DISCRETIZATION OF THE LOTKA-VOLTERRA SYSTEM

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Abstract. We revisit the Kahan-Hirota-Kimura discretization of a quadratic vector field. The corresponding discrete system is generated by successive iterations of a birational map. We include a proof of a formula for the Jacobian of this map. We essentially focus on the case of the Lotka-Volterra system. We show that in this case, the formula is equivalent to the preservation of a singular volume form as shown before by Sanz-Serna. It is not clear for the moment if the discretization of the Lotka-Volterra is an integrable system for all values of h. We show that the KHK-map for $h = 1$ preserves a pencil of conics (generic hyperbolas). We then propose numerical simulations for several values of h.

1. Introduction

The Kahan discretization was introduced in the unpublished lecture notes of a AMS congress organized at the Fields Institute in 1993 ([13]). The next appearance of this discretization was in two articles of Hirota and Kimura in 2000 ([15, 16]) where it was shown that in several cases the method preserves integrability. According to a proposal of T. Ratiu, discretizations of KHK type should be considered for numerous integrable systems ([19]). We mainly focus in our article to Quadratic Planar Vector Fields.

$$\dot{z} = f(z) = Q(z) + B(z) + c \\ z = (x, y) \in \mathbb{R}^2. \tag{1}$$

Each component of $Q : \mathbb{R}^2 \to \mathbb{R}^2$ is a quadratic form, while $B \in Gl(2, \mathbb{R})$ and $c \in \mathbb{R}^2$.

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The Kahan-Hirota-Kimura (KHK) discretization of the Quadratic Planar Vector Fields is the mapping $z \mapsto z'$ defined as:

$$\frac{z'-z}{h} = Q(z,z') + \frac{1}{2}B(z+z') + c, \tag{2}$$

where $Q(z,z') = \frac{1}{2}[Q(z+z') - Q(z) - Q(z')]$ is the symmetric bilinear form corresponding to the quadratic form Q.

In the case of quadratic vector field, this mapping can be identified with the (implicit) Runge-Kutta discretization:

$$\begin{aligned} \frac{z'-z}{h} = -\frac{1}{2}f(z) + 2f(\frac{z+z'}{2}) - \frac{1}{2}f(z'), \\ z = (x,y), z' = (x',y'). \end{aligned} \tag{3}$$

Expanding the Taylor series about z shows that:

$$\frac{z'-z}{h} = f(z) + \frac{1}{2}Df(z)(z'-z) \tag{4}$$

This yields an explicit KHK rational map:

$$z' = F_h(z) = z + h(I - \frac{h}{2}Df(z))^{-1}f(z), \tag{5}$$

where $Df(z)$ is the Jacobi matrix of $f(z)$.

Another remarkable point is that:

$$F_h^{-1}(z) = F_{-h}(z),$$

and thus, in particular, the KHK-map is birational. Further references on the subject include ([5, 6, 7, 8, 9, 12, 14, 17, 18, 22]).

2. A key formula for the Jacobian

Theorem 1. *Consider a KHK-map of a quadratic vector field in any dimension n :*

$$F_h : z = (x_1, ...x_n) \mapsto z' = (x'_1, ...x'_n)$$

Set $\Delta = \Delta(z,h) = \mathrm{Det}(I - \frac{h}{2}Df(z))$ and denote $\Delta' := \Delta(z', -h)$; the following formula can be shown:

$$\frac{dx_1 \wedge ... \wedge x_n}{\Delta} = \frac{dx'_1 \wedge ... \wedge x'_n}{\Delta'}. \tag{6}$$

Be careful that this relation cannot be interpreted as the conservation of a volume.

Proof. Denote $A = \frac{\partial z'}{\partial z}$ the Jacobian matrix of the coordinates z' relatively to z.

From the formula (3), we deduce

$$A - I = -\frac{h}{2}Df(z) + \frac{h}{2}2Df(\frac{z+z'}{2})(I+A) - \frac{h}{2}Df(z')A.$$

Using the fact that $Df()$ is linear, this yields

$$A = I - \frac{h}{2}Df(z) + \frac{h}{2}(Df(z) + Df(z'))(I+A) - \frac{h}{2}Df(z')A,$$

which displays:

$$(I - \frac{h}{2}Df(z))A = (I + \frac{h}{2}Df(z')),$$

this implies:

$$\Delta \mathrm{Det} A = \Delta'.$$

□

3. The Discrete Lotka-Volterra and the Theorem of Sanz-Serna

After some scaling, the famous Lotka-Volterra system modeling the interaction of predator with prey can be written as

$$\begin{aligned} \dot{x} &= x(1-y) \\ \dot{y} &= y(x-1). \end{aligned} \tag{7}$$

This system is not Hamiltonian for the usual symplectic form but it is "generalized Darboux" integrable with $H = xye^{-(x+y)}$.

