Then there exists a constant $C(D, q, r, \alpha)$ such that*

$$\|(1+|x|)^\alpha e^{-tA} P_D f\|_{r,D} \leq C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \|(1+|x|)^\alpha f\|_{q,D}$$

for all $t > 0$ and $f \in L^q_{(1+|x|)^{\alpha q}}(D)$.

In order to study the nonlinear problem, we next deduce the estimate of the first derivative of e^{-tA_a}. For the Stokes semigroup e^{-tA} in the L^q framework, it was proved by Maremonti and Solonnikov [28] and Hishida [19] that the restriction $1 < q \le r \le 3$ is optimal in the sense that we cannot have

$$\|\nabla e^{-tA} P_D f\|_{r,D} \le C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|f\|_{q,D}$$

for $1 < q \le r < q_0$ with $q_0 > 3$. However, the next theorem yields that the range of exponent is enlarged in weighted L^q space than the one in L^q space. In particular, for the Stokes semigroup in $L^q_{(1+|x|)^{\alpha q}}(D)$, it is proved that the condition (2.9) below is optimal.

Theorem 2.3. *1. Given $a_0 > 0$ arbitrarily, we assume $a \in [0, a_0]$. Let $1 < r < \infty$, $1 < q_i < \infty$ $(i = 1, 2, 3, 4), \alpha, \beta > 0$ satisfy (2.4). If $\alpha < 2/3$ (resp. $\alpha \ge 2/3$), we suppose*

$$1 < q_4 \le q_i \le q_1 \le r < \min\left\{\frac{3}{1-\alpha-\beta}, \frac{3}{1-\frac{3\alpha}{2}}\right\} \quad (i = 2, 3)$$
$$\left(\text{resp. } 1 < q_4 \le q_i \le q_1 \le r < \frac{3}{1-\alpha-\beta} \quad (i = 2, 3)\right).$$

Then there exists a constant $C(D, a_0, q_1, q_2, q_3, q_4, r, \alpha, \beta)$, independent of a, such that

$$\begin{aligned}&\|(1+|x|)^\alpha (1+|x|-x_1)^\beta \nabla e^{-tA_a} P_D f\|_{r,D} \\ &\le C \sum_{i=1}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})-\frac{1}{2}+\eta_i} \|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} f\|_{q_i,D}\end{aligned} \tag{2.8}$$

for all $t \ge 3$ and $f \in \bigcap_{i=1}^{3} L^{q_i}_{\rho_i}(D) \cap L^{q_4}(D)$.

2. Let $a = 0$. Let $1 < q \le r \le \infty$ $(q \ne \infty)$ and $\alpha \ge 0$ satisfy $0 \le \alpha < \min\{3(1 - 1/q), 1\}$. We also suppose

$$1 < q \le r \le \frac{3}{1-\alpha}. \tag{2.9}$$

Then there exists a constant $C(D, q, r, \alpha)$ such that

$$\|(1+|x|)^\alpha \nabla e^{-tA} P_D f\|_{r,D} \le C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|(1+|x|)^\alpha f\|_{q,D} \tag{2.10}$$

for all $t > 0$ and $f \in L^q_{(1+|x|)^{\alpha q}}(D)$.

Let us proceed to the nonlinear problems. To study the stability of a stationary solution u_s, we consider the integral equation

$$v(t) = e^{-tA_a} b - \int_0^t e^{-(t-\tau)A_a} P_D\Big[v \cdot \nabla v + v \cdot \nabla u_s + u_s \cdot \nabla v\Big] d\tau. \tag{2.11}$$

For the problem (2.11), we have the following. The proof of this theorem is accomplished by adapting the argument due to Enomoto and Shibata [7].

Theorem 2.4. *Let α, β satisfy $\alpha \geq 0$, $0 \leq \beta < 1/3$, $\alpha+\beta < 1$. Then there exist constants $\kappa = \kappa(\alpha, \beta) > 0$ and $\varepsilon = \varepsilon(\alpha, \beta, a) > 0$ such that if $0 < a^{1/4} < \kappa$ and if $b \in L^3_{(1+|x|)^{3\alpha}(1+|x|-x_1)^{3\beta}}(D)$ fulfills $\|b\|_{3,D} < \varepsilon$, then the problem (2.11) admits a solution v which enjoys*

$$\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} v(t)\|_{q,D} = o(t^{-\frac{1}{2}+\frac{3}{2q}+\gamma_i+\frac{\delta_i}{2}}),$$
$$\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} \nabla v(t)\|_{3,D} = o(t^{-\frac{1}{2}+\gamma_i+\frac{\delta_i}{2}})$$

as $t \to \infty$ for all $q \in [3, \infty]$ and $i = 1, 2, 3, 4$, where γ_i, δ_i are given by (2.7).

3 Outline of proof of Theorem 2.2 and Theorem 2.3

This section is devoted to the anisotropically weighted L^q-L^r decay estimates of the Oseen semigroup in the exterior domain. We fitst prepare the anisotropically weighted L^q-L^r estimates of solutions to the Oseen equation in $\mathbb{R}^3$:

$$\partial_t u - \Delta u + a\partial_{x_1} u + \nabla p = 0, \quad \nabla \cdot u = 0, \quad x \in \mathbb{R}^3,\ t > 0, \qquad u(x, 0) = g, \quad x \in \mathbb{R}^3. \tag{3.1}$$

Given $g \in L^q(\mathbb{R}^3)$, we denote a solution to the heat equation by $e^{t\Delta}g$, which is given by the formula:

$$(e^{t\Delta}g)(x) = \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

If in particular, $g \in L^q_\sigma(\mathbb{R}^3)$, then we see that $\nabla p = 0$ and that

$$u(x, t) = (S_a(t)g)(x) := (e^{t\Delta}g)(x - ate_1) \tag{3.2}$$

solves the problem (3.1).

Proposition 3.1. *1. Given $a_0 > 0$ arbitrarily, we assume $a \in [0, a_0]$. Let q, r satisfy $1 < q \leq r \leq \infty$ $(q \neq \infty)$ and let $\alpha, \beta \geq 0$ satisfy*

$$\beta < 1 - \frac{1}{q}, \quad \alpha + \beta < 3\left(1 - \frac{1}{q}\right). \tag{3.3}$$

For multi-index k satisfying $|k| \leq 1$, there exists a constant $C = C(a_0, q, r, \alpha, \beta, k)$, independent of a, such that

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \partial_x^k S_a(t) P_{\mathbb{R}^3} f\|_{r,\mathbb{R}^3} \leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|k|}{2}} \|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{q,\mathbb{R}^3} \tag{3.4}$$

for all $t \leq 1$, $f \in L^q_\rho(\mathbb{R}^3)$.

