DISTRIBUTIONS OF SUMS OF THE DIVISOR FUNCTION OVER FUNCTION FIELDS

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Abstract. We discuss some recent results due to Keating, Rodgers, Roditty-Gershon, and Rudnick [KRRGR18] and Kuperberg and Lalín [KL22b, KL22a] on the distribution of certain sums of the divisor function $d_k(f)$ over the function field $\mathbb{F}_q[T]$ when q goes to infinity. The results show that the main-squares of such sums are related to integrals over the ensembles of unitary, unitary symplectic, and orthogonal matrices. We also discuss some conjectures over the number field case that can be derived from the function field statements.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements, where q is a prime power. In $\mathbb{F}_q[T]$, one can define the divisor function $d_k(f)$ analogously to the construction over number fields. More precisely, for a monic polynomial f,

$$d_k(f) := \#\{(f_1, \dots, f_k) \,:\, f = f_1 \cdots f_k,\ f_j \text{ monic}\},$$

while $d_k(cf) := d_k(f)$ for $c \in \mathbb{F}_q^*$ and f monic. As in the number field case, this function arises by considering the coefficients of powers of the zeta function. The zeta function for $\mathbb{F}_q[T]$ is given by

$$\zeta_q(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{1}{|P|^s}\right)^{-1} = \frac{1}{1-q^{1-s}},$$

where the sum is taken over the set $\mathcal{M}$ of monic polynomials in $\mathbb{F}_q[T]$ and the product is taken over the subset $\mathcal{P}$ of monic irreducible polynomials. The norm of a nonzero element $f \in \mathbb{F}_q[T]$ is given by $|f| := \#(\mathbb{F}_q[T]/(f)) = q^{\deg(f)}$. The initial sum and Euler product converge for $\mathrm{Re}(s) > 1$, however, the right-hand side equality gives a meromorphic continuation with single poles at $s = 1 + \frac{2\pi i k}{\log q}$ for k integer. Then we have

$$\zeta_q(s)^k = \sum_{f \in \mathcal{M}}^{\infty} \frac{d_k(f)}{|f|^s}.$$

In [KRRGR18], Keating, Rodgers, Roditty-Gershon, and Rudnick study the distribution of

$$\mathcal{S}_{d_k;n;Q}(A) := \sum_{\substack{f \in \mathcal{M}_n \\ f \equiv A \,(\mathrm{mod}\, Q)}} d_k(f),$$

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where $\mathcal{M}_n$ denotes the set of monic polynomials in $\mathbb{F}_q[T]$ with degree n, and prove the following result.

Theorem 1.1. [KRRGR18, Theorem 3.1] *If Q is square free and $n \leq k(\deg(Q)-1)$, then the variance of $\mathcal{S}_{d_k;n;Q}$ is given by*

$$\lim_{q\to\infty} \frac{\operatorname{Var}(\mathcal{S}_{d_k;n;Q})}{q^n/|Q|} = \int_{\mathrm{U}(\deg(Q)-1)} \Big| \sum_{\substack{j_1+\cdots+j_k=n \\ 0\leq j_1,\ldots,j_k\leq \deg(Q)-1}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_k}(U)\Big|^2 dU, \tag{1}$$

where the integral takes place over the unitary matrices, and the $\mathrm{Sc}_j(U)$ are the secular coefficients, defined for a $N\times N$ matrix U by

$$\det(I+xU) = \sum_{j=0}^{N} \mathrm{Sc}_j(U)x^j.$$

A key ingredient to obtain such result is given by an equidistribution theorem of Katz [Kat13a].

The work of Keating et al. also includes a similar theorem involving the same integral in the case of the short interval regime. Let

$$\mathcal{N}_{d_k,h,n}(A) := \sum_{\substack{f\in\mathcal{M}_n \\ |f-A|\leq q^h}} d_k(f).$$

Theorem 1.2. [KRRGR18, Theorem 1.2] *For $0 \leq h \leq \min\{n-5, (1-\frac{1}{k})n-2\}$, as $q\to\infty$,*

$$\operatorname{Var}(\mathcal{N}_{d_k,h,n}) = q^h \int_{\mathrm{U}(n-h-2)} \left| \sum_{\substack{j_1+\cdots+j_k=n \\ 0\leq j_1,\ldots,j_k\leq n-h-2}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_k}(U)\right|^2 dU + O(q^{h-\frac{1}{2}}). \tag{2}$$

An ingredient in the proof of the above theorem is an equidistribution result of Katz [Kat13b].

