Stability of Relaxed Navier-Stokes Equations

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1 Introduction

We report on the formation of singularities in one-dimensional hyperbolic compressible Navier-Stokes equations, a model proposing a relaxation leading to a hyperbolization through a nonlinear Cattaneo law for heat conduction as well as through the constitutive Maxwell type relations for the stress tensor. There are in general no global C^1 solutions for the studied system, for some large initial data. This is in contrast to the global large well-posedness for the non-relaxed, classical system. Relations to incompressible Navier-Stokes equations, and possible higher-dimensional situations are also addressed For detailed results, proofs and references see the joint work with Y. Hu and N. Wang [2].

Consider the one-dimensional non-isentropic compressible Navier-Stokes equations,

$$\rho_t + (\rho u)_x = 0, \tag{1}$$
$$\rho u_t + \rho u u_x + p_x = S_x, \tag{2}$$
$$\rho e_t + \rho u e_x + p u_x + q_x = S u_x, \tag{3}$$

ρ, u, e, p, S, q: fluid density, velocity, specific internal energy per unit mass, pressure, stress tensor, heat flux.

Classical relations:

$$q = -\kappa\theta_x, \qquad S = \mu u_x, \qquad (\kappa,\ \mu > 0).$$

Here: Nonlinear Cattaneo law of heat conduction

$$\tau_1(q_t + u \cdot q_x) + q + \kappa\theta_x = 0, \tag{4}$$

and the Maxwell type constitutive relations for the stress tensor

$$\tau_2(S_t + u \cdot S_x) + S = \mu u_x. \tag{5}$$

e and p satisfy

$$e = C_v\theta + \frac{\tau_1}{\kappa\theta\rho}q^2 + \frac{\tau_2}{\mu\rho}S^2,$$
$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta}q^2 - \frac{\tau_2}{2\mu}S^2, .$$

with positive constants τ_1, τ_2, C_v, R. We consider the Cauchy problem for

$$(\rho, u, \theta, S, q) : \mathbb{R} \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R},$$

with initial condition

$$(\rho(x,0), u(x,0), \theta(x,0), S(x,0), q(x,0)) = (\rho_0, u_0, \theta_0, S_0, q_0). \tag{6}$$

Local existence and global existence for small data are known. In the singular limit $\tau_1 = \tau_2 \equiv \tau \to 0$, smooth solutions converge to solutions of the classical Navier-Stokes equations on finite time intervals. For the classical Navier-Stokes system ($\tau_1 = \tau_2 = 0$), smooth solutions exist for arbitrary initial data away from vacuum (Kazhikov [3]). Here we have a blow-up for some large data. What is the "correct" model?

2 Effects of Relaxation

– 2nd-order thermoelasticity: relaxed/classical; linear/nonlinear, small/large data: same behavior

– Thermoelastic plate:

$$\begin{aligned} w_{tt} + \Delta^2 w + \Delta\theta &= 0, \\ \theta_t + \operatorname{div} q - \Delta u_t &= 0, \\ \tau q_t + q + \nabla\theta &= 0. \end{aligned}$$

$\tau = 0$: Exponential stability in bounded domains, no regularity loss for the Cauchy problem

$\tau > 0$: No exponential stability in bounded domains, regularity loss for the Cauchy problem

Here we will have an effect of the relaxation only in the *nonlinear* system, having the same behavior for the linearized system.

– *Isentropic* compressible Navier-Stokes equations with linearized constitutive equations: same effect (blow-up for large data while linearized similar)

– Semilinear heat resp. damped wave equation: Relaxed systems, linear or nonlinear, behave similar.

– *Incompressible* Navier-Stokes equations:

$$\begin{gathered} u_t + (u \cdot \nabla)u + \nabla p = \operatorname{div} S \qquad \operatorname{div} u = 0, \\ \tau S_t + S = \mu(\nabla u + (\nabla u)'), \end{gathered}$$

leading to

$$\begin{aligned} \tau u_{tt} - \mu\Delta u + u_t + \nabla p + \tau\nabla p_t \quad &= \\ &-(u \cdot \nabla)u - (\tau u_t \cdot \nabla)u - (\tau u \cdot \nabla)u_t, \end{aligned}$$

with a quasilinear nonlinearity. Known results: Global small (smooth) solutions; linearized systems behave similar.

