Remarks on two new estimates for the plane stationary Navier-Stokes equations

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May, 2023

Abstract

In this note, we recall two new estimates for plane stationary Navier-Stokes solutions in annulus type domains established in the recent paper [6]. Then, we explain how to use them to deduce various classical and recent results on the plane stationary Navier-Stokes equations in exterior domains.

1 Introduction

Notations. We use the notation $z = x + yi = re^{i\theta}$ for an arbitrary point $(x, y) \in \mathbb{R}^2$; The open discs and circles centered at the origin will be denoted by $B_r = \{|z| < r\}$ and $S_r = \partial B_r = \{|z| = r\}$. We write $\Omega_{r_1,r_2} = \{z \in \mathbb{R}^2 : r_1 < |z| < r_2\}$.

We study the stationary Navier-Stokes equations in two dimensions, *i.e.*,

$$\begin{cases} -\Delta \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p = 0, \\ \nabla \cdot \mathbf{w} = 0, \end{cases} \tag{SNS}$$

where $\mathbf{w}, p$ are the unknown velocity and pressure fields respectively. With no loss of generality, we have set the viscosity coefficient of the fluid to be 1. The key open problem in the field is to prove the existence of solutions to the 2D flow around obstacle problem:

$$\begin{cases} -\Delta \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p = 0 \ \text{ in } \Omega, \\ \nabla \cdot \mathbf{w} = 0 \ \text{ in } \Omega, \\ \mathbf{w}|_{\partial\Omega} = \mathbf{0}, \\ \mathbf{w} \to \mathbf{w}_\infty = \lambda \mathbf{e}_1 \ \text{ as } \ |z| \to \infty. \end{cases} \tag{OBS}$$

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Here $\Omega = \mathbb{R}^2 \setminus \bar{U}$ is an exterior plane domain, U is the corresponding bounded open set (not necessarily connected) with smooth boundary in $\mathbb{R}^2$; $\mathbf{w}_\infty$ is the far field constant velocity. The parameter $\lambda > 0$ will be referred to as the Reynolds number. $\mathbf{e}_1 = (1, 0)$ is the unit vector along x-axis. Physically, the system (OBS) describes the stationary motion of a viscous incompressible fluid flowing past a rigid cylindrical body. The existence of solutions to (OBS) with arbitrary λ was included by Professor V.I. Yudovich in the list of "Eleven Great Problems in Mathematical Hydrodynamics" [13].

We are also interested in the exterior problem with more general boundary data:

$$\begin{cases} -\Delta \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p = 0, \\ \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}|_{\partial\Omega} = \mathbf{a}, \\ \mathbf{w}(z) \to \mathbf{w}_\infty = \lambda \mathbf{e}_1 \quad \text{as} \quad r = |z| \to +\infty. \end{cases} \tag{GEN}$$

Here, the boundary datum $\mathbf{a}$ is an arbitrary (smooth) vector-valued function on the finite curve Ω. Another closely related problem is the whole-plane forced system:

$$\begin{cases} -\Delta \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p = \mathbf{f} \quad \text{in } \mathbb{R}^2, \\ \nabla \cdot \mathbf{w} = 0 \quad \text{in } \mathbb{R}^2, \\ \mathbf{w} \to \mathbf{w}_\infty = \lambda \mathbf{e}_1 \quad \text{as} \quad |z| \to \infty. \end{cases} \tag{FOR}$$

We often assume that $\mathbf{f}$ has compact support and enjoys $W^{-1,2}$ local regularity.

The above three problems are hard in 2D mainly due to the fact that the Dirichlet integral $\int_{\mathbb{R}^2} |\nabla f|^2 \,\mathrm{d}x\mathrm{d}y$ alone is not sufficient to control the asymptotic behaviour of functions at spatial infinity. The elegant nonlinear structures of (SNS) are crucial for our research. There are many classical papers that study the problem (OBS), see, *e.g.*, [5, 1, 4]. For the basic tools and the up-to-date results concerning the systems (OBS), (GEN) and (FOR), we refer the readers to professor Galdi's book [3] and the forthcoming survey [11].