The authors further study the integrals in (1) and (2) and obtain the following result.

Theorem 1.3. [KRRGR18, Theorem 1.5] *Let $c=\frac{m}{N}$. Then for $c\in[0,k]$,*

$$\int_{\mathrm{U}(N)} \Big| \sum_{\substack{j_1+\cdots+j_k=m \\ 0\leq j_1,\ldots,j_k\leq N}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_k}(U)\Big|^2 dU = \gamma_k(c)N^{k^2-1} + O_k(N^{k^2-2}),$$

where

$$\gamma_k(c) = \frac{1}{k!G(1+k)^2}\int_{[0,1]^k} \delta_c(w_1+\cdots+w_k)\prod_{i<j}(w_i-w_j)^2 d^k w. \tag{3}$$

Here $\delta_c(w) = \delta(w-c)$ is the delta distribution translated by c, and G is the Barnes G-function, given for positive integers k by $G(1+k) = 1!\cdot 2!\cdots(k-1)!$.

Keating, Rodgers, Roditty-Gershon, and Rudnick study various properties of $\gamma_k(c)$. They prove that it is a continuous piecewise polynomial function of c supported in the interval $[0,k]$. More specifically, it is a fixed polynomial for $r\leq c< r+1$ (with r an integer), and each time the value of c passes through an integer, it becomes a different polynomial, of degree k^2-1 if $[r,r+1]\subset[0,k]$.

Based on their result over function fields and their study of the integral in (1), Keating et al. are able to conjecture the following.

Conjecture 1.4. [KRRGR18, Conjecture 3.3] *Let Q be prime and define*

$$\mathcal{S}_{d_k;X;Q}(A) := \sum_{\substack{n\leq X\\ n\equiv A \,(\mathrm{mod}\, Q)}} d_k(n).$$

Then, if $Q^{1+\varepsilon} < X < Q^{k-\varepsilon}$, as $X\to\infty$,

$$\mathrm{Var}(\mathcal{S}_{d_k;X;Q}) \sim \frac{X}{Q} a_k \gamma_k \left(\frac{\log X}{\log Q}\right) (\log Q)^{k^2-1},$$

where a_k is given by

$$a_k = \prod_p \left(\left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \binom{k-1}{j}^2 \frac{1}{p^j}\right)$$

and γ_k is given by (3)*.*

An analogous conjecture is formulated in the short interval regime.

2. Results in the symplectic and orthogonal worlds

In [KL22b], we consider the distribution of $d_k(f)$ when restricted to quadratic residues modulo an irreducible polynomial P. In other words, we define

$$\mathcal{S}^S_{d_k,n}(P) := \sum_{\substack{f\in\mathcal{M}_n\\ f\equiv\square \,(\mathrm{mod}\, P)\\ P\nmid f}} d_k(f),$$

where P is a monic irreducible polynomial of degree $2g+1$. Let $\mathcal{P}_n$ denote the set of monic irreducible polynomials of degree n. We prove

Theorem 2.1. [KL22b, Theorem 1.1] *Let $n\leq 2gk$. As $q\to\infty$*

$$\mathcal{S}^S_{d_k,n}(P) \sim \frac{1}{2}\sum_{\substack{f\in\mathcal{M}_n\\ P\nmid f}} d_k(f) \sim \frac{q^n}{2}\binom{k+n-1}{k-1},$$

and

$$\mathrm{Var}^*(\mathcal{S}^S_{d_k,n}) := \frac{1}{\#\mathcal{P}_{2g+1}} \sum_{P\in\mathcal{P}_{2g+1}} \left| \mathcal{S}^S_{d_k,n}(P) - \frac{1}{2}\sum_{\substack{f\in\mathcal{M}_n\\ P\nmid f}} d_k(f)\right|^2$$

$$\sim \frac{q^n}{4}\int_{\mathrm{Sp}(2g)} \left| \sum_{\substack{j_1+\cdots+j_k=n\\ 0\leq j_1,\ldots,j_k\leq 2g}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_k}(U)\right|^2 dU. \tag{4}$$

We remark that this result, while analogous to Theorems 1.1 and 1.3, involves an integral over the set of unitary symplectic matrices.