2. *Given $a_0 > 0$ arbitrarily, we assume $a \in (0, a_0]$. Let q_i $(i = 1, 2, 3, 4), r$ satisfy $1 < q_i \leq r \leq \infty$ $(q_i \neq \infty, i = 1, 2, 3, 4)$ and let $\alpha, \beta \geq 0$ satisfy*

$$0 \leq \alpha < 3\left(1 - \frac{1}{q_3}\right), \quad 0 \leq \beta < \min\left\{1 - \frac{1}{q_1}, 1 - \frac{1}{q_2}\right\}, \quad \alpha + \beta < 3\left(1 - \frac{1}{q_1}\right).$$

For multi-index k satisfying $|k| \leq 1$, there exists a constant $C = C(a_0, q_1, q_2, q_3, q_4, r, \alpha, \beta, k)$, independent of a, such that

$$\begin{aligned}
&\|(1 + |x|)^\alpha (1 + |x| - x_1)^\beta \partial_x^k S_a(t) P_{\mathbb{R}^3} f\|_{r,\mathbb{R}^3} \\
&\leq C \sum_{i=1}^{4} t^{-\frac{3}{2}(\frac{1}{q_i} - \frac{1}{r}) - \frac{|k|}{2} + \eta_i} \|(1 + |x|)^{\gamma_i} (1 + |x| - x_1)^{\delta_i} f\|_{q_i,\mathbb{R}^3}
\end{aligned} \tag{3.5}$$

for all $t \geq 1$, $f \in \bigcap_{i=1}^{3} L^{q_i}_{\rho_i}(\mathbb{R}^3) \cap L^{q_4}(\mathbb{R}^3)$.

3. *Let $a = 0$. Let $1 < q \leq r \leq \infty$ $(q \neq \infty)$ and $0 \leq \alpha < 3(1 - 1/q)$. For multi-index k satisfying $|k| \leq 1$, there exists a constant $C = C(q, r, \alpha, k)$, such that*

$$\|(1 + |x|)^\alpha \partial_x^k S_0(t) P_{\mathbb{R}^3} f\|_{r,\mathbb{R}^3} \leq C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{|k|}{2}} \|(1 + |x|)^\alpha f\|_{q,\mathbb{R}^3} \tag{3.6}$$

for all $t > 0$ and $f \in L^q_{(1+|x|)^{\alpha q}}(\mathbb{R}^3)$.

Proof. From

$$\begin{aligned}
(1 + |x|)^\alpha (1 + |x| - x_1)^\beta \leq C\big\{&(1 + |y|)^\alpha (1 + |y| - y_1)^\beta + (1 + |x - y|)^\alpha (1 + |y| - y_1)^\beta \\
&+ (1 + |y|)^\alpha (1 + |x - y| - (x_1 - y_1))^\beta \\
&+ (1 + |x - y|)^\alpha (1 + |x - y| - (x_1 - y_1))^\beta\big\},
\end{aligned}$$

we have

$$\begin{aligned}
&(1 + |x|)^\alpha (1 + |x| - x_1)^\beta |\partial_x^k S_a(t) P_{\mathbb{R}^3} f(x)| \\
&\quad \leq C\Big\{G_{1,k} * \big((1 + |y|)^\alpha (1 + |y| - y_1)^\beta |P_{\mathbb{R}^3} f|\big)(x) + G_{2,k} * \big((1 + |y| - y_1)^\beta |P_{\mathbb{R}^3} f|\big)(x) \\
&\qquad + G_{3,k} * \big((1 + |y|)^\alpha |P_{\mathbb{R}^3} f|\big)(x) + G_{4,k} * |P_{\mathbb{R}^3} f|(x)\Big\},
\end{aligned} \tag{3.7}$$

where

$$G_{1,k}(x, t) = \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \left(\frac{1}{2\sqrt{t}}\right)^{|k|} \left|h_k\left(\frac{x - ate_1}{2\sqrt{t}}\right)\right|, \tag{3.8}$$

$$G_{2,k}(x, t) = \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \left(\frac{1}{2\sqrt{t}}\right)^{|k|} \left|h_k\left(\frac{x - ate_1}{2\sqrt{t}}\right)\right| (1 + |x|)^\alpha, \tag{3.9}$$

$$G_{3,k}(x,t) = \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \left(\frac{1}{2\sqrt{t}}\right)^{|k|} \left| h_k\left(\frac{x-ate_1}{2\sqrt{t}}\right)\right| (1+|x|-x_1)^{\beta}, \tag{3.10}$$

$$G_{4,k}(x,t) = \left(\frac{1}{4\pi t}\right)^{\frac{3}{2}} \left(\frac{1}{2\sqrt{t}}\right)^{|k|} \left| h_k\left(\frac{x-ate_1}{2\sqrt{t}}\right)\right| (1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}, \tag{3.11}$$

$$h_k(z) = \partial_z^k e^{-|z|^2}. \tag{3.12}$$

By changing variables $z = (x-ate_1)/(2\sqrt{t})$, we get

$$\|G_{1,k}(t)\|_{s,\mathbb{R}^3} \le Ct^{-\frac{3}{2}-\frac{|k|}{2}+\frac{3}{2s}} \tag{3.13}$$

for $t>0$ and $s \in [1,\infty)$. Moreover, it holds that

$$\begin{aligned}\|G_{2,k}(t)\|_{s,\mathbb{R}^3}^s &\le Ct^{-\frac{3}{2}s-\frac{|k|}{2}s+\frac{3}{2}} \left((1+at)^{\alpha s}\int_{\mathbb{R}^3} |\partial_z^k e^{-|z|^2}|^s\,dz + t^{\frac{\alpha}{2}s}\int_{\mathbb{R}^3} |\partial_z^k e^{-|z|^2}|^s\,|z|^{\alpha s}\,dz\right) \\ &\le Ct^{-\frac{3}{2}s-\frac{|k|}{2}s+\frac{3}{2}}\{(1+at)^{\alpha s} + t^{\frac{\alpha}{2}s}\},\end{aligned}$$

and that

$$\begin{aligned}\|G_{3,k}(t)\|_{s,\mathbb{R}^3}^s &\le Ct^{-\frac{3}{2}s-\frac{|k|}{2}s+\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\partial_z^k e^{-|z|^2}|^s\,dz + (\sqrt{t})^{\beta s}\int_{\mathbb{R}^3} |\partial_z^k e^{-|z|^2}|^s|z|^{\beta s}\,dz\right) \\ &\le Ct^{-\frac{3}{2}s-\frac{|k|}{2}s+\frac{3}{2}}(1+t^{\frac{\beta}{2}s}),\end{aligned}$$