In order to avoid the appearance of various powers of 2, we change $h/2$ into h (cf. [21]). The KKS discretization yields to:

$$\begin{aligned} x' - x &= h[(x'+x) - (x'y + xy')] \\ y' - y &= h[(x'y + xy') - (y'+y)], \end{aligned} \tag{8}$$

and this displays:

$$\Delta := \Delta(x,y,h) = 1 - h^2 - h(1-h)x + h(1+h)y, \tag{9}$$

$$
\begin{aligned}
x' &= \frac{x}{\Delta}[(1+h)^2 - h(1+h)x - h(1-h)y] \\
y' &= \frac{y}{\Delta}[(1-h)^2 + h(1+h)x + h(1-h)y].
\end{aligned} \tag{10}
$$

Denote the mapping defined above by:

$$(x', y') = F_h(x, y) = (A_h(x, y), B_h(x, y).$$

It should be noted that this KHK map leaves invariant both $x = 0$ and $y = 0$ (for $h \neq 1$).

We denote the three lines:

$$
\begin{aligned}
D &: \Delta = 0 \\
D_1 &: (1+h)^2 - h(1+h)x - h(1-h)y = 0 \\
D_2 &: (1-h)^2 + h(1+h)x + h(1-h)y = 0.
\end{aligned} \tag{11}
$$

The straight lines D, D_1 and D_2 respectively correspond to the cancellation of the denominator and the numerators of x' and y' in (10). Let define the points $B(1 + \frac{1}{h}, 0)$ and $C(0, 1 - \frac{1}{h})$, then

$$B = D \cap D_1$$

and

$$C = D \cap D_2.$$

They are called focal points in the sense of Bischi-Gardini-Mira [1, 2, 3, 10, 11]. Such points can play a specific role in the dynamics of the maps with denominator.

We include some results on the fixed points of this mapping.

Proposition 2. *The fixed points of F_h are $(0, 0)$, $(1, 1)$. The point $(0, 0)$ is a saddle, the point $(1, 1)$ is a center.*

Proof. The list of fixed points can be easily found by direct analysis of the equations $F_h(x, y) = (x, y)$. To study the nature of the fixed points, it is quite convenient to use the equation (after changing h into $2h$) :

$$(I - hDf(z))A = (I + hDf(z')).$$

In case of a fixed point $z' = z = z_0$, this yields:

$$det(A - \lambda I) = det([I - hDf(z_0)]^{-1}[I + hDf(z_0)] - \lambda I),$$

so that the eigenvalues λ of the jacobian matrix of F_h at a fixed point are solutions of

$$det[(1-\lambda)I + (1+\lambda)hDf(z_0)] = 0,$$

and the result follows from this formula.

□

We can now consider the theorem of Sanz-Serna

Theorem 3. *The KHK map of the Lotka-Volterra system preserves the (singular) volume form:*

$$\Omega = \frac{dx \wedge dy}{xy}. \tag{12}$$

Proof. Consider

$$\Delta := \Delta(x, y, h) = 1 - h^2 - h(1-h)x + h(1+h)y, \tag{13}$$

and change in Δ, both (x, y) into (x', y') and h into $-h$. This yields

$$\Delta' := \Delta(x', y', -h) = 1 - h^2 + h(1+h)x' - h(1-h)y'. \tag{14}$$

Next compute directly from (18) the differentials and obtain:

$$\frac{dx \wedge dy}{\Delta} = \frac{dx' \wedge dy'}{\Delta'}. \tag{15}$$

Now from (20), it follows:

$$(\frac{x'}{x} + \frac{y'}{y})\Delta = (1+h)^2 + (1-h)^2 = 2(1+h^2). \tag{16}$$

By the same transformation, we obtain:

$$(\frac{x'}{x} + \frac{y'}{y})\Delta = (\frac{x}{x'} + \frac{y}{y'})\Delta', \tag{17}$$

and this yields:

$$\frac{dx \wedge dy}{xy} = \frac{dx' \wedge dy'}{x'y'}. \tag{18}$$

□

The set of birational transformations of the plane which preserves the volume form $\frac{dx\wedge dy}{xy}$ is more shortly called "symplectic birational transformations of the plane" in the litterature. For instance, this is the terminology used in the article [4]. In this article the author proves the following remarkable result which looks in particular useful for further studies of discretized Lotka-Volterra of the plane:

Proposition 4. *The group of symplectic birational transformations of the plane is generated by* $SL(2,\mathbb{Z})$*, the torus* $\mathbb{C}*^2$ *and a special map of order* 5*:* $P : (x, y) - - > (y, (y + 1)/x)$.

The mapping P is a special case of the so-called Lyness map ([8, 12, 11]).