In the recent paper [6], together with Julien Guillod, we established two new estimates for D-solutions[1] in annulus type domains, which are called the First and Second Basic Estimates respectively. We recall these estimates as follows.

Theorem 1 (First Basic Estimate). *Let $\mathbf{w}$ be the D-solution to the Navier–Stokes system*

$$\begin{cases} \Delta \mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{w} - \nabla p = \mathbf{0}, \\ \nabla \cdot \mathbf{w} = 0 \end{cases} \tag{1.1}$$

in the annulus type domain $\Omega_{r_1,r_2} = \{z \in \mathbb{R}^2 : r_1 \le |z| \le r_2\}$. Then

$$|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| \le C_* \sqrt{\log(2+\mu)\, D(r_1, r_2)}, \tag{1.2}$$

where

$$\mu = \frac{1}{r_1 m}, \quad m := \max\{|\bar{\mathbf{w}}(r_1)|, |\bar{\mathbf{w}}(r_2)|\}, \quad D(r_1, r_2) := \int_{\Omega_{r_1,r_2}} |\nabla \mathbf{w}|^2 \tag{1.3}$$

and C_* is some universal positive constant (does not depend on $\mathbf{w}, r_i$, etc.).

[1] Solutions with finite Dirichlet integrals, *i.e.*, $\int |\nabla \mathbf{w}|^2 \,\mathrm{d}x\mathrm{d}y < +\infty$.

Remark 2. Actually, in [6], instead of (1.2) we proved a weaker inequality

$$|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| \le C_*(1+\mu)\sqrt{D(r_1, r_2)}. \tag{1.4}$$

(1.2) can be deduced from (1.4) using the simple arguments in Section 2.

Remark 3. (1.2) is qualitatively precise, since for a solution to (OBS) in the case of small λ (first constructed by Finn and Smith [2] in 1967), the opposite inequality

$$\int\limits_{\Omega} |\nabla \mathbf{w}|^2 \le C(\Omega)\, \frac{\lambda^2}{\log(2+\frac{1}{\lambda})} \tag{1.5}$$

holds, see [9].

Theorem 4 (Second Basic Estimate)**.** *Let $\mathbf{w}_k$ be a sequence of D-solutions to the Navier–Stokes system*

$$\begin{cases} \Delta \mathbf{w}_k - (\mathbf{w}_k \cdot \nabla)\mathbf{w}_k - \nabla p_k = \mathbf{0}, \\ \nabla \cdot \mathbf{w}_k = 0 \end{cases} \tag{1.6}$$

in the annulus type domains $\Omega_{r_{1k}, r_{2k}}$. Suppose, in addition, that

$$r_{1k} \to +\infty, \qquad \frac{r_{2k}}{r_{1k}} \to +\infty, \tag{1.7}$$

and there exist two vectors $\mathbf{w}_0$, $\mathbf{w}_\infty \in \mathbb{R}^2$ such that

$$\max_{z \in S_{r_{1k}}} |\mathbf{w}_k(z) - \mathbf{w}_0| \to 0, \qquad \max_{z \in S_{r_{2k}}} |\mathbf{w}_k(z) - \mathbf{w}_\infty| \to 0 \qquad \text{as } k \to \infty. \tag{1.8}$$

Then

$$|\mathbf{w}_0 - \mathbf{w}_\infty| \le C_{**} \frac{D_*}{m}, \tag{1.9}$$

where $m := \max\{|\mathbf{w}_0|, |\mathbf{w}_\infty|\}$, $D_* = \varliminf\limits_{k\to\infty} \int_{\Omega_{r_{1k}, r_{2k}}} |\nabla \mathbf{w}_k|^2$, and C_{**} is some universal positive constant (does not depend on $\mathbf{w}_k$, etc.).

Remark 5. For an arbitrary function f with bounded Dirichlet integral, it is easy to show that

$$|\bar{f}(r_2) - \bar{f}(r_1)| \le \frac{1}{\sqrt{2\pi}} \left(\int_{r_1<|z|<r_2} |\nabla f|^2 \, dxdy \right)^{\frac{1}{2}} \left(\ln \frac{r_2}{r_1} \right)^{\frac{1}{2}}. \tag{1.10}$$

(See, *e.g.*, [7, Lemma 2.1].) We emphasize that Theorem 1 improves the trivial estimate (1.10) when $r_2 \gg r_1$. Theorem 4 further improves Theorem 1 in the asymptotic case (1.7) if $D_* \ll m^2$.