3 Blow-up Result

For some small $\delta > 0$, (ρ, θ, q, S) take values in

$$\Omega := (1-\delta, 1+\delta) \times (1-\delta, 1+\delta) \times (-\delta, \delta) \times (-\delta, \delta).$$

Theorem 3.1 (Local Existence)**.** *Let* $(\rho_0, u_0, \theta_0, q_0, S_0) : \mathbb{R} \to \mathbb{R}$ *be given with*

$$(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0) \in H^2, \forall x \in \mathbb{R}, \quad (\rho_0, \theta_0, q_0, S_0) \in \Omega.$$

Then, the initial value problem (1)-(6) has a unique solution (ρ, u, θ, q, S) *on a maximal time interval* $[0, T_0)$*, for some* $T_0 > 0$*, with*

$$(\rho - 1, u, \theta - 1, q, S) \in C^0([0, T_0), H^2) \cap C^1([0, T_0), H^1)$$

and

$$\forall x \in \mathbb{R}, \quad \forall t \in [0, T_0), \quad (\rho(x,t), \theta(x,t), q(x,t), S(x,t)) \in \Omega.$$

Lemma 3.2 (Finite Propagation Speed)**.** *Let* (ρ, u, θ, q, S) *be a local solution to* (1)-(6) *on* $[0, T_0)$*. Let* $M > 0$*. Assume the initial data* $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)$ *are compactly supported in* $(-M, M)$ *and* $(\rho_0, \theta_0, q_0, S_0) \in \Omega$*. Then, there exists a constant* σ *such that*

$$(\rho(\cdot,t), u(\cdot,t), \theta(\cdot,t), q(\cdot,t), S(\cdot,t) = (1,0,1,0,0) := (\bar\rho, \bar u, \bar\theta, \bar q, \bar S)$$

on $D(t) = \{x \in \mathbb{R} \,|\, |x| \geq M + \sigma t\}, \quad 0 \leq t < T_0$.

Let

$$F(t) := \int_{\mathbb{R}} x\rho(x,t)u(x,t)dx,$$
$$G(t) := \int_{\mathbb{R}} (E(x,t) - \bar E)dx,$$

where

$$E(x,t) := \rho(e + \frac{1}{2}u^2)$$

is the total energy and

$$\bar E := \bar\rho(\bar e + \frac{1}{2}\bar u^2) = C_v.$$

Theorem 3.3 (Blow-up Result)**.** *We assume that the data*

$$(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)$$

are compactly supported in $(-M, M)$*, and that*

$$G(0) > 0.$$

Then, there exists u_0 *satisfying*

$$F(0) > \max\left\{\frac{32\sigma \max\rho_0}{3-\gamma}, \frac{4\sqrt{\max\rho_0}}{\sqrt{3-\gamma}}\right\} M^2,$$
$$1 < \gamma := 1 + \frac{R}{C_v} < 3$$

such that the length T_0 *of the maximal interval of existence of a smooth solution* (ρ, u, θ, q, S) *of* (1)-(6) *is finite, provided the compact support of the initial data is sufficiently large.*

Ingredients of the proof:
- exploit Sideris' ideas used for compressible Euler equations
- derive a Riccati-type differential inequality for the functional F, starting with

$$F'(t) \geq \frac{c_3}{2(1+c_2 t)^3}F^2 - \frac{\tau_1(2\gamma-1)}{\kappa\bar\theta}\int_{\mathbb{R}} q^2 dx - \frac{\tau_2(2\gamma-1)+\mu}{2\mu}\int_{\mathbb{R}} S^2 dx$$

- use an entropy dissipation equation
- choose

$$u_0(x) := \begin{cases} 0, & x \in (-\infty, -M], \\ \frac{L}{2}\cos(\pi(x+M)) - \frac{L}{2}, & x \in (-M, -M+1], \\ -L, & x \in (-M+1, -1], \\ L\cos(\frac{\pi}{2}(x-1)), & x \in (-1, 1], \\ L, & x \in (1, M-1], \\ \frac{L}{2}\cos(\pi(x-M+1)) + \frac{L}{2}, & x \in (M-1, M], \\ 0, & x \in (M, \infty), \end{cases}$$

where $L > 0$ is determined in the proof.

4 Linear Stability

The linearized version of (1)–(5) is

$$\begin{aligned}
\rho_t + u_x &= 0,\\
u_t - S_x + R\theta_x + R\rho_x &= 0,\\
C_v\theta_t + Ru_x + q_x &= 0,\\
\tau_1 q_t + q + \kappa\theta_x &= 0,\\
\tau_2 S_t + S - \mu u_x &= 0.
\end{aligned}$$

Case 1: Bounded domain, $x \in (0,1)$.
Boundary conditions

$$u(t,0) = u(t,1) = 0, \quad q(t,0) = q(t,1) = 0.$$

Without loss of generality,

$$\int_0^1 \rho_0(x)dx = \int_0^1 \theta_0(x)dx = 0.$$

$$E_1(t) := \int_0^1 \left(\frac{R}{2}\rho^2 + \frac{1}{2}u^2 + \frac{C_v}{2}\theta^2 + \frac{\tau_1}{2\kappa}q^2 + \frac{\tau_2}{2\mu}S^2 \right) dx \equiv E_1(t;\rho,\dots,S),$$

$$E_2(t) := E_1(t;\rho_t,\dots,S_t),$$

$$E(t) := E_1(t) + E_2(t.$$

Theorem 4.1 (Exponential Stability in Bounded Domains)**.** *There are constants $C, d > 0$ such that for all $t \geq 0$ we have*

$$E(t) \leq CE(0)\mathrm{e}^{-dt}.$$

The proof is obtained by a standard multiplier method.