We also consider the sum

$$\mathcal{N}^S_{d_\ell,k,n}(v) := \sum_{\substack{f\in\mathcal{M}_n\\ f(0)\neq 0\\ U(f)\in \mathrm{Sect}(v,k)}} d_\ell(f),$$

where the sum is taken over monic polynomials of fixed degree with a condition over function fields that models an analogue of having the argument of a complex number lying in certain specific sector of the unit circle. This follows a construction of Gaussian integers in the function field context that was initially developed by Bary-Soroker, Smilansky, and Wolf in [BSSW16] to study Landau's counting of integers that are sum of two squares, and it was later considered by Rudnick and Waxman in [RW19] to model the distribution of Gaussian primes in circle sectors. The idea is as follows. For a $P(T) \in \mathcal{P}$, there exist $A(T), B(T) \in \mathbb{F}_q[T]$ such that

$$P(T) = A(T)^2 + TB(T)^2 \tag{5}$$

if and only if $P(0)$ is a square in $\mathbb{F}_q$. Set $S := \sqrt{-T}$ so that $\mathbb{F}_q[T] \subseteq \mathbb{F}_q[S]$. Now equation (5) becomes

$$P(T) = (A + BS)(A - BS) = \mathfrak{p}\overline{\mathfrak{p}}$$

in $\mathbb{F}_q[S]$. There are two automorphisms of $\mathbb{F}_q[S]$ fixing $\mathbb{F}_q[T]$: the identity and $S \to -S$, which can be thought of as the analogue of complex conjugation. It can be extended to the ring of formal power series:

$$\sigma : \mathbb{F}_q[[S]] \to \mathbb{F}_q[[S]], \qquad \sigma(S) = -S.$$

The norm map is then defined by

$$\mathrm{Norm} : \mathbb{F}_q[[S]]^\times \to \mathbb{F}_q[[T]]^\times, \qquad \mathrm{Norm}(f) = f\sigma(f) = f(S)f(-S).$$

The group of formal power series with constant term 1 and unit norm is

$$\mathbb{S}^1 := \{g \in \mathbb{F}_q[[S]]^\times \ : \ g(0) = 1, \mathrm{Norm}(g) = 1\},$$

and can be thought of as an analogue of the unit circle in this setting.

For $f \in \mathbb{F}_q[[S]]$, let $\mathrm{ord}(f) = \max\{j \ : \ S^j \mid f\}$ and $|f|_\infty := q^{-\mathrm{ord}(f)}$, the absolute value associated with the place at infinity. This finally leads to the definition of a sector in the unit circle:

$$\mathrm{Sect}(v; k) := \{w \in \mathbb{S}^1 \ : \ |w - v|_\infty \leq q^{-k}\}.$$

We have the following result.

Theorem 2.2. [KL22b, Theorem 1.2] *Let $n \leq \ell(2\kappa - 2)$ with $\kappa = \left\lfloor \frac{k}{2} \right\rfloor$. As $q \to \infty$, the average of $\mathcal{N}^S_{d_\ell,k,n}$ is given by*

$$\langle \mathcal{N}^S_{d_\ell,k,n} \rangle \sim q^{n-\kappa} \binom{\ell + n - 1}{\ell - 1},$$

and the variance $\mathrm{Var}(\mathcal{N}^S_{d_\ell,k,n})$ is given by

$$\frac{1}{q^\kappa} \sum_{u \in \mathbb{S}^1_k} \left| \mathcal{N}^S_{d_\ell,k,n}(u) - \langle \mathcal{N}^S_{d_\ell,k,n} \rangle \right|^2 \sim \frac{q^n}{q^\kappa} \int_{\mathrm{Sp}(2\kappa-2)} \Bigg| \sum_{\substack{j_1+\cdots+j_\ell=n \\ 0 \leq j_1,\ldots,j_\ell \leq 2\kappa-2}} \mathrm{Sc}_{j_1}(U) \cdots \mathrm{Sc}_{j_\ell}(U) \Bigg|^2 dU.$$

Once again, the integral involved in the above statement is over the set of unitary symplectic matrices.

Let χ_2 be the character on $\mathbb{F}_q$ defined by

$$\chi_2(x) = \begin{cases} 0 & x = 0, \\ 1 & x \text{ is a nonzero square in } \mathbb{F}_q, \\ -1 & \text{otherwise.} \end{cases}$$

The character χ_2 can be extended to $\mathcal{M}$ by defining $\chi_2(f) := \chi_2(f(0))$. Then equation (5) is solvable if and only if $\chi_2(P) = 1$.