thereby,

$$\|G_{2,k}(t)\|_{s,\mathbb{R}^3} \le Ct^{-\frac{3}{2}-\frac{|k|}{2}+\frac{3}{2s}}, \qquad \|G_{3,k}(t)\|_{s,\mathbb{R}^3} \le Ct^{-\frac{3}{2}-\frac{|k|}{2}+\frac{3}{2s}} \tag{3.14}$$

for all $t \le 1$, $s \in [1,\infty)$,

$$\|G_{2,k}(t)\|_{s,\mathbb{R}^3} \le Ct^{-\frac{3}{2}-\frac{|k|}{2}+\frac{3}{2s}+\alpha}, \qquad \|G_{3,k}(t)\|_{s,\mathbb{R}^3} \le Ct^{-\frac{3}{2}-\frac{|k|}{2}+\frac{3}{2s}} \tag{3.15}$$

for all $t \ge 1$, $s \in [1,\infty)$, where constant C is independent of a. From

$$\begin{aligned}\|G_{4,k}(t)\|_{s,\mathbb{R}^3}^s \le Ct^{-\frac{3}{2}s-\frac{|k|}{2}s+\frac{3}{2}} \Big\{&(1+at)^{\alpha s}\int_{\mathbb{R}^3} |\partial_z^k e^{-|z|^2}|^s\,dz \\ &+(1+at)^{\alpha s}(\sqrt{t})^{\beta s}\int_{\mathbb{R}^3} |\partial_z^k e^{-|z|^2}|^s\,|z|^{\beta s}\,dz + (\sqrt{t})^{(\alpha+\beta)s}\int_{\mathbb{R}^3} |\partial_z^k e^{-|z|^2}|^s\,|z|^{\alpha s+\beta s}\,dz\Big\} \\ \le Ct^{-\frac{3}{2}s-\frac{|k|}{2}s+\frac{3}{2}}\big\{&(1+at)^{\alpha s}(1+t^{\frac{\beta}{2}s}) + t^{\frac{\alpha}{2}s+\frac{\beta}{2}s}\big\},\end{aligned}$$

we find

$$\|G_{4,k}(t)\|_{s,\mathbb{R}^3} \le Ct^{-\frac{3}{2}-\frac{|k|}{2}+\frac{3}{2s}} \tag{3.16}$$

for all $t \le 1, s \in [1,\infty)$,

$$\|G_{4,k}(t)\|_{s,\mathbb{R}^3} \le C a_0^\alpha t^{-\frac{3}{2}-\frac{|k|}{2}+\frac{3}{2s}+\alpha+\frac{\beta}{2}} \tag{3.17}$$

for all $t \ge 1, s \in [1,\infty)$, where constant C is independent of a. Given $q, q_i \le r$ $(i = 1,2,3,4)$, we set $1/s_0 := 1 - 1/q + 1/r \in (0,1]$, $1/s_i := 1 - 1/q_i + 1/r \in (0,1]$ $(i = 1,2,3,4)$. Then collecting (3.7)–(3.17) and using the weighted L^q boundedness of $P_{\mathbb{R}^3}$ imply that

$$\begin{aligned}
&\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \partial_x^k S_a(t) P_{\mathbb{R}^3} f(x)\|_{r,\mathbb{R}^3} \\
&\le C\Big(\|G_{1,k}\|_{s_0,\mathbb{R}^3}\|\big((1+|y|)^\alpha(1+|y|-y_1)^\beta P_{\mathbb{R}^3} f\|_{q,\mathbb{R}^3} + \|G_{2,k}\|_{s_0,\mathbb{R}^3}\|(1+|y|-y_1)^\beta P_{\mathbb{R}^3} f\|_{q,\mathbb{R}^3} \\
&\qquad + \|G_{3,k}\|_{s_0,\mathbb{R}^3}\|(1+|y|)^\alpha P_{\mathbb{R}^3} f\|_{q,\mathbb{R}^3} + \|G_{4,k}\|_{s_0,\mathbb{R}^3}\|P_{\mathbb{R}^3} f\|_{q,\mathbb{R}^3}\Big) \\
&\le C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|k|}{2}}\|(1+|y|)^\alpha(1+|y|-y_1)^\beta f\|_{q,\mathbb{R}^3}
\end{aligned}$$

for $t \le 1$, $f \in L^q_\rho(\mathbb{R}^3)$ and that

$$\begin{aligned}
&\|(1+|x|)^\alpha(1+|x|-x_1)^\beta \partial_x^k S_a(t) P_{\mathbb{R}^3} f(x)\|_{r,\mathbb{R}^3} \\
&\le C\Big(\|G_{1,k}\|_{s_1,\mathbb{R}^3}\|\big((1+|y|)^\alpha(1+|y|-y_1)^\beta P_{\mathbb{R}^3} f\|_{q_1,\mathbb{R}^3} + \|G_{2,k}\|_{s_2,\mathbb{R}^3}\|(1+|y|-y_1)^\beta P_{\mathbb{R}^3} f\|_{q_2,\mathbb{R}^3} \\
&\qquad + \|G_{3,k}\|_{s_3,\mathbb{R}^3}\|(1+|y|)^\alpha P_{\mathbb{R}^3} f\|_{q_3,\mathbb{R}^3} + \|G_{4,k}\|_{s_4,\mathbb{R}^3}\|P_{\mathbb{R}^3} f\|_{q_4,\mathbb{R}^3}\Big) \\
&\le C\sum_{i=1}^4 t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})-\frac{|k|}{2}+\eta_i}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} f\|_{q_i,\mathbb{R}^3}
\end{aligned}$$

for all $t \ge 1$, $f \in \bigcap_{i=1}^3 L^{q_i}_{\rho_i}(\mathbb{R}^3) \cap L^{q_4}(\mathbb{R}^3)$. The proofs of the assertion 1 and the assertion 2 are complete. We next prove the assertion 3 by using

$$(1+|x|)^\alpha|\partial_x^k S_0(t) P_{\mathbb{R}^3} f(x)| \le C\big\{G_{1,k} * \big((1+|y|)^\alpha|P_{\mathbb{R}^3} f|\big)(x) + G_{2,k} * |P_{\mathbb{R}^3} f|(x)\big\}.$$

If $P_{\mathbb{R}^3} f \in L^q_{(1+|x|)^{\alpha q}}(\mathbb{R}^3)$, then Lorentz-Hölder inequality yields $P_{\mathbb{R}^3} f \in L^{(3q)/(3+\alpha q),q}(\mathbb{R}^3)$ with

$$\|P_{\mathbb{R}^3} f\|_{L^{\frac{3q}{3+\alpha q},q}(\mathbb{R}^3)} \le \|(1+|y|)^{-\alpha}\|_{L^{\frac{3}{\alpha},\infty}(\mathbb{R}^3)}\|(1+|y|)^\alpha P_{\mathbb{R}^3} f\|_{q,\mathbb{R}^3}. \tag{3.18}$$