The proof of Theorem 1 is based on ideas from [8, 5, 1], in particular, the topological structure of Φ-level sets is involved. ($\Phi = \frac{|\mathbf{w}|^2}{2} + p$ is the Bernoulli function.) The proof of Theorem 4 is based on ideas from [10], in particular, we need a blow-down argument which uses fine estimates of the Euler solutions to control asymptotic behaviour of the Navier-Stokes solutions.

These two Basic Estimates are very useful in the study of (OBS), (GEN) and (FOR), for example:

a) Using Theorem 1, we give a new proof for the boundedness and convergence of D-solutions in exterior domains. This result was first obtained by Amick [1] under the zero-total-flux and the axi-symmetry conditions, and recently proved by Korobkov, Pileckas and Russo [7] in the general case.

b) Using Theorem 1, we prove a pointwise estimate for small Reynolds number D-solutions to (OBS). This estimate was originally proved in [9] as a key step in the proof of the unconditional uniqueness theorem for (OBS).

c) Using Theorem 1, we can apply Leray's invading domain method [12] to (FOR) and construct a D-solution $\mathbf{w}_L$ to $(\text{FOR})_{1,2}$ for arbitrary compactly supported $W^{-1,2}$ force. This is one of the main results in [6]. The square root on $D(r_1, r_2)$ in (1.2) plays a key role in obtaining certain uniform bounds for the invading domain solutions.

d) Using Theorem 4, we are able to determine the limit of Leray solutions[2] to (OBS) with small Reynolds numbers and to (FOR) in two scenarios. These are treated in [10] and [6]. It is crucial that in (1.9) there is no square root on D_*.

In this note, we shall explain the proof of (1.2) using (1.4), and the above items a) and b). For items c) and d), we refer to the recent papers [10], [6] and the forthcoming survey [11]. The content of this note will also be included in the doctoral thesis (in Chinese) of the second author.

2 Proof of (1.2) using (1.4)

Notice that (1.2) and (1.4) are equivalent in the case $\mu < 1$. Hence, using (1.4), for the case $\mu \leq 1$ Theorem 1 is already proved. We ony need to consider the case $\mu > 1$, $r_1 m < 1$. There are a few subcases. (1) If $r_2 m \geq 1$, we further consider two subcases: (1a) If $\max\{|\bar{\mathbf{w}}(m^{-1})|, |\bar{\mathbf{w}}(r_2)|\} \leq \frac{m}{2}$, then $|\bar{\mathbf{w}}(r_1)| = m$. Hence, by (1.10),

$$
\begin{aligned}
|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| &\leq Cm \leq C|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(m^{-1})| && (2.1)\\
&\leq C\sqrt{\log \mu}\sqrt{D(r_1, m^{-1})} && (2.2)\\
&\leq C\sqrt{\log(2+\mu)}\sqrt{D(r_1, r_2)}. && (2.3)
\end{aligned}
$$

(1b) If $\max\{|\bar{\mathbf{w}}(m^{-1})|, |\bar{\mathbf{w}}(r_2)|\} > \frac{m}{2}$, then by (1.4) and (1.10), we get

$$
\begin{aligned}
|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| &\leq |\bar{\mathbf{w}}(m^{-1}) - \bar{\mathbf{w}}(r_2)| + |\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(m^{-1})| && (2.4)\\
&\leq C\sqrt{D(m^{-1}, r_2)} + C\sqrt{\log \mu}\sqrt{D(r_1, m^{-1})} && (2.5)\\
&\leq C\sqrt{\log(2+\mu)}\sqrt{D(r_1, r_2)}. && (2.6)
\end{aligned}
$$

(2) If $r_2 m \leq 1$, then again by (1.10),

$$
\begin{aligned}
|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| &\leq C\sqrt{\log \frac{r_2}{r_1}}\sqrt{D(r_1, r_2)} && (2.7)\\
&\leq C\sqrt{\log(2+\mu)}\sqrt{D(r_1, r_2)}. && (2.8)
\end{aligned}
$$

In conclusion, Theorem 1 holds for $\mu > 1$ case as well.