In [KL22a], we study the following sum:

$$\mathcal{N}^{O}_{d_\ell,k,n}(v) := \sum_{\substack{f \in \mathcal{M}_n \\ f(0) \neq 0 \\ U(f) \in \mathrm{Sect}(v,k)}} d_\ell(f) \left(\frac{1+\chi_2(f)}{2} \right).$$

We have the following result.

Theorem 2.3. *Let $n \leq \ell(2\kappa - 1)$ with $\kappa = \lfloor \frac{k}{2} \rfloor$. As $q \to \infty$, the average of $\mathcal{N}^{O}_{d_\ell,k,n}$ is given by*

$$\langle \mathcal{N}^{O}_{d_\ell,k,n} \rangle \sim \frac{q^{n-\kappa}}{2} \binom{\ell+n-1}{\ell-1},$$

and the variance $\mathrm{Var}(\mathcal{N}^{O}_{d_\ell,k,n})$ is given by

$$(6)\quad \frac{1}{q^\kappa} \sum_{u \in \mathbb{S}^1_k} \left| \mathcal{N}^{O}_{d_\ell,k,n}(u) - \langle \mathcal{N}^{O}_{d_\ell,k,n} \rangle \right|^2 \sim \frac{q^n}{4q^\kappa} \int_{\mathrm{O}(2\kappa-1)} \Big| \sum_{\substack{j_1+\cdots+j_\ell=n \\ 0 \leq j_1,\ldots,j_\ell \leq 2\kappa-1}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_\ell}(U) \Big|^2 \mathrm{d}U.$$

We remark that now the integral takes place over the set of orthogonal matrices.

The three Theorems 2.1, 2.2, and 2.3 rest on equidistribution results due to Katz [KL22b, Kat17].

Naturally, the first step in understanding results in the style of Theorems 2.1, 2.2, and 2.3 consists of studying the integrals involved in (4) and (6). To this end, set

$$I^{S}_{d_k,2}(n;N) := \int_{\mathrm{Sp}(2N)} \Big| \sum_{\substack{j_1+\cdots+j_k=n \\ 0 \leq j_1,\ldots,j_k \leq 2N}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_k}(U) \Big|^2 dU$$

and

$$I^{O}_{d_k,2}(n;N) := \int_{\mathrm{O}(2N+1)} \Big| \sum_{\substack{j_1+\cdots+j_k=n \\ 0 \leq j_1,\ldots,j_k \leq 2N}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_k}(U) \Big|^2 \mathrm{d}U.$$

The simplest case of $k = 1$ for $I^{S}_{d_k,2}(n;N)$ follows from work by Corey, Farmer, Keating, Rubinstein, and Snaith [CFK+03].

Proposition 2.4. [KL22b, Corollary 5.12] [1] *If $k = 1$, then $I^{S}_{d_1,2}(n,N)$ is given by*

$$\int_{\mathrm{Sp}(2N)} \left| \mathrm{Sc}_n(U) \right|^2 dU = \begin{cases} \lfloor \frac{n+2}{2} \rfloor & 0 \leq n \leq N, \\ \lfloor \frac{2N-n+2}{2} \rfloor & N+1 \leq n \leq 2N. \end{cases}$$

As $N \to \infty$, $I^{S}_{d_1,2}(n,N)$ is asymptotic to

$$I^{S}_{d_1,2}(n,N) \sim \gamma^{S}_{d_1,2}(c) 2N,$$

where $c = \frac{n}{2N}$ and

$$\gamma^{S}_{d_1,2}(c) = \begin{cases} \frac{c}{2} & 0 \leq c \leq \frac{1}{2}, \\ \frac{1-c}{2} & \frac{1}{2} \leq c \leq 1. \end{cases}$$

[1] A different normalization was used for c in the statement of [KL22b, Corollary 5.12].

It can be also seen that $I^O_{d_1,2}(n;N)=0$.