Moreover, we define s_5 by $1/s_5 = -1/q + 1/r - \alpha/3 + 1$, which satisfies $1 < s_5 < \infty$ and $1 \le (3s_5)/(3+\alpha s_5) < \infty$ due to $\alpha < 3(1-1/q)$ and $q \le r$. Then we get $G_{2,k}(t) \in L^{s_5,(3s_5)/(3+\alpha s_5)}(\mathbb{R}^3)$ with

$$\|G_{2,k}(t)\|_{L^{s_5,\frac{3s_5}{3+\alpha s_5}}(\mathbb{R}^3)} \le C\|G_{2,k}(t)\|^{1-\theta}_{\kappa_1,\mathbb{R}^3}\|G_{2,k}(t)\|^\theta_{\kappa_2,\mathbb{R}^3}, \tag{3.19}$$

where $1 < \kappa_1 < s_5 < \kappa_2 < \infty, 0 < \theta < 1$ satisfy $1/s_5 = (1-\theta)/\kappa_1 + \theta/\kappa_2$. From (3.18), (3.19), the weighted L^q boundedness of $P_{\mathbb{R}^3}$ together with Young's inequality for convolution in Lorentz spaces, we have

$$\|G_{2,k} * |P_{\mathbb{R}^3} f|\big)\|_{r,\mathbb{R}^3} \le C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|k|}{2}}\|(1+|x|)^\alpha f\|_{q,\mathbb{R}^3}$$

for all $t > 0$ and $f \in L^q_{(1+|x|)^{\alpha q}}(\mathbb{R}^3)$, which completes the proof. □

We next consider the decay estimates of $e^{-tA_a}f$ near the boundary of D. Given $f \in L^q(D)$, Kobayashi and Shibata [25] derived the decay estimate with the rate $t^{-3/(2q)}$, however, given $f \in L^q_\rho(D)$, we use the better spatial decay structure of f at infinity to get the better decay rate $t^{-3/(2q)-\varepsilon}$, where ρ is given by (2.2). Let us start with the so-called local energy decay estimates derived by [25].

Proposition 3.2 ([25, Theorem 1.1])**.** *Let $R > 0$ such that $\mathbb{R}^3 \setminus D \subset B_{R-1}(0)$. We set $D_R := D \cap B_R(0)$. Let $1 < q < \infty$, $a_0 > 0$ and assume $|a| \le a_0$. Then there exists a constant $C > 0$, independent of a, such that*

$$\|\partial_t e^{-tA_a} f\|_{q,D_R} + \|e^{-tA_a} f\|_{W^{2,q}(D_R)} \le Ct^{-\frac{3}{2}}\|f\|_{q,D_R}$$

for all $t \ge 1$ and $f \in \{f \in L^q_\sigma(D) \mid f(x) = 0 \text{ for } |x| \ge R\}$.

Combining Proposition 3.2 and the L^q-L^r estimates of $S_a(t)$ implies the following estimates of $e^{-tA_a}f$ near the boundary when $f \in L^q_\sigma(D)$.

Proposition 3.3 ([25, (6.18)])**.** *Let $R > 0$ such that $\mathbb{R}^3 \setminus D \subset B_{R-1}(0)$ and set $D_R = D \cap B_R(0)$. Let $1 < q < \infty$, $a_0 > 0$ and assume $|a| \le a_0$. Then there exists a constant $C > 0$, independent of a, such that*

$$\|\partial_t e^{-tA_a} f\|_{q,D_R} + \|e^{-tA_a} f\|_{W^{2,q}(D_R)} \le Ct^{-\frac{3}{2q}}\|f\|_{q,D}$$

for all $t \ge 1$ and $f \in L^q_\sigma(D)$.

Hölder inequality tells us that $f \in L^r(D)$ with some $r < q$ if $f \in L^q_\rho(D)$. By making use of this, we next derive the better decay rate $t^{-3/2q-\varepsilon}$ than the one in Proposition 3.3.

Proposition 3.4. *Let $1 < q < \infty$. We take $R > 0$ such that $\mathbb{R}^3 \setminus D \subset B_{R-1}(0)$ and set $D_R = D \cap B_R(0)$. Fix $a_0 > 0$ and assume $a \in [0, a_0]$.*

1. *Let $\alpha, \beta > 0$ satisfy $\alpha+\beta < 3(1-1/q)$ and let $s \in (\max\{3q/(3+\alpha q+\beta q), 2q/(2+\alpha q)\}, q]$. Then there exists a constant $C(D, a_0, q, s, \alpha, \beta)$ such that*

$$\|\partial_t e^{-tA_a} P_D f\|_{q,D_R} + \|e^{-tA_a} P_D f\|_{W^{2,q}(D_R)} \le Ct^{-\frac{3}{2s}}\|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{q,D} \tag{3.20}$$

   for all $t \ge 3$ and $f \in L^q_\rho(D)$, where ρ is given by (2.2).

2. *Let $0 \le \alpha < 3(1-1/q)$. Then there exists a constant $C(D, a_0, q, \alpha)$ such that*

$$\|\partial_t e^{-tA_a} P_D f\|_{q,D_R} + \|e^{-tA_a} P_D f\|_{W^{2,q}(D_R)} \le Ct^{-\frac{3}{2q}-\frac{\alpha}{2}}\|(1+|x|)^\alpha f\|_{q,D} \tag{3.21}$$

   for all $t \ge 3$ and $f \in L^q_{(1+|x|)^{\alpha q}}(D)$.

Proof. Under the assumption in the assertion 1, Hölder inequality yields

$$\|f\|_{s,D} \le \|(1+|x|)^{-\alpha}(1+|x|-x_1)^{-\beta}\|_{\frac{qs}{q-s},D}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{q,D} \tag{3.22}$$

for $f \in L^q_\rho(D)$. Let ζ be a function on $\mathbb{R}^3$ satisfying

$$\zeta \in C^\infty(\mathbb{R}^3), \quad \zeta(x) = 0 \quad |x| \le R-1, \quad \zeta(x) = 1 \quad |x| \ge R \tag{3.23}$$

and denote the Bogovskiĭ operator on $A_{R-1,R} = \{x \in \mathbb{R}^3\,;\, R-1 < |x| < R\}$ by $\mathbb{B}_{A_{R-1,R}}$, see Bogovskiĭ [2], Borchers and Sohr [3] and Galdi [16]. Note that given bounded domain G with Lipschitz boudary, $1 < q < \infty$ and integer $k \ge 0$, $\mathbb{B}_G$ is a bounded operator from $W^k_0(G)$ to $W^{k+1}_0(G)^3$, thus