[2] Solutions constructed by the invading domains method.

3 The boundedness and convergence of arbitrary D-solutions in an exterior domain

Theorem 6 ([7]). *Let* $\mathbf{w}$ *be a D-solution to the Navier-Stokes equations in the exterior domain* $\Omega_{1,+\infty}$, *then there exists a constant vector* $\mathbf{w}_0 \in \mathbb{R}^2$ *such that* $\mathbf{w}$ *converges uniformly to* $\mathbf{w}_0$ *at infinity. In particular,* $\mathbf{w}$ *is uniformly bounded in a neighbourhood of infinity.*

The proof of this theorem went through a rather long path, see [5, 1, 7]. A discussion on the history can be found in the forthcoming survey [11]. Here, we present a rather short proof using Theorem 1. Note that this short proof does not mean Theorem 6 is trivial, since Theorem 1 itself is highly nontrivial.

Proof. Let $r_2 > r_1 \geq 1$. For any $\varepsilon > 0$, By Theorem 1 we have, when $\max\{|\bar{\mathbf{w}}(r_1)|, |\bar{\mathbf{w}}(r_2)|\} > \varepsilon$,

$$|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| \leq C\sqrt{\log\left(\frac{1}{\varepsilon} + 2\right) D(r_1, r_2)}. \tag{3.1}$$

When $\max\{|\bar{\mathbf{w}}(r_1)|, |\bar{\mathbf{w}}(r_2)|\} < \varepsilon$, there clearly holds

$$|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| \leq 2\varepsilon. \tag{3.2}$$

Hence, for any $\varepsilon > 0$,

$$|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| \leq C(\varepsilon)\sqrt{D(r_1, r_2)} + 2\varepsilon. \tag{3.3}$$

Since as $r_1 \to +\infty$, $D(r_1, r_2) \to 0$, we know that $\bar{\mathbf{w}}(r)$ is a Cauchy sequence as $r \to +\infty$. Hence, there exists a vector $\mathbf{w}_0 \in \mathbb{R}^2$ such that

$$\lim_{r \to +\infty} \bar{\mathbf{w}}(r) = \mathbf{w}_0. \tag{3.4}$$

Denote $\bar{\mathbf{w}}(r, z) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{w}(z + re^{i\theta})d\theta$. By [9, Lemma 6] (the technique of finding *good circles*), for any $z \in \Omega_{10,+\infty}$, there exists $\tilde{r} \in \left(\frac{|z|}{3}, \frac{|z|}{2}\right)$, such that on the circle $S_{\tilde{r}}(z)$,

$$|\mathbf{w} - \bar{\mathbf{w}}(\tilde{r}, z)| \leq C\sqrt{D\left(\frac{|z|}{2}, 2|z|\right)}. \tag{3.5}$$

Again by [9, Lemma 6], there exists $\hat{r} \in \left(\frac{9|z|}{10}, \frac{11|z|}{10}\right)$, such that on the circle $S_{\hat{r}}(0)$,

$$|\mathbf{w} - \bar{\mathbf{w}}(\hat{r})| \leq C\sqrt{D\left(\frac{|z|}{2}, 2|z|\right)}. \tag{3.6}$$

Observe that $S_{\hat{r}}(0) \cap S_{\tilde{r}}(z) \neq \emptyset$. By the definition of D-solutions, as $|z| \to +\infty$,

$$D\left(\frac{|z|}{2}, 2|z|\right) \to 0. \tag{3.7}$$

Hence, according to (3.3), if $|z|$ is sufficiently large,

$$|\bar{\mathbf{w}}(1,z) - \bar{\mathbf{w}}(\tilde{r},z)| \le C(\varepsilon)\sqrt{D\left(\frac{|z|}{2}, 2|z|\right)} + 2\varepsilon \le 3\varepsilon. \tag{3.8}$$