In [KL22a] we do a systematic study of $I^S_{d_k,2}(n;N)$ and $I^O_{d_k,2}(n;N)$. Before stating the next result, we recall that a quasi-polynomial $P(m)$ of period r is a function on integer numbers for which there exist polynomials $P_0(m),\dots,P_{r-1}(m)$ such that

$$P(m)=\begin{cases}P_0(m) & m\equiv 0 \,(\mathrm{mod}\, r),\\ P_1(m) & m\equiv 1 \,(\mathrm{mod}\, r),\\ \vdots & \\ P_{r-1}(m) & m\equiv r-1 \,(\mathrm{mod}\, r).\end{cases}$$

Proposition 2.5. [KL22a, Propositions 4.2 and 8.2]

(1) *For $n\le N+\frac{1+k}{2}$, we have*

$$I^S_{d_k,2}(n;N)=\frac{1}{G(1+k)}\sum_{\substack{\ell=0\\ \ell\equiv n\,(\mathrm{mod}\,2)}}^{n}\binom{\frac{n-\ell}{2}+\binom{k+1}{2}-1}{\binom{k+1}{2}-1}^2\binom{\ell+k^2-1}{k^2-1}.$$

Moreover, $I^S_{d_k,2}(n;N)$ is a quasi-polynomial in n of period 2 and degree $2k^2+k-2$ (provided that $n\le N+\frac{1+k}{2}$).

(2) *For $n\le N+\frac{k}{2}$,*

$$I^O_{d_k,2}(n;N)=\frac{2}{G(1+k)}\sum_{\substack{\ell=0\\ \ell\equiv n\,(\mathrm{mod}\,2)}}^{n}\binom{\frac{n-\ell}{2}+\binom{k}{2}-1}{\binom{k}{2}-1}^2\binom{\ell+k^2-1}{k^2-1}.$$

Moreover, $I^O_{d_k,2}(n;N)$ is a quasi-polynomial in n of period 2 and degree $2k^2-k-2$ (provided that $n\le N+\frac{k}{2}$).

The above result shows that

$$I^S_{d_k,2}(n;N)\sim\gamma^S_{d_k,2}(c)(2N)^{2k^2+k-2}$$

for $c=\frac{n}{2N}$ and

$$I^O_{d_k,2}(n;N)\sim\gamma^O_{d_k,2}(c)(2N+1)^{2k^2-k-2}$$

for $c=\frac{n}{2N+1}$, and where $\gamma^S_{d_k,2}(c)$ is a polynomial of degree $2k^2+k-2$ (respectively $\gamma^O_{d_k,2}(c)$ is a polynomial of degree $2k^2-k-2$) in the interval $0\le c\le\frac12$. However, this does not completely describe the functions since $\gamma^S_{d_k,2}(c)$ and $\gamma^O_{d_k,2}(c)$ are non-trivial in the interval $0\le c\le k$. We provide evidence that $\gamma^S_{d_k,2}(c)$ (resp. $\gamma^O_{d_k,2}(c)$) is given by polynomials of degree $2k^2+k-2$ (resp. $2k^2-k-2$) in each of the intervals $\left[\frac{m}{2},\frac{m+1}{2}\right]\subset[0,k]$.

Theorem 2.6. [KL22a, Theorems 1.2 and 1.5] *Let $c=\frac{a}{b}$ be a fixed rational number and k be a fixed integer.*

(1) *If $2N$ is a multiple of b, then $I^S_{d_k,2}(c2N;N)$ is a polynomial of degree $2k^2+k-2$ in N.*
(2) *If $2N+1$ is a multiple of b, then $I^O_{d_k,2}(c(2N+1);N)$ is a polynomial of degree $2k^2-k-2$ in N.*

This result is achieved by expressing a generating function involving $I^S_{d_k,2}(n;N)$ (resp. $I^O_{d_k,2}(n;N)$) in terms of a sum of Schur functions over certain even partitions (resp. even conjugate partitions). This allows us to interpret $I^S_{d_k,2}(n;N)$ (resp. $I^O_{d_k,2}(n;N)$) as a function counting points inside a polytope, and combining this with Ehrhart theory we can deduce the degree of $\gamma^S_{d_k,2}(c)$ (resp. $\gamma^O_{d_k,2}(c)$) for certain rational values of n. The techniques from [KL22a] are inspired by those from [KRRGR18], but they are more involved. The symmetric function theory in the unitary case of [KRRGR18] requires the consideration of semi-standard Young tableaux arising from a single rectangular Ferrer diagram, while in [KL22a] we must consider a sum including more general shapes.