$$\|\mathbb{B}_G f\|_{W^{k+1}(G)} \le C\|f\|_{W^k(G)} \tag{3.24}$$

with some constant $C = C(G,q,k)$. Set $g := \zeta e^{-A_a}P_D f - \mathbb{B}_{A_{R-1,R}}[(\nabla\zeta)\cdot e^{-A_a}P_D f]$. We then find $\nabla \cdot g = 0$ in $\mathbb{R}^3$ and

$$\|S_a(t)g\|_{W^{3,q}(D_R)} \le Ct^{-\frac{1}{2}}(1+t)^{\frac{1}{2}-\frac{3}{2s}}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{q,D} \tag{3.25}$$

for all $t > 0$ due to (3.22), (3.24) and analyticity of e^{-tA_a}. We take a function $\widetilde{\zeta}$ such that $\widetilde{\zeta} \in C^\infty(\mathbb{R}^3)$, $\widetilde{\zeta}(x) = 0$ for $|x| \le R$, $\widetilde{\zeta}(x) = 1$ for $|x| \ge R+1$ and set $v(t) := u(t) - \widetilde{\zeta}S_a(t)g + \mathbb{B}_{A_{R,R+1}}[(\nabla\widetilde{\zeta})\cdot S_a(t)g]$, $u(t) := e^{-(t+1)A_a}P_D f$, then the pair $(v(t), p)$, where $p(t)$ is the pressure associated with $u(t)$, obeys

$$\begin{aligned}&\partial_t v - \Delta v + a\partial_{x_1}v + \nabla p = K, \quad \nabla\cdot v = 0, \quad x \in D,\ t > 0 \qquad v|_{\partial D} = 0, \quad t > 0,\\ &v(x,0) = (1-\widetilde{\zeta})e^{-A_a}P_D f + \mathbb{B}_{A_{R,R+1}}[(\nabla\widetilde{\zeta})\cdot e^{-A_a}P_D f], \quad x \in D,\end{aligned}$$

where $K(t)$ fulfills supp $K(t) \subset D_{R+1}$ and

$$\|K(t)\|_{W^{2,q}(D)} \le Ct^{-\frac{1}{2}}(1+t)^{\frac{1}{2}-\frac{3}{2s}}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{q,D} \tag{3.26}$$

for all $t > 0$. By Duhamel's principle, v satisfies the integral equation

$$v(t) = e^{-tA_a}v(0) + \int_0^t e^{-(t-\tau)A_a}K(\tau)\,d\tau. \tag{3.27}$$

Since supp $v(0) \subset D_{R+1}$, applying Proposition 3.2 and (3.24) leads to

$$\|\partial_t e^{-tA_a}v(0)\|_{q,D_{R+1}} + \|e^{-tA_a}v(0)\|_{W^{2,q}(D_{R+1})} \le Ct^{-\frac{3}{2}}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{q,D} \tag{3.28}$$

for all $t \ge 1$. On the other hand, it follows from $K(\tau) \in \mathscr{D}(A_a)$ and Proposition 3.2 that

$$\|\partial_t e^{-(t-\tau)A_a}K(\tau)\|_{q,D_{R+1}} + \|e^{-(t-\tau)A_a}K(\tau)\|_{W^{2,q}(D_{R+1})} \le C(1+t-\tau)^{-\frac{3}{2}}\|K(\tau)\|_{W^{2,q}(D)}$$

for all $0 < \tau < t$, which combined with (3.26) yields

$$\int_0^t \|\partial_t e^{-(t-\tau)A_a} K(\tau)\|_{q,D_{R+1}} \, d\tau + \int_0^t \|e^{-(t-\tau)A_a} K(\tau)\|_{W^{2,q}(D_{R+1})} \, d\tau \\ \leq Ct^{-\frac{3}{2s}} \|(1+|x|)^\alpha (1+|x|-x_1)^\beta f\|_{q,D} \tag{3.29}$$

for $t \geq 2$. Due to

$$\partial_t v(t) = \partial_t e^{-tA_a} v(0) + \int_0^t \partial_t e^{-(t-\tau)A_a} K(\tau) \, d\tau + K(t)$$

and $v|_{D_R} = u(t)$, collecting (3.26)–(3.29) completes the proof of the assertion 1.

We next prove the assertion 2. We know $f \in L^{(3q)/(3+\alpha q),q}(D)$ with $\|f\|_{L^{(3q)/(3+\alpha q),q}(D)} \leq C\|(1+|x|)^\alpha f\|_{q,D}$ for $f \in L^q_{(1+|x|)^{\alpha q}}(D)$. Moreover, from

$$\|\partial_x^k S_a(t) P_{\mathbb{R}^3} h\|_{\infty,\mathbb{R}^3} \leq Ct^{-\frac{3}{2q}-\frac{\alpha}{2}-\frac{|k|}{2}} \|h\|_{L^{\frac{3q}{3+\alpha q},q}(\mathbb{R}^3)}$$

for all $t > 0$, $|k| \leq 3$ and $h \in L^{3q/(3+\alpha q),q}(\mathbb{R}^3)$, we find that (3.25) is replaced by

$$\|S_a(t) g\|_{W^{3,q}(D_R)} \leq Ct^{-\frac{1}{2}} (1+t)^{\frac{1}{2}-\frac{3}{2q}-\frac{\alpha}{2}} \|(1+|x|)^\alpha f\|_{q,D}.$$

By applying this estimate, we can obtain (3.26) and (3.29) with $\beta = 0, s = (3q)/(3+\alpha q)$. We thus conclude the assertion 2. The proof is complete. □

To prove Theorem 2.2 and Theorem 2.3, it is convenient to prepare the following lemma.

Lemma 3.5. *1. Let $\lambda, s > 0$, $j \geq 0$, $\eta \in \mathbb{R}$ and $r \in (0,\infty]$ satisfy $\lambda \leq 3/2$ and $\lambda < s$. Then there exists a constant C independent of t such that*

$$\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})-\frac{j}{2}+\eta} (1+\tau)^{-\frac{3}{2s}} \, d\tau \leq Ct^{-\frac{3}{2}(\frac{1}{s}-\frac{1}{r})-\frac{j}{2}+\eta} \tag{3.30}$$

for $t \geq 1$.

2. Let $s > 0$, $0 < \lambda < 3/2$, $\eta \in \mathbb{R}$ and $r \in (0,\infty]$. Then there exists a constant C independent of t such that

$$\int_{\frac{t}{2}}^{t-1} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})+\eta} (1+\tau)^{-\frac{3}{2s}} \, d\tau \leq Ct^{-\frac{3}{2}(\frac{1}{s}-\frac{1}{r})+\eta} \tag{3.31}$$

for $t \geq 2$.

