ON LARGE PRIME FACTORS OF FOURIER COEFFICIENTS OF NEWFORMS

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ABSTRACT. This is an expository article showcasing some existing results about large Fourier coefficients of normalized Hecke eigenforms which are non CM forms. We also allude to some very recent works in this direction.

1. INTRODUCTION

Throughout the article, let p, q, ℓ denote rational primes, m, n be natural numbers and $k \geq 2$ an even integer. Also let $\mathcal{H}$ be the Poincaré upper half plane defined as

$$\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$$

and for any $z \in \mathcal{H}$, we have $q = e^{2\pi i z}$. Let Γ denote the full modular group given by

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}.$$

This group acts on the upper half plane $\mathcal{H}$ via fractional linear transformations and the quotient space $\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})$ is the set of isomorphism classes of elliptic curves over $\mathbb{C}$.

Definition 1. *A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ if f satisfies the transformation property*

$$f(\gamma.z) = (cz+d)^k f(z) \tag{1}$$

for any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and also f is holomorphic at ∞ which is the only cusp up to $\mathrm{SL}_2(\mathbb{Z})$ equivalence.

Due to the transformation property satisfied by modular forms, they are periodic functions with period one and hence have a Fourier expansion

$$f(z) = \sum_{n \geq 0} a_f(n) q^n, \quad z \in \mathcal{H}.$$

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Holomorphicity at ∞ ensured that the above power series has no negative terms. We say that $a_f(n)$ is the nth Fourier coefficient of f.

Definition 2. *A modular form f is called a cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ if $a_f(0) = 0$.*

The set of modular forms of weight k for Γ forms a finite dimensional complex vector space denoted by M_k. The set of cusp forms, denoted by S_k, forms a co-dimension one subspace of M_k.

One can define an inner product, the *Petersson inner product*, on the space of cusp forms S_k. This is given by

$$< f, g >= \int_{\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

In fact, existence of the integral is ensured if at least one of f and g, say f is a cusp form as $f(z) = O(e^{-2\pi y})$ as $y \to \infty$. Further, since the integrand is $\mathrm{SL}_2(\mathbb{Z})$ invariant, choice of $\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})$ does not matter.

For integers $n \geq 1$, the n-th Hecke operator T_n on the space of cusp forms (one can define it for all modular forms, but it preserves the subspace of cusp forms and we will restrict our attention to cusp forms) is defined as follows: For $f \in S_k$, $T_n(f) \in S_k$ is given by

$$T_n(f)(z) = n^{k-1} \sum_{\substack{a \geq 1, ad=n \\ 0 \leq b < d}} d^{-k} f\left(\frac{az+b}{d}\right).$$

The family of Hecke operators $T_n, n \geq 1$ are Hermitian with respect to the Petersson inner product. This implies that each T_n is diagonalisable and their eigenvalues are real. Further these Hecke operators $T_n, n \geq 1$ are commuting and hence the space S_k has a basis consisting of cusp forms which are simultaneous eigen vectors for all these Hecke operators.

If $f = \sum_{n=1}^{\infty} a_f(n) q^n$ is a Hecke eigen form, i.e. is an eigen vector for all Hecke operators $T_n,\ n \geq 1$, then its first Fourier coefficient $a_f(1)$ is necessarily non-zero. We say that f is normalised if $a_f(1) = 1$. For such a normalised eigen form, the nth Fourier coefficient $a_f(n)$ is an eigen value of T_n.

Let f be a normalized Hecke eigen cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. The first such non-trivial normalized Hecke eigenform belongs to S_{12} and is called the Delta function

$$\Delta(z) \ = \ q \prod_{n=1}^{\infty} (1-q^n)^{24} \ = \ \sum_{n \geq 1} \tau(n) q^n.$$

Fourier coefficients of Δ are called Ramanujan tau function. Ramanujan predicted (see [18]) three important properties of τ function. The first two such properties are

$$\begin{aligned} \tau(mn) &= \tau(m)\tau(n) \quad \text{for} \quad (m,n)=1 \\ \text{and} \quad \tau(p^r) &= \tau(p^{r-1})\tau(p) - p^{11}\tau(p^{r-2}), \end{aligned} \tag{2}$$

for any prime number p and integer $r \geq 2$. These were proved by Mordell (see [13, 8, 9] for further details). The third prediction was about size of $|\tau(p)|$. More precisely, Ramanujan predicted that

$$|\tau(p)| < 2p^{11/2} \tag{3}$$

which is equivalent to the property that the polynomial

$$x^2 - \tau(p)x + p^{11}$$

does not have any real root.