Case 2: Cauchy problem, $x \in \mathbb{R}$.
$V := (\rho, u, \theta, q, S)$, $\hat{V}$: Fourier transform of V. Using Kawashima's approach we obtain:

Theorem 4.2 (Pointwise Energy Estimate)**.** *There are positive constants C and C_1 such that*

$$|\hat{V}(t,\xi)|^2 \leq Ce^{-C_1 h(|\xi|)t}|\hat{V}(0,\xi)|^2, \qquad \text{for} \quad (t,\xi) \in \mathbb{R}^+ \times \mathbb{R},$$

where $h(r) := \frac{r^2}{1+r^2}$.

Theorem 4.3 (Decay Rates without Loss)**.** *Let $l \geq 0$, and $0 \leq k \leq l$ be integers, and let $p \in [1,2]$. Assume that $V(0) \in H^l(\mathbb{R}) \cap L^p(\mathbb{R})$. Then we have*

$$\|\partial_x^l V(t)\|^2 \leq C\left\{ e^{-C_1 t}\|\partial_x^l V(0)\|^2 + (1+t)^{-(2\lambda+l-k)}\|\partial_x^k V(0)\|_{L^p}^2 \right\}, \tag{7}$$

where $\lambda = \frac{1}{2p} - \frac{1}{4}$.

The number of derivatives necessary on the right-hand side of (7) is not more than the number estimated on the left-hand side ("no loss").

5 Isentropic Case

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ \rho u_t + \rho u u_x + p_x &= S_x, \\ \tau S_t + S &= \mu u_x. \end{aligned}$$

$\tau = 0$: global large solutions exist, $\tau > 0$: blow-up for large data, i.e. a similar situation as in the non-isentropic case.

Linearized Isentropic Case:

$$\begin{aligned} \rho_t + u_x &= 0, \\ u_t + R_1 \rho_x - S_x &= 0, \\ \tau S_t + S - \mu u_x &= 0. \end{aligned}$$

$\tau = 0$:

$$u_{tt} - R_1 u_{xx} - \mu u_{txx} = 0,$$

with some $R_1 > 0$.
Bounded domains: exponential stability, Cauchy problem: no loss of regularity.

$\tau > 0$:

$$\tau u_{ttt} + u_{tt} - R_1 u_{xx} - (\tau R_1 + \mu) u_{txx} = 0.$$

(Jordan-Moore-Gibson-Thompson type), again: exponential stability in bounded domains and no loss of regularity for the Cauchy problem.

6 Remarks on Higher Dimensions:

Higher-dimensional case ($n = 2, 3$), first classical:

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho u) &= 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p &= \mathrm{div}(S), \\ \rho \partial_t e + \rho u \cdot \nabla e + p \mathrm{div} u + \mathrm{div} q &= S : \nabla u, \end{aligned}$$

with the constitutive law for a Newtonian fluid,

$$S = \mu \left(\nabla u + \nabla u^T - \frac{2}{n} \mathrm{div}\, u I_n \right) + \lambda \mathrm{div} u I_n$$

and heat conduction given by Fourier's law,

$$q = -\kappa \nabla \theta.$$

Maxwell splitting: $S = S_1 + S_2 I_n$, with

$$\begin{aligned} S_1 &= \mu(\nabla u + \nabla u^T - \frac{2}{n} \mathrm{div} u I_n), \\ S_2 &= \lambda \mathrm{div} u. \end{aligned}$$

Relaxation in S_2 (only) and q:

$$\begin{aligned}
\partial_t \rho + \mathrm{div}(\rho u) &= 0, \\
\rho \partial_t u + \rho u \cdot \nabla u + \nabla p &= \mu \mathrm{div}(\nabla u + \nabla u^T - \frac{2}{n} \mathrm{div} u I_n) + \nabla S_2, \\
\rho \partial_t e + \rho u \cdot \nabla e + p \mathrm{div} u + \mathrm{div} q &= \mu(\nabla u + (\nabla u)^T - \frac{2}{n} \mathrm{div} u I_n) : \nabla u + S_2 \mathrm{div} u, \\
\tau_1(\partial_t q + u \cdot \nabla q) + q + \kappa \nabla \theta &= 0, \\
\tau_3(\partial_t S_2 + u \cdot \nabla S_2) + S_2 &= \lambda \mathrm{div} u,
\end{aligned}$$

Results: local existence, global small existence for $\mu > 0$, blow-up for large data for $\mu = 0$ (joint work with Y. Hu [1]).

References

[1] Hu, Y., Racke, R.: Global existence versus blow-up of solutions for multi-dimensional hyperbolic compressible Navier-Stokes equations. *Konstanzer Schriften Math.* **405** (2022), 26 pp.

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[3] Kazhikhov, A.V.: Cauchy problem for viscous gas equations, *Siberian Mathematical Journal* **23** (1982), 44-49.

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