3. Let $\lambda, r, s > 0$, $\eta, \kappa \in \mathbb{R}$ satisfy

$$\frac{1}{\lambda} - \frac{1}{r} \neq \frac{1-2\eta}{3}, \quad \frac{1+2\kappa-2\eta}{3} \leq \frac{1}{r}, \quad \frac{2+2\kappa}{3} \leq \frac{1}{\lambda}. \tag{3.32}$$

Then there exists a constant C independent of t such that

$$\int_{\frac{t}{2}}^{t-1} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})-\frac{1}{2}+\eta}(1+\tau)^{-\frac{3}{2s}+\kappa}\,d\tau \le Ct^{-\frac{3}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}+\eta} \tag{3.33}$$

for $t \ge 2$.

4. *Let $\lambda, s > 0$, $j \ge 0$ and $r \in (0,\infty]$ satisfy $1/\lambda - 1/r < (2-j)/3$. Then there exists a constant C independent of t such that*

$$\int_{t-1}^{t} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})-\frac{j}{2}}(1+\tau)^{-\frac{3}{2s}}\,d\tau \le Ct^{-\frac{3}{2s}} \tag{3.34}$$

for $t \ge 1$.

Proof. We have

$$\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})-\frac{j}{2}+\eta}(1+\tau)^{-\frac{3}{2s}}\,d\tau \le Ct^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})-\frac{j}{2}+\eta} \times \begin{cases} t^{-\frac{3}{2s}+1} & \text{if } -\dfrac{3}{2s} > -1, \\ \log t & \text{if } -\dfrac{3}{2s} = -1, \\ 1 & \text{if } -\dfrac{3}{2s} < -1 \end{cases}$$

for $t \ge 1$, which combined with $\lambda \le 3/2$, $\lambda < s$ yields (3.30). It holds that

$$\int_{\frac{t}{2}}^{t-1} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})+\eta}(1+\tau)^{-\frac{3}{2s}}\,d\tau \le Ct^{-\frac{3}{2s}} \times \begin{cases} t^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})+\eta+1} & \text{if } -\dfrac{3}{2}\left(\dfrac{1}{\lambda}-\dfrac{1}{r}\right)+\eta > -1, \\ \log t & \text{if } -\dfrac{3}{2}\left(\dfrac{1}{\lambda}-\dfrac{1}{r}\right)+\eta = -1, \\ 1 & \text{if } -\dfrac{3}{2}\left(\dfrac{1}{\lambda}-\dfrac{1}{r}\right)+\eta < -1 \end{cases}$$

for $t \ge 2$. By $\lambda < 3/2$, we find $\min\{3/(2s)+3(1/\lambda-1/r)/2-\eta-1, 3/(2s)\} > 3(1/s-1/r)/2-\eta$ except for the case $r = \infty, \eta = 0$, thus conclude (3.31) except for the case $r = \infty, \eta = 0$. If $r = \infty, \eta = 0$, then $\lambda < 3/2$ yields $-3(1/\lambda - 1/r)/2 + \eta < -1$, which combined with $-3(1/s-1/r)/2+\eta = -3/(2s)$ implies (3.31). Since $-3(1/\lambda-1/r)/2-1/2+\eta \ne -1$ follows from (3.32), we can derive (3.33) in the same way. We use $1/\lambda - 1/r < (2-j)/3$ to deduce

$$\int_{t-1}^{t} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})-\frac{j}{2}}(1+\tau)^{-\frac{3}{2s}}\,d\tau \le Ct^{-\frac{3}{2s}} \int_{t-1}^{t} (t-\tau)^{-\frac{3}{2}(\frac{1}{\lambda}-\frac{1}{r})-\frac{j}{2}}\,d\tau \le Ct^{-\frac{3}{2s}}$$

for $t \ge 1$, which asserts (3.34). The proof is complete. □

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. We derive the estimate on $\mathbb{R}^3 \setminus B_R(0)$. Let ζ be a function on

$\mathbb{R}^3$ satisfying (3.23). Given $f \in \bigcap_{i=1}^{3} L^{q_i}_{\rho_i}(D) \cap L^{q_4}(D)$, we define $u(t) := e^{-(t+1)A_a}P_D f$, $w(t) := \zeta u(t) - \mathbb{B}_{A_{R-1,R}}[\nabla\zeta\cdot u(t)]$, $\pi(t) := \zeta p(t)$, where $p(t)$ is the pressure associated with $u(t)$ that satisfies $\int_{A_{R-1,R}} p(t)\,dx = 0$ for all $t > 0$. Then w obeys

$$w(t) = S_a(t)w(0) + \int_0^t S_a(t-\tau)P_{\mathbb{R}^3}L(\tau)\,d\tau, \tag{3.35}$$

where

$$\begin{aligned}
&w(0) = \zeta e^{-A_a}P_D f - \mathbb{B}_{A_{R-1,R}}[\nabla\zeta\cdot e^{-A_a}P_D f],\\
&L(x,t) = -2(\nabla\zeta\cdot\nabla)u - (\Delta\zeta)u + a(\partial_{x_1}\zeta)u - (\partial_t - \Delta + a\partial_{x_1})\mathbb{B}_{A_{R-1,R}}[\nabla\zeta\cdot u(t)] + (\nabla\zeta)p.
\end{aligned}$$

It follows that

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta S_a(t)w(0)\|_{r,\mathbb{R}^3} \le C\sum_{i=1}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})+\eta_i}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i}f\|_{q_i,D}$$

for $t > 0$ and $f \in \bigcap_{i=1}^{3} L^{q_i}_{\rho_i}(D) \cap L^{q_4}(D)$, where $\gamma_i, \delta_i, \eta_i$ are given by (2.7). We use Proposition 3.3 and the Poincaré inequality to find that $(\nabla\zeta)p(t) \in L^\kappa(\mathbb{R}^3)$ for all $t > 0$, $\kappa = q_i$ $(i = 1,2,3,4)$ and that

$$\|(\nabla\zeta)p(t)\|_{\kappa,\mathbb{R}^3} \le C(1+t)^{-\frac{3}{2\kappa}}\|f\|_{\kappa,D} \tag{3.36}$$

for all $t > 0$ and $\kappa = q_i$ $(i = 1,2,3,4)$. Given $1 < q_i < \infty$ $(i = 1,2,3,4)$, $1 < r \le \infty$ and α,β subject to (2.3)–(2.4), we take λ_i $(i = 1,2,3,4)$ so that $1 < \lambda_i < \min\{3/2, q_i\}$, $\beta < 1 - 1/\lambda_1$, $\beta < 1-1/\lambda_2$, $\alpha < 3(1-1/\lambda_3)$, $\alpha+\beta < 3(1-1/\lambda_1)$ for $i = 1,2,3,4$. Then due to (3.36), Proposition 3.3, $\operatorname{supp} L(t) \subset A_{R-1,R}$ and (3.24), we get $L(t) \in L^{\lambda_i}_{(1+|x|)^{\gamma_i\lambda_i}(1+|x|-x_1)^{\delta_i\lambda_i}}(\mathbb{R}^3)$ with

$$\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i}P_{\mathbb{R}^3}L(t)\|_{\lambda_i,\mathbb{R}^3} \le C(1+t)^{-\frac{3}{2q_i}}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i}f\|_{q_i,D} \tag{3.37}$$

for $t > 0$ and $i = 1,2,3,4$. Similarly, we have

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta P_{\mathbb{R}^3}L(t)\|_{q_1,\mathbb{R}^3} \le C(1+t)^{-\frac{3}{2q_1}}\|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{q_1,D}. \tag{3.38}$$