Combining (3.4)–(3.8), we get that if $|z|$ is sufficiently large,

$$|\bar{\mathbf{w}}(1,z) - \mathbf{w}_0| \le 4\varepsilon. \tag{3.9}$$

Since ε can be taken arbitrarily small, we get, as $|z| \to +\infty$,

$$|\bar{\mathbf{w}}(1,z) - \mathbf{w}_0| \to 0. \tag{3.10}$$

By the local Stokes estimates (see, *e.g.*, [9, Lemma 8]) and (3.10), (3.7), as $|z| \to +\infty$, we have

$$|\nabla \mathbf{w}(z)| \to 0. \tag{3.11}$$

(For explicit decay estimates of $|\nabla \mathbf{w}|$, see [5].) By (3.10)–(3.11), as $|z| \to +\infty$, we have $|\mathbf{w}(z) - \mathbf{w}_0| \to 0$. □

4 Uniqueness of D-solutions to (OBS)

In [9], we proved that, when λ is small (and nonzero), (OBS) is *uniquely* solvable[3] in the class of D-solutions. The key step is the following pointwise estimate, see [9, Lemma 16 and eq. (5.8)]. Without loss of generality, we assume that $\Omega_{1,+\infty} \subset \Omega$. In the sequel, the constants C may depend on Ω.

Lemma 7. *Suppose* $\mathbf{w}$ *is a D-solution to* (OBS) *with sufficiently small and nonzero* λ*, then* $|\mathbf{w}(z) - \lambda\mathbf{e}_1| \le C\varepsilon\lambda\sqrt{\log\left(\frac{1}{\lambda|z|} + 2\right)}$.

Here, we give a different proof of Lemma 7 using the First Basic Estimate.

Proof. Let $\varepsilon = \frac{1}{\sqrt{|\log\lambda|}} \ll 1$. First of all, we recall a result proved in [9],

$$D := \int_\Omega |\nabla \mathbf{w}|^2 \, dxdy \le C\varepsilon^2\lambda^2. \tag{4.1}$$

Apply Theorem 1 to $\mathbf{w}$, with $r_1 = r, r_2 \to +\infty$, we have

$$\limsup_{r_2\to+\infty} \mu \le \frac{1}{r\lambda}$$

and

$$|\bar{\mathbf{w}}(r) - \lambda\mathbf{e}_1| \le C\sqrt{\log\left(\limsup_{r_2\to+\infty}\mu + 2\right) D} \le C\varepsilon\lambda\sqrt{\log\left(\frac{1}{\lambda r} + 2\right)} \tag{4.2}$$

In particular, as $r \ge \lambda^{-1}$,

$$|\bar{\mathbf{w}}(r) - \lambda\mathbf{e}_1| \le C\varepsilon\lambda \tag{4.3}$$

[3] Solvability was already shown by Finn and Smith [2] in 1967.

By [9, Lemma 6], there exists a sequence of numbers $r_k \in [2^k, 2^{k+1}]$, $k = 1, 2, 3, \cdots$, such that on the circle S_{r_k},

$$|\mathbf{w} - \bar{\mathbf{w}}(r_k)| \le C D^{\frac{1}{2}} \le C\varepsilon\lambda \tag{4.4}$$

By (4.2) and (4.4), on the circle S_{r_k},

$$|\mathbf{w} - \lambda \mathbf{e}_1| \le C\varepsilon\lambda \sqrt{\log\left(\frac{1}{\lambda r_k} + 2\right)}. \tag{4.5}$$

For any $z \in \Omega_{10,+\infty}$, by [9, Lemma 6], we can find $r_* \in \left(\frac{2|z|}{3}, \frac{3|z|}{4}\right)$, such that on $S_{r_*}(z)$, $|\mathbf{w} - \bar{\mathbf{w}}(r_*, z)| \le C D^{\frac{1}{2}} \le C\varepsilon\lambda$. Observe that there exists $k \ge 1$, such that $S_{r_k}(0) \cap S_{r_*}(z) \ne \emptyset$. Hence,

$$|\bar{\mathbf{w}}(r_*, z) - \lambda \mathbf{e}_1| \le C\varepsilon\lambda \sqrt{\log\left(\frac{1}{\lambda |z|} + 2\right)}. \tag{4.6}$$