A famous conjecture of Lehmer predicts that

$$\tau(n) \neq 0 \qquad \text{for all} \quad n.$$

Lehmer himself and later independently Kowalski-Robert-Wu [10] and Murty-Murty [15] showed that

$$\tau(n) \neq 0 \quad \Longleftrightarrow \quad \tau(p) \neq 0$$

for all primes p. Let me a sketch a proof of this which will showcase the importance of Ramanujan's conjectures (2) and (3) in this context. Suppose that $\tau(p) \neq 0$ for all p. Let α and β be the roots of the polynomial $x^2 - \tau(p)x + p^{11}$. Using induction and applying (2), we can write

$$\tau(p^{n-1}) = \frac{\alpha_p^n - \beta_p^n}{\alpha_p - \beta_p}. \tag{4}$$

Expanding then right hand side, we see that

$$\tau(p^r) = \tau(p)^{\delta_r} \prod_{1 \leq j < \frac{r+1}{2}} \left(\tau(p)^2 - (\zeta_{r+1}^j + \zeta_{r+1}^{-j} + 2)p^{11} \right),$$

where $\zeta_{r+1} = e^{\frac{2i\pi}{r+1}}$ and

$$\delta_r = \begin{cases} 0 & \text{when } r \text{ is even;} \\ 1 & \text{when } r \text{ is odd.} \end{cases}$$

Now applying (3), we can see that $\frac{\tau(p)^2}{p^{11}} < 4$ and is a rational number which is not an integer. However $\zeta_{r+1}^{j} + \zeta_{r+1}^{-j} + 2$ is an algebraic integer. Thus $\tau(p^r) \neq 0$ for all $r \geq 1$.

Let us fix few notations before proceeding further. For a subset S of primes, we shall define the lower and the upper densities of S to be

$$\liminf_{x\to\infty} \frac{\#\{p \leq x : p \in S\}}{\pi(x)} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\#\{p \leq x : p \in S\}}{\pi(x)}$$

respectively. Here $\pi(x)$ denotes the number of rational primes less than or equal to x.

If both upper and lower density of a subset S of primes are equal, say to $\mathcal{D}$, we say that S has density $\mathcal{D}$. We say a property A holds for almost all primes if the set

$$\{p : p \text{ has property } A\}$$

has density one.

Now coming back to the conjecture of Lehmer, the best known result in this direction is by Serre [20, 21] who showed that $\tau(p) \neq 0$ for almost all primes in the sense of density. It is now natural to ask

Question. *When $\tau(p) \neq 0$, what can we say about the lower bound of $|\tau(p)|$?*

It seems Atkin and Serre thought about this question and predicted that for any $\epsilon > 0$, there exists a constant $c(\epsilon) > 0$ such that

$$|\tau(p)| \ \geq \ c(\epsilon) p^{9/2-\epsilon}$$

for all primes p. In fact, their conjecture was for newforms of weight $k \geq 4$ for

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv 0 \bmod n \right\}.$$

Let f be a normalized cuspidal Hecke eigenform of even weight $k \geq 4$ for $\Gamma_0(N)$ with trivial character lying in the newform space and having Fourier coefficients $\{a_f(n)\}_{n\geq 1}$. It is well known that $a_f(n)$'s are real algebraic integers and $\mathrm{K}_f = \mathbb{Q}(\{a_f(n) : n \in \mathbb{N}\})$ is a number field (see [22]). We say that f has complex multiplication (or f is a CM form) if there exists an imaginary quadratic field K such that $a_f(p) = 0$ for all primes $p \nmid N$ which are inert in K. Otherwise f is called a non-CM form. Serre, using Roth's theorem, showed that if f is a CM form and $a_f(p) \neq 0$, then $a_f(p)$ satisfies the desired lower bound. When f is a non CM form, Atkin-Serre conjectured the following;

Conjecture 1. (Atkin-Serre) [19] *For any $\epsilon > 0$,*

$$|a_f(p)| \gg_\epsilon p^{(k-3)/2-\epsilon}$$

From now on, we assume that f is a non CM form and $a_f(n)$'s are rational integers. In 1987, Murty, Murty and Shorey [16] using Diophantine techniques, proved the following theorem when $f = \Delta$.

Theorem 1. (Murty-Murty-Shorey) *There exists a constant* $c > 0$ *such that*

$$|\tau(n)| \geq (\log n)^c$$

provided $\tau(n)$ *is odd.*

Even though it is a very nice result, it does not provide any information about Atkin-Serre conjecture. It can be easily seen, either by using Jacobi's triple product identity or by non existence of non-trivial Galois representations

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_2)$$

which are ramified only at 2, that

$$\tau(p) \equiv 0 \pmod{2}$$

for all primes p. One year later, Murty, Murty and Saradha [16], using effective Chebotarev density theorem, proved the following result;

Theorem 2. (Murty-Murty-Saradha) *There exists a constant* $c > 0$ *such that*

$$|a_f(p)| \geq (\log p)^c$$

for a set of primes p *of density one.*

Applying effective Sato-Tate theorem, this result was improved in 2021 by Gafni-Thorner-Wong [4].