From (3.4), (3.5), (3.37) and (3.38), we apply (3.30) to $(\lambda, r, s, j, \eta) = (\lambda_i, r, q_i, 0, \eta_i)$ $(i = 1,2,3,4)$, (3.31) to $(\lambda, r, s, \eta, \kappa) = (\lambda_i, r, q_i, \eta_i, 0)$ $(i = 1,2,3,4)$, (3.34) to $(\lambda, r, s, j) = (q_1, r, q_1, 0)$, then obtain

$$\int_0^t \|(1+|x|)^\alpha(1+|x|-x_1)^\beta S_a(t-\tau)P_{\mathbb{R}^3}L(\tau)\|_{r,\mathbb{R}^3}\,d\tau$$

$$\leq C\sum_{i=1}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})+\eta_i}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i}f\|_{q_i,D}$$

for $t \geq 2$, $f \in \bigcap_{i=1}^{3} L^{q_i}_{\rho_i}(D) \cap L^{q_4}(D)$ provided that $1/q_1 - 1/r < 2/3$. Due to $w|_{\mathbb{R}^3\setminus B_R(0)} = e^{-(t+1)A_a}P_D f$, we deduce

$$\begin{aligned}&\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}e^{-tA_a}P_D f\|_{r,\mathbb{R}^3\setminus B_R(0)}\\&\quad\leq C\sum_{i=1}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})+\eta_i}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i}f\|_{q_i,D}\end{aligned}$$

for $t \geq 3$, $f \in \bigcap_{i=1}^{3} L^{q_i}_{\rho_i}(D)\cap L^{q_4}(D)$ if $1/q_1 - 1/r < 2/3$. On the other hand, if $1/q_1 - 1/r < 1/3$, then we also have

$$\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}e^{-tA_a}P_D f\|_{r,D_R} \leq Ct^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{r})}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{q_1,D}$$

by the Sobolev embedding and Proposition 3.3, thus (2.6) holds if $1/q_1 - 1/r < 1/3$. The restriction $1/q_1 - 1/r < 1/3$ is eliminated by the semigroup property and by (2.6). The proof of the assertion 1 of Theorem 2.2 is complete.

Under the assumption in the assertion 2, we take λ so that $1 < \lambda < \min\{3/2, q\}$, $\alpha < 3(1 - 1/\lambda)$, then the same calculation as above yields

$$\|(1+|x|)^{\alpha}P_{\mathbb{R}^3}L(t)\|_{\kappa,\mathbb{R}^3} \leq C(1+t)^{-\frac{3}{2q}}\|(1+|x|)^{\alpha}f\|_{q,D}$$

for $t > 0, \kappa = \lambda, q$. Taking this and (3.6) into account and applying (3.30) to $(\lambda, r, s, j, \eta) = (\lambda, r, q, 0, 0)$, (3.31) to $(\lambda, r, s, \eta, \kappa) = (\lambda, r, q, 0, 0)$, (3.34) to $(\lambda, r, s, j) = (q, r, q, 0)$ lead us to

$$\|(1+|x|)^{\alpha}e^{-tA}P_D f\|_{r,\mathbb{R}^3\setminus B_R(0)} \leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}\|(1+|x|)^{\alpha}f\|_{q,D}$$

for $t > 0, f \in L^q_{(1+|x|)^{\alpha q}}(D)$. The estimate of e^{-tA} near the boundary is also derived by the Sobolev embedding and Proposition 3.3. The proof is complete. □

Let us close the paper with completion of the proof of Theorem 2.3.

Proof of Theorem 2.3. Let $\alpha, \beta > 0$ and $1 < q_4 \leq q_i \leq r$ $(i = 2, 3)$ satisfy

$$\alpha < \min\left\{3\left(1-\frac{1}{q_3}\right), 1\right\}, \quad \beta < \min\left\{1-\frac{1}{q_2}, \frac{1}{3}\right\}, \quad \alpha+\beta < \min\left\{3\left(1-\frac{1}{r}\right), 1\right\}$$

and we also suppose

$$1 < q_4 \leq q_i \leq r < \min\left\{\frac{3}{1-\alpha-\beta}, \frac{3}{1-\frac{3\alpha}{2}}\right\} \quad \left(\text{resp. } 1 < q_4 \leq q_i \leq r < \frac{3}{1-\alpha-\beta}\right)$$

for $i = 2, 3$ if $\alpha < 2/3$ (resp. $\alpha \geq 2/3$). In view of the semigroup property and the assertion 1 of Theorem 2.2, to prove the assertion 1 of Theorem 2.3, it is enough to derive

$$\begin{aligned}
&\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}\nabla e^{-tA_a}P_D f\|_{r,D} \\
&\leq Ct^{-\frac{1}{2}}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{r,D} + C\sum_{i=2}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})-\frac{1}{2}+\eta_i}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i}f\|_{q_i,D}
\end{aligned} \tag{3.39}$$

for $t \geq 3$ and $f \in \bigcap_{i=2}^{3} L^{q_i}_{\rho_i}(D) \cap L^r_{\widetilde{\rho}}(D) \cap L^{q_4}(D)$, where $\widetilde{\rho} = (1+|x|)^{\alpha r}(1+|x|-x_1)^{\beta r}$. Let s_0 satisfy

$$\max\left\{\frac{3r}{3+\alpha r+\beta r}, \frac{2r}{2+\alpha r}\right\} < s_0 < \min\{3, r\}.$$