Consider the case $|z| \ge 10\lambda^{-1}$ first. Apply Theorem 1 with z as the center and take $r_1 = \lambda^{-1}$, $r_2 = r_*$, we get

$$|\bar{\mathbf{w}}(\lambda^{-1}, z) - \bar{\mathbf{w}}(r_*, z)| \le C\varepsilon\lambda \tag{4.7}$$

By (4.6)–(4.7), we have

$$|\bar{\mathbf{w}}(\lambda^{-1}, z) - \lambda \mathbf{e}_1| \le C\varepsilon\lambda \tag{4.8}$$

Consider the rescaled solutions

$$\widetilde{\mathbf{w}}(\cdot) = \lambda^{-1}\mathbf{w}(\lambda^{-1}\cdot + z), \quad \widetilde{p} = \lambda^{-2} p(\lambda^{-1}\cdot + z), \tag{4.9}$$

then

$$|\bar{\widetilde{\mathbf{w}}}(1) - \mathbf{e}_1| \le C\varepsilon, \tag{4.10}$$

$$\int_{B_2} |\nabla \widetilde{\mathbf{w}}|^2 dxdy = \lambda^{-2} \int_{B_{2\lambda^{-1}}(z)} |\nabla \mathbf{w}|^2 dxdy \le \lambda^{-2} D \le C\varepsilon^2. \tag{4.11}$$

By the local Stokes estimates ([9, Lemma 8]) and (4.10)–(4.11),

$$\sup_{B_1} |\nabla \widetilde{\mathbf{w}}| \le C\varepsilon \tag{4.12}$$

By (4.12) and (4.10), $|\widetilde{\mathbf{w}}(0) - \mathbf{e}_1| \le C\varepsilon$, hence for $|z| \ge 10\lambda^{-1}$, we have the uniform estimate $|\mathbf{w}(z) - \lambda \mathbf{e}_1| \le C\varepsilon\lambda$.

Next, consider the case $3 < |z| < 10\lambda^{-1}$ This time, we define the rescaled solutions

$$\widetilde{\mathbf{w}}(\cdot) = |z| \mathbf{w}(|z|\cdot + z), \quad \widetilde{p} = |z|^2 p(|z|\cdot + z) \tag{4.13}$$

then

$$|\bar{\widetilde{\mathbf{w}}}(r_*/|z|) - |z|\lambda \mathbf{e}_1| \le C\varepsilon |z| \lambda \sqrt{\log\left(\frac{1}{\lambda |z|} + 2\right)} \lesssim 1, \tag{4.14}$$

$$\int_{B_{\frac{3}{4}}} |\nabla \widetilde{\mathbf{w}}|^2 dxdy = |z|^2 \int_{B_{\frac{3}{4}|z|}(z)} |\nabla \mathbf{w}|^2 dxdy \le |z|^2 D \le C\varepsilon^2 \lambda^2 |z|^2. \tag{4.15}$$

By the local Stokes estimates [9, Lemma 8] and (4.14)–(4.15), we get

$$\sup_{B_{\frac{1}{2}}} |\nabla \widetilde{\mathbf{w}}| \leq C\varepsilon |z| \lambda. \tag{4.16}$$

By (4.16) and (4.14), $|\widetilde{\mathbf{w}}(0) - |z|\lambda \mathbf{e}_1| \leq C\varepsilon|z|\lambda\sqrt{\log\left(\frac{1}{\lambda|z|}+2\right)}$, hence for $10 < |z| < 10\lambda^{-1}$, we have the uniform estimate $|\mathbf{w}(z) - \lambda\mathbf{e}_1| \leq C\varepsilon\lambda\sqrt{\log\left(\frac{1}{\lambda|z|}+2\right)}$.

Finally, combined with [9, Lemma 11], the desired pointwise estimate follows. □

Acknowledgement We would like to thank Cheng Yuan for pointing out various typos in the manuscript.

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