We also study the generating functions of $I^S_{d_k,2}(n;N)$ and $I^O_{d_k,2}(n;N)$ with complex analysis techniques, allowing us to describe $\gamma^S_{d_k,2}(c)$ and $\gamma^O_{d_k,2}(c)$ as piecewise polynomial functions of degree *at most* $2k^2+k-2$ and $2k^2-k-2$ respectively for any real number. More precisely, we have the following result.

Theorem 2.7. [KL22a, Theorems 1.3 and 1.6]

(1) *The function* $\gamma^S_{d_k,2}(c)$ *is given by*

$$\gamma^S_{d_k,2}(c) = \sum_{\substack{0\le b\le c\\ 0\le a\le 2c-b}} (2c-k)^{2k^2+k-a(k-a)-b(k-b)-2} g^S_{a,b}(2c-k),$$

and each $g^S_{a,b}(t)$ *is a polynomial of degree* $a(k-a)+b(k-b)$.

(2) *The function* $\gamma^O_{d_k,2}(c)$ *is given by*

$$\gamma^O_{d_k,2}(c) = \sum_{\substack{0\le b\le c\\ 0\le a\le 2c-b}} (2c-k)^{2k^2-k-a(k-a)-b(k-b)-2} g^O_{a,b}(2c-k),$$

and each $g^O_{a,b}(t)$ *is a polynomial of degree* $a(k-a)+b(k-b)$.

As in the case of the previous results, the techniques for proving the above statements are similar to techniques employed in [KRRGR18]. However, considerable new challenges arise when considering the square of the absolute value inside the integrals for $I^S_{d_k,2}(n;N)$ and $I^O_{d_k,2}(c(2N+1);N)$. This square is natural in the unitary case, where the eigenvalues are complex and the absolute value is necessary. In the symplectic and orthogonal cases, the square comes from considering the variance, but it is less natural in the random matrix theory context and poses many technical difficulties.

3. Some conjectures in the number field case

The understanding of $\gamma^S_{d_k,2}(c)$ and $\gamma^O_{d_k,2}(c)$ can be applied to formulate conjectures in the number field setting that are analogous to the statements of Theorems 2.1, 2.2, and 2.3.

Conjecture 3.1. [KL22a, Conjecture 1.1] *Let p be a prime and define*

$$\mathcal{S}^S_{d_k;x}(p) := \sum_{\substack{n\le x\\ n\equiv\square \,(\mathrm{mod}\, p)\\ p\nmid n}} d_k(n).$$

Let $x^{1/k}\le y$. For $y\le p\le 2y$,

$$\mathrm{Var}_{p\in[y,2y]}\left(\mathcal{S}^S_{d_k;x}\right) \sim a^S_k \frac{x}{4}\gamma^S_{d_k,2}\left(\frac{\log x}{\log y - 1}\right)(\log y - 1)^{2k^2+k-2}, \tag{7}$$

where a_k^S is certain arithmetic constant and $\gamma_{d_k,2}^S(c)$ is a piecewise polynomial function of degree $2k^2+k-2$ given by

$$\gamma_{d_k,2}^S(c) = \frac{2^{-2k+1}}{G(1+k)} \int_{[0,1]^{\frac{3}{2}k^2+\frac{3}{2}k}} \delta_c(u_1^{(k)}+\cdots+u_k^{(k)})\delta_{2c}(u_1^{(2k)}+\cdots+u_{2k}^{(2k)})$$

$$\times \mathbb{1}\begin{bmatrix} & & u_1^{(k)} & \leq & \dots & \leq & u_1^{(2k)} \\ & \iddots & & & & \iddots & \\ u_k^{(k)} & \leq & \dots & \leq & u_k^{(2k)} & & \\ \vdots & & \iddots & & & & \\ u_{2k}^{(2k)} & & & & & & \end{bmatrix} \Delta(u_1^{(k)}, u_2^{(k)}, \dots, u_k^{(k)}) d^{\frac{3}{2}k^2+\frac{3}{2}k}u, \tag{8}$$

where $\mathbb{1}_X$ denotes the characteristic function of the set X and Δ denotes the Vandermonde determinant.