It follows from (3.20) with $s = s_0, q = r$ that

$$\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}\nabla e^{-tA_a}P_D f\|_{r,D_R} \leq Ct^{-\frac{1}{2}}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{r,D}$$

for all $t \geq 3$ and $f \in L^r_{\widetilde{\rho}}(D)$. We use

$$\nabla w(t) = \nabla S_a(t)w(0) + \int_0^t \nabla S_a(t-\tau)P_{\mathbb{R}^3}L(\tau)\,d\tau \tag{3.40}$$

to derive the estimate on $\mathbb{R}^3 \setminus B_R(0)$, see (3.35). Applying (3.5) leads us to

$$\begin{aligned}
&\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}\nabla S_a(t)w(0)\|_{r,\mathbb{R}^3} \\
&\leq Ct^{-\frac{1}{2}}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{r,D} + C\sum_{i=2}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})-\frac{1}{2}+\eta_i}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i}f\|_{q_i,D}
\end{aligned}$$

for $t \geq 1$ and $f \in \bigcap_{i=2}^{3} L^{q_i}_{\rho_i}(D) \cap L^r_{\widetilde{\rho}}(D) \cap L^{q_4}(D)$. Let $\{\widetilde{\lambda}_i\}_{i=1}^4$ satisfy

$$1 < \widetilde{\lambda}_1 < \min\left\{\frac{3}{2}, r\right\}, \quad 1 < \widetilde{\lambda}_i < \min\left\{\frac{3}{2}, q_i\right\} \quad (i = 2, 3, 4), \quad \beta < 1 - \frac{1}{\widetilde{\lambda}_1},$$
$$\beta < 1 - \frac{1}{\widetilde{\lambda}_2}, \quad \alpha < 3\left(1 - \frac{1}{\widetilde{\lambda}_3}\right), \quad \alpha + \beta < 3\left(1 - \frac{1}{\widetilde{\lambda}_1}\right), \quad \frac{1}{\widetilde{\lambda}_i} - \frac{1}{r} \neq \frac{1-2\eta_i}{3} \quad (i = 1, 2, 3, 4).$$

Then by Proposition 3.3, (3.20), (3.21) and (3.24), we carry out the same calculation as in (3.36), (3.37) to get

$$\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}P_{\mathbb{R}^3}L(t)\|_{\widetilde{\lambda}_1,\mathbb{R}^3} \leq C(1+t)^{-\frac{3}{2s_0}}\|(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}f\|_{r,D},$$

$$\|(1+|x|)^\alpha(1+|x|-x_1)^\beta P_{\mathbb{R}^3}L(t)\|_{r,\mathbb{R}^3} \le C(1+t)^{-\frac{3}{2s_0}}\|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{r,D},$$
$$\|(1+|x|)^\alpha P_{\mathbb{R}^3}L(t)\|_{\widetilde{\lambda}_3,\mathbb{R}^3} \le C(1+t)^{-\frac{3}{2q_3}-\frac{\alpha}{2}}\|(1+|x|)^\alpha f\|_{q_3,D},$$
$$\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} P_{\mathbb{R}^3}L(t)\|_{\widetilde{\lambda}_i,\mathbb{R}^3} \le C(1+t)^{-\frac{3}{2q_i}}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} f\|_{q_i,D}$$

for $t>0, i=2,4$. Taking these estimates, (3.4), (3.5) and $s_0<\min\{3,r\}$ into account and applying (3.30) to $(\lambda,r,s,j,\eta)=(\widetilde{\lambda}_1,r,r,1,0),(\widetilde{\lambda}_i,r,q_i,1,\eta_i)$ $(i=2,3,4)$, (3.33) to $(\lambda,r,s,\eta,\kappa)=(\widetilde{\lambda}_1,r,r,0,-3(1/s_0-1/r)/2),(\widetilde{\lambda}_3,r,q_3,\beta/2,-\alpha/2),(\widetilde{\lambda}_i,r,q_i,\eta_i,0)$ $(i=2,4)$, (3.34) to $(\lambda,r,s,j)=(r,r,s_0,1)$ assert

$$\int_0^t \|(1+|x|)^\alpha(1+|x|-x_1)^\beta\nabla S_a(t-\tau)P_{\mathbb{R}^3}L(\tau)\|_{r,\mathbb{R}^3}\,d\tau$$
$$\le Ct^{-\frac{1}{2}}\|(1+|x|)^\alpha(1+|x|-x_1)^\beta f\|_{r,D} + C\sum_{i=2}^{4} t^{-\frac{3}{2}(\frac{1}{q_i}-\frac{1}{r})-\frac{1}{2}+\eta_i}\|(1+|x|)^{\gamma_i}(1+|x|-x_1)^{\delta_i} f\|_{q_i,D}$$

for $t\ge 2, f\in \bigcap_{i=2}^{3} L^{q_i}_{\rho_i}(D)\cap L^r_{\widetilde{\rho}}(D)\cap L^{q_4}(D)$. We thus obtain (3.39), from which the assertion 1 follows. In order to prove the assertion 2, let r,α satisfy $1<r\le 3/(1-\alpha)$ and $0\le\alpha<\min\{3(1-1/r),1\}$. We take $\widetilde{\lambda}$ so that $1<\widetilde{\lambda}<\min\{3/2,r\},\alpha<3(1-1/\widetilde{\lambda}),1/\widetilde{\lambda}-1/r\neq 1/3$. Then in view of (3.21), the same calculation as in (3.36), (3.37) asserts

$$\|(1+|x|)^\alpha P_{\mathbb{R}^3}L(t)\|_{\kappa,\mathbb{R}^3} \le C(1+t)^{-\frac{3}{2r}-\frac{\alpha}{2}}\|(1+|x|)^\alpha f\|_{r,D}$$

for $t>0,\kappa=\widetilde{\lambda},r$. From this and (3.40), applying (3.30) to $(\lambda,r,s,j,\eta)=(\widetilde{\lambda},r,r,1,0)$, (3.33) to $(\lambda,r,s,\eta,\kappa)=(\widetilde{\lambda},r,r,0,-\alpha/2)$, (3.34) to $(\lambda,r,s,j)=(r,r,(3q)/(3+\alpha q),1)$ leads us to

$$\|(1+|x|)^\alpha e^{-tA}P_Df\|_{r,\mathbb{R}^3\setminus B_R(0)} \le Ct^{-\frac{1}{2}}\|(1+|x|)^\alpha f\|_{r,D}$$

for $t>0, f\in L^r_{(1+|x|)^{\alpha r}}(D)$. The estimate of ∇e^{-tA} near the boundary is also derived by the Sobolev embedding and (3.21), which yields

$$\|(1+|x|)^\alpha e^{-tA}P_Df\|_{r,D} \le Ct^{-\frac{1}{2}}\|(1+|x|)^\alpha f\|_{r,D}$$

for $t>0, f\in L^r_{(1+|x|)^{\alpha r}}(D)$. This with the assertion 2 of Theorem 2.2 completes the proof. □

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