4. Integrability of the discretized Lotka-Volterra in the case $h = 1$

In this section we focus in the special case where the parameter $h = 1$. Replacing $h = 1$ in the formula 10 yields to the map:

$$\Delta := 2y, \tag{19}$$

$$\begin{aligned} x' &= \frac{x}{y}(2-x) \\ y' &= \frac{y}{y}(x) = x. \end{aligned} \tag{20}$$

We consider then

$$x' + y' - 2 = \frac{(x-y)(2-x)}{y}, \tag{21}$$

$$x' - y' = \frac{x(2-x-y)}{y}, \tag{22}$$

$$\frac{(x'+y'-2)(x'-y')}{x'y'} = -\frac{(x+y-2)(x-y)}{xy}, \tag{23}$$

so that we have checked that the mapping preserves the pencil of conics:

$$[(x+y-2)(x-y)]^2 = \mu[xy]^2. \tag{24}$$

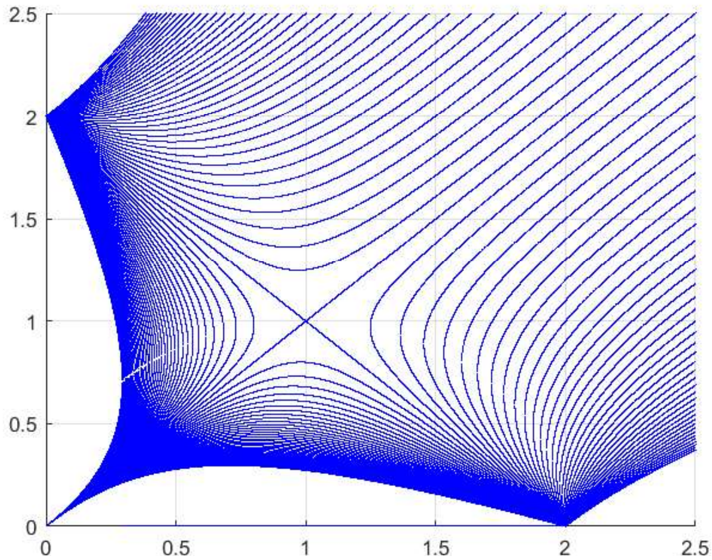

FIGURE 1. The pencil of conics

Figure 1 shows the pencil of conics for μ varying from -2 to 2 with step 0.05.

Existence of cycles of order 4 can be shown in the case $h = 1$:

Proposition 5. *For all x real, the points $(x, x), (2 - x, x), (2 - x, 2 - x), (x, 2 - x)$ form a cycle of order 4.*

This can be easily checked by direct computation. There are numerical evidences that the map is not periodic of period 4. Indeed, some orbits turn successively around the fixed point $A(1, 1)$ and go towards infinity.

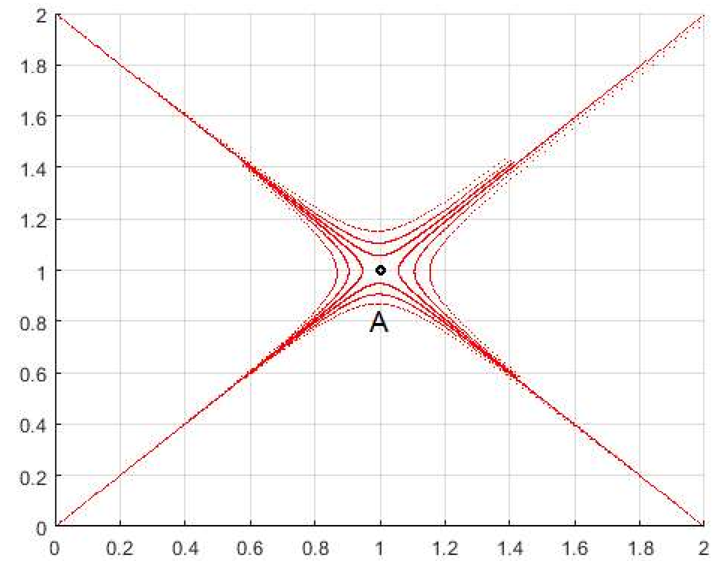

FIGURE 2. Three orbits obtained numerically for $h = 1$, initial conditions are (0.58,1.4), (0.58,1.41), (0.58,1.417).

5. NUMERICAL SIMULATIONS IN THE CASE $0 < h < 1$

Numerical simulations in the case $0 < h < 1$ are proposed in Figures 3-10. They have been plotted using Matlab. In Figure 4, h is small and the plotted ovals (or invariant curves) around the center fixed point $A(1, 1)$ look very similar to the trajectories of the Lotka-Volterra system. When h increases between 0 and 1 (see Figures 4-10), the ovals change their shape and some of them become cyclic. In Figure 5, an order 20 cyclic oval is obtained and in Figure 6, an order 55 cyclic oval and an order 9 cyclic one are obtained from different initial conditions.