Here the variance is defined by

$$\mathrm{Var}_{p\in[y,2y]}\left(\mathcal{S}_{d_k;x}^S\right) := \frac{1}{y}\sum_{y<p\leq 2y}\Bigg(\sum_{\substack{n\leq x \\ n\equiv\square \,(\mathrm{mod}\, p) \\ p\nmid n}} d_k(n) - \left\langle \mathcal{S}_{d_k;x}^S \right\rangle\Bigg)^2,$$

where $\langle\cdot\rangle$ denotes the average, given by

$$\left\langle \mathcal{S}_{d_k;x}^S \right\rangle := \frac{1}{y}\sum_{y<p\leq 2y}\sum_{\substack{n\leq x \\ n\equiv\square \,(\mathrm{mod}\, p) \\ p\nmid n}} d_k(n) = \frac{1}{y}\sum_{n\leq x} d_k(n) \sum_{\substack{y<p\leq 2y \\ n\equiv\square \,(\mathrm{mod}\, p) \\ p\nmid n}} 1.$$

To reach Conjecture 3.1, one can rewrite (4) as

$$\mathrm{Var}^*(\mathcal{S}_{d_k,n}^S) \sim \frac{q^n}{4}\gamma_{d_k,2}^S\left(\frac{n}{2g}\right)(2g)^{2k^2+k-2}$$

and compare the sums

$$\mathcal{S}_{d_k,n}^S(P) = \sum_{\substack{f\in\mathcal{M}_n \\ f\equiv\square \,(\mathrm{mod}\, P) \\ P\nmid f}} d_k(f) \qquad \text{and} \qquad \mathcal{S}_{d_k;x}^S(p) = \sum_{\substack{m\leq x \\ n\equiv\square \,(\mathrm{mod}\, p) \\ p\nmid m}} d_k(m).$$

We see that q^n represents the size of f on the left side. Hence, it corresponds to the size of m given by x on the right side. Similarly q^{2g+1} represents the size of P on the left side. To control the size of p, it is natural to consider a condition of type $y\leq p\leq 2y$. Replacing q^n by x, n by $\log x$ and $2g$ by $\log y$, we reach formula (7), except for the arithmetic factor a_k^S. Formula (8) comes from the interpretation of $I_{d_k,2}^S(n;N)$ as a function counting points inside a polytope ([KL22a, Proposition 3.5]).

Following Rudnick and Waxman [RW19], each ideal $\mathfrak{a}=(\alpha)\subset\mathbb{Z}[i]$ can be associated to a direction vector $u(\mathfrak{a})=u(\alpha):=\left(\frac{\alpha}{\overline{\alpha}}\right)^2$ in the unit circle, and we can write $u(\mathfrak{a})=e^{i4\theta_\mathfrak{a}}$. For a given θ let $I_K(\theta)=[\theta-\frac{\pi}{4K},\theta+\frac{\pi}{4K}]$ be a neighborhood of θ. We consider

$$\mathcal{N}_{d_\ell,K;x}^S(\theta) := \sum_{\substack{\mathfrak{a}\text{ ideal} \\ N(\mathfrak{a})\leq x \\ \theta_\mathfrak{a}\in I_K(\theta)}} d_\ell(\mathfrak{a})$$

and obtain the following conjecture by replacing q^n by x and q^κ by K in Theorem 2.2.

Conjecture 3.2. [KL22a, Conjecture 10.2]*Let $x \le K^\ell$. Then, there is a constant $a_\ell^S \in \mathbb{Q}$ depending on ℓ such that*

$$\mathrm{Var}\left(\mathcal{N}^S_{d_\ell,K;x}\right) \sim a_\ell^S \frac{x}{K}\gamma^S_{d_\ell,2}\left(\frac{\log x}{2\log K - 2}\right)(2\log K - 2)^{2\ell^2+\ell-2}.$$

Here the variance is given by

$$\mathrm{Var}\left(\mathcal{N}^S_{d_\ell,K;x}\right) := \frac{2}{\pi}\int_0^{\pi/2}\Big(\sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\le x\\ \theta_\mathfrak{a}\in I_K(\theta)}} d_\ell(\mathfrak{a}) - \langle \mathcal{N}^S_{d_\ell,K;x}\rangle\Big)^2\mathrm{d}\theta,$$

where

$$\langle \mathcal{N}^S_{d_\ell,K;x}\rangle := \frac{2}{\pi}\int_0^{\pi/2}\sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\le x\\ \theta_\mathfrak{a}\in I_K(\theta)}} d_\ell(\mathfrak{a})d\theta = \sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\le x}} d_\ell(\mathfrak{a})\frac{2}{\pi}\int_0^{\pi/2}\mathbb{1}_{\theta_\mathfrak{a}\in I_K(\theta)}\mathrm{d}\theta = \frac{1}{K}\sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\le x}} d_\ell(\mathfrak{a}).$$