Theorem 3. (Gafni-Thorner-Wong) *We have*

$$|a_f(p)| \; > \; 2p^{11/2}\frac{\log\log p}{\sqrt{\log p}}$$

for a set of primes p *of density one.*

It is now natural to ask, what can one say about the largest prime factor of $\tau(p)$? Before proceeding further, let us fix some notation. For an integer n, let $P(n)$ denote the largest prime factor of n with the convention that $P(0) = P(\pm 1) = 1$.

For $p \nmid N$, let α_p, β_p be the roots of the polynomial $x^2 - a_f(p)x + p^{k-1}$ and $\gamma_p = \alpha_p/\beta_p$. For any prime ideal $\mathfrak{P}$ of the ring of integers of $\mathbb{Q}(\gamma_p)$, let $\nu_{\mathfrak{P}}$ denote the $\mathfrak{P}$-adic valuation. Also let φ denote the Euler-phi function and $\omega(n)$ denote the number of distinct prime factors of n. From the recurrence formula (2), for any integer $r \geq 2$ and primes $p \nmid N$, we can see that

$$\tau(p^{r-1}) \;=\; \prod_{\substack{d|r \\ d>1}} \Phi_d(\alpha_p, \beta_p),$$

where

$$\Phi_d(X, Y) \;=\; \prod_{\substack{j=1 \\ (j,n)=1}}^{n} \left(X - \zeta_d^j Y\right) \in \mathbb{Z}[X, Y],\; d \geq 2$$

is the d-th cyclotomic polynomial and $\zeta_d = e^{\frac{2\pi i}{d}}$. In 2013, Stewart [23] proved the following theorem.

Theorem 4. *If $\alpha, \beta \in \mathbb{C}$ are such that $(\alpha+\beta)^2, \alpha\beta$ are non-zero rational integers and γ_p is not a root of unity, then*

$$P\left(\Phi_d(\alpha, \beta)\right) \;>\; d \exp\left(\frac{\log d}{104 \log\log d}\right)$$

for sufficiently large n_0 depending on discriminant of $\mathbb{Q}(\gamma_p)$ and $\omega(\gamma_p)$.

Applying Stewart's theorem, for $p \nmid N$ and $r \geq 2$, we see that

$$a_f(p^{r-1}) \;>\; r \exp\left(\frac{\log r}{104 \log\log r}\right),$$

where γ_p is not a root of unity. The lower bound for the largest prime factor of $a_f(p^{r-1})$ in the result of Stewart is a function of r and not of p. One would like to have a lower bound where both p and r appear. Using a "super Wieferich" criterion, one can derive such desired lower bound. Let us recall the notion of super Wieferich primes.

Definition 3. *Let α be a non-zero element of a number field K which is not a root of unity and $\mathfrak{p}$ be a prime ideal in the ring of integers $\mathcal{O}_K$ such that $\nu_{\mathfrak{p}}(\alpha) = 0$. We say that $\mathfrak{p}$ is a Wieferich prime for α in K if $\nu_{\mathfrak{p}}\left(\alpha^{\mathcal{N}(\mathfrak{p})-1} - 1\right) \geq 2$ and is called a super Wieferich prime for α in K if $\nu_{\mathfrak{p}}\left(\alpha^{\mathcal{N}(\mathfrak{p})-1} - 1\right) \geq 3$. Here $\mathcal{N}$ denotes the absolute norm on K.*

Using an effective number field analogue of a result of Murty and Séguin [14], in a joint work with Bilu and Naik [2], we prove the following theorem.

Theorem 5. *Let f be as before with rational integer Fourier coefficients and p be a prime with $p \nmid N$. Also let γ_p is not a root of unity,* $\mathrm{K} = \mathbb{Q}(\gamma_p)$ *and there exists a integer $n \geq 2$ such that*

$$\nu_{\mathfrak{P}}\left(\gamma_p^{\mathcal{N}(\mathfrak{P})-1} - 1\right) \leq n$$

for any prime ideal $\mathfrak{P}$ of the ring of integers of K. *Then for primes* $p > 3$, *we have*

$$P(a_f(p^{r-1})) \;>\; \frac{(k-1-2\nu_{f,p})\log p}{52n} \cdot \frac{\varphi(r)^2}{2^{\omega(r)}}$$

for all sufficiently large r *(depending on* f *and* p*). Here* $\nu_{f,p}$ *is the* p*-adic valuation of* $a_f(p)$, d_f *is the degree of* K_f *over* $\mathbb{Q}$, $h_{f,p}$ *is the class number of* K *and* $\mathcal{N}$ *is the absolute norm on* K.