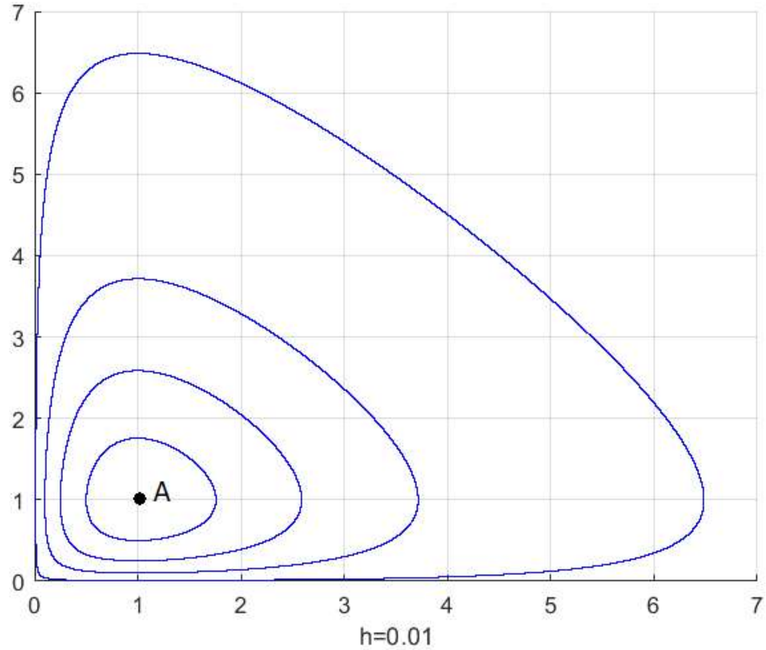

FIGURE 3. h=0.01, ovals obtained with 4 different initial conditions (1,0.01), (1,0.1), (1,0.25), (1,0.5)

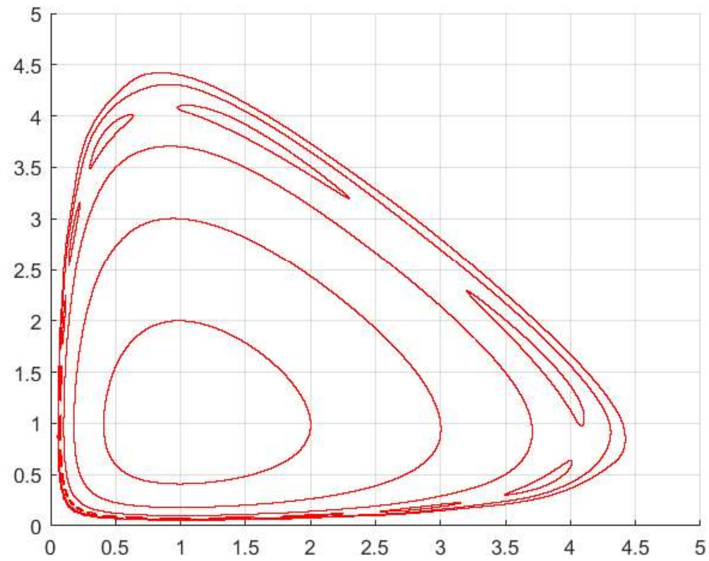

FIGURE 4. h=0.2, invariant curves or ovals obtained with 6 different initial conditions (1,2), (1,3), (1,3.7), (1,4.1), (1, 4.3), (1,4.4)

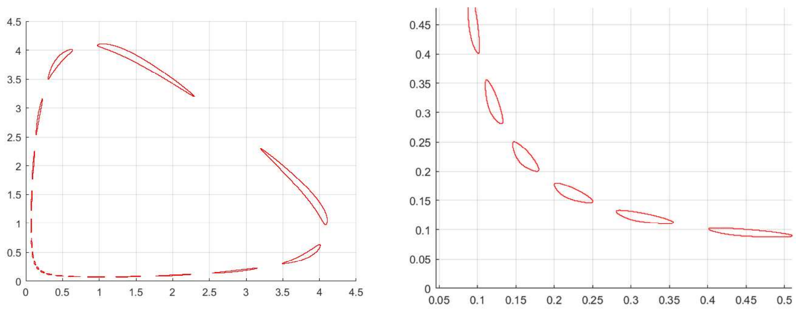

FIGURE 5. h=0.2, focus on the order 20 cyclic oval obtained with the initial condition (1,4.1). The Figure on the right shows a magnification in the square $[0, 0.5]^2$.

Figure 8 shows the basin of initial conditions giving rise to ovals. Its shape looks "like a bat", as the oval obtained in Figure 7 with the initial condition $(1, 0.328)$. The initial conditions taken outside this basin give rise to unbounded trajectories in the plane (x, y). This basin corresponds to the domain of stability of the map (10) in the sense of Lagrange.