Now consider the character defined for $\pi \in \mathbb{Z}[i]$ prime as

$$\chi_2(\pi) = \begin{cases} 1 & \pi = a^2 + ib^2 \text{ for } a, b \in \mathbb{Z}[i],\\ -1 & \text{otherwise,}\end{cases}$$

and extended multiplicatively to $\mathbb{Z}[i]$. Define the sum

$$\mathcal{N}^O_{d_\ell,K;x}(\theta) := \sum_{\substack{\mathfrak{a}=(\alpha)\text{ ideal}\\ N(\mathfrak{a})\le x\\ \theta_\mathfrak{a}\in I_K(\theta)}} d_\ell(\mathfrak{a})\left(\frac{1+\chi_2(\alpha)}{2}\right).$$

The following conjecture is obtained by replacing q^n by x and q^κ by K in Theorem 2.3.

Conjecture 3.3. [KL22a, Conjecture 10.3]*Let $x \le K^\ell$. Then there is a constant $a_\ell^O \in \mathbb{Q}$ depending on ℓ such that*

$$\mathrm{Var}\left(\mathcal{N}^O_{d_\ell,K;x}\right) \sim a_\ell^O \frac{x}{4K}\gamma^O_{d_\ell,2}\left(\frac{\log x}{2\log K - 1}\right)(2\log K - 1)^{2\ell^2-\ell-2},$$

where

$$\gamma^O_{d_k,2}(c) = \frac{2}{G(1+k)}\int_{[0,1]^{\frac{3}{2}k^2-\frac{k}{2}}}\delta_c(u_1^{(k)}+\cdots+u_k^{(k)})\delta_c(u_1^{(2k-1)}+\cdots+u_{2k-1}^{(2k-1)})$$

$$\times\mathbb{1}\begin{bmatrix} & & u_1^{(k)} & \le & \dots & \le & u_1^{(2k)}\\ & \iddots & & & & \iddots & \\ u_k^{(k)} & \le & \dots & \le & u_k^{(2k)} & & \\ \vdots & & \iddots & & & & \\ u_{2k}^{(2k)} & & & & & & \end{bmatrix}\Delta(u_1^{(k)}, u_2^{(k)}, \dots, u_k^{(k)})d^{\frac{3}{2}k^2-\frac{k}{2}}u. \tag{9}$$

Here the variance is given by

$$\mathrm{Var}\left(\mathcal{N}^O_{d_\ell,K;x}\right) := \frac{2}{\pi}\int_0^{\pi/2}\Big(\sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\text{ sum of two squares}\\ N(\mathfrak{a})\le x\\ \theta_\mathfrak{a}\in I_K(\theta)}} d_\ell(\mathfrak{a}) - \langle \mathcal{N}^O_{d_\ell,K;x}\rangle\Big)^2\mathrm{d}\theta,$$

where

$$\left\langle \mathcal{N}^{O}_{d_\ell,K;x} \right\rangle := \frac{2}{\pi}\int_0^{\pi/2} \sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\text{ sum of two squares}\\ N(\mathfrak{a})\le x\\ \theta_{\mathfrak{a}}\in I_K(\theta)}} d_\ell(\mathfrak{a}) d\theta$$

$$= \sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\text{ sum of two squares}\\ N(\mathfrak{a})\le x}} d_\ell(\mathfrak{a}) \frac{2}{\pi}\int_0^{\pi/2} \mathbb{1}_{\theta_{\mathfrak{a}}\in I_K(\theta)} \mathrm{d}\theta = \frac{1}{K} \sum_{\substack{\mathfrak{a}\text{ ideal}\\ N(\mathfrak{a})\text{ sum of two squares}\\ N(\mathfrak{a})\le x}} d_\ell(\mathfrak{a}).$$

Formula (9) comes from the interpretation of $I^{O}_{d_k,2}(n;N)$ as a function counting points inside a polytope ([KL22a, Proposition 7.4]).

To have a whole understanding of the above conjectures, it remains to formulate precise descriptions for the arithmetic factors a^S_k, a^S_ℓ, and a^O_ℓ. We hope to deduce these arithmetic factors in the near future by following the work of Conrey, Farmer, Keating, Rubinstein, and Snaith [CFK+05, CFK+03] giving a heuristic by comparing the number field setting with the random matrix theory arising from the function field setting.

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