In 2021, when $f = \Delta$, Bennett, Gherga, Patel and Siksek [1] unconditionally proved the following theorem

Theorem 6. *There exists a computable constant* $c > 0$ *such that for any prime* p *with* $\tau(p) \neq 0$ *and for any integer* $r \geq 2$, *we have*

$$P(\tau(p^r)) \;>\; c\, \exp\left(\frac{\log\log p^r}{\log\log\log p^r}\right).$$

In 2023, in a joint work with Bilu and Naik [2], using effective Chebotarev density theorem, effective Sato-Tate theorem, Brun-Titchmarsh theorem, we proved the following theorem.

Theorem 7. *Let* f *be a non-CM normalized cuspidal Hecke eigenform of even weight* $k \geq 2$ *for* $\Gamma_0(N)$ *having integer Fourier coefficients* $\{a_f(n)\}_{n\geq 1}$ *and let* $\epsilon > 0$ *be a real number. Then we have*

$$P(a_f(p)) > (\log p)^{1/8}(\log\log p)^{3/8-\epsilon}$$

for a set of primes p *of density one.*

Remark 1.1. *Instead of* $a_f(p)$, *if we consider product of* $a_f(p)$ *and* $a_f(p^2)$, *using recurrence formula and analytic techniques, finding lower bound becomes relatively easy. When* $f = \Delta$, *Luca and Shparlinski* [12] *showed that for a set of primes* p *of density one, we have*

(5) $$P(\tau(p)\tau(p^2)) \;>\; (\log p)^{\frac{33}{31}+o(1)}.$$

The exponent in the lower bound (5) *was further refined to* $13/11$ *by Garaev, Garcia and Konyagin* [5], *but for infinitely many primes* p.

Now suppose that the Generalized Riemann hypothesis (GRH), i.e., the Riemann hypothesis for all Artin L-series is true. Conditionally on GRH, we prove the following result.

Theorem 8. *Suppose that GRH is true and let* f *be as in Theorem 7. There exists a positive constant* c *depending on* f *such that the set of primes* p *for which* $a_f(p) \neq 0$ *and*

$$P(a_f(p)) \;>\; c\, p^{1/14}(\log p)^{2/7}$$

has lower density at least $1 - \frac{2}{13(k-1)}$.

As a consequence of Theorem 7 and Theorem 8, we have the following results.

Theorem 9. *Let f be as in Theorem 7 and $\epsilon > 0$ be a real number. Then for a set of primes p of density one and for all $r \geq 1$, we have*

$$P(a_f(p^{2r+1})) > (\log p^{2r+1})^{1/8} (\log\log p^{2r+1})^{3/8-\epsilon}.$$

Further, suppose that GRH is true and let f be as in Theorem 7. Then for a set of primes of lower density at least $1 - \frac{2}{13(k-1)}$ and for all $r \geq 1$, we have

$$P(a_f(p^{2r+1})) \; > \; c\, p^{1/14} (\log p^{2r+1})^{2/7},$$

where c is a positive constant depending on f.

In a joint work with Naik [6], using symmetric powers of a Galois representation attached to Ramanujan Delta function and divisibility properties of cyclotomic polynomials, we prove the following theorem.

Theorem 10. *Let $r \geq 1$ be an integer and $\epsilon > 0$ be a real number. Then for a set of primes p of density one, we have*

$$P(a_f(p^{2r})) \; > \; (\log p^{2r})^{1/8} (\log\log p^{2r})^{3/8-\epsilon}.$$

Further, if $q = P(2r+1)$ is sufficiently large, then the set of primes p such that

$$P(a_f(p^{2r})) \; > \; q^{-\epsilon} (\log p^{2r})^{1/8} (\log\log p^{2r})^{3/8}$$

has positive lower density.

Combining Theorem 7 and Theorem 10, we get

Corollary 11. *For any $\epsilon > 0$ and any integer $r \geq 1$, we have*

$$P(a_f(p^r)) > (\log p^r)^{1/8} (\log\log p^r)^{3/8-\epsilon}$$

for a set of primes p of density one.

2. Future Directions and new results

Analogous to the question of finding large prime factors of an integer in a short interval (for instance in intervals of length strictly less than $x/\log x$), one can also ask the question of finding large prime factors of Fourier coefficients of newforms in short intervals i.e. Atkin-Serre question in short intervals. In a recent work [7], along with Sunil Naik, we are able to show existence of large prime factors of Fourier coefficients of newforms in an interval of length $\frac{x}{(\log x)^A}$ for any $A > 0$. It would be nice to show large prime factors in a shorter intervals. Under GRH, we can prove similar theorems [7] in intervals of length as small as $x^{1/2+\epsilon}$ for any $\epsilon > 0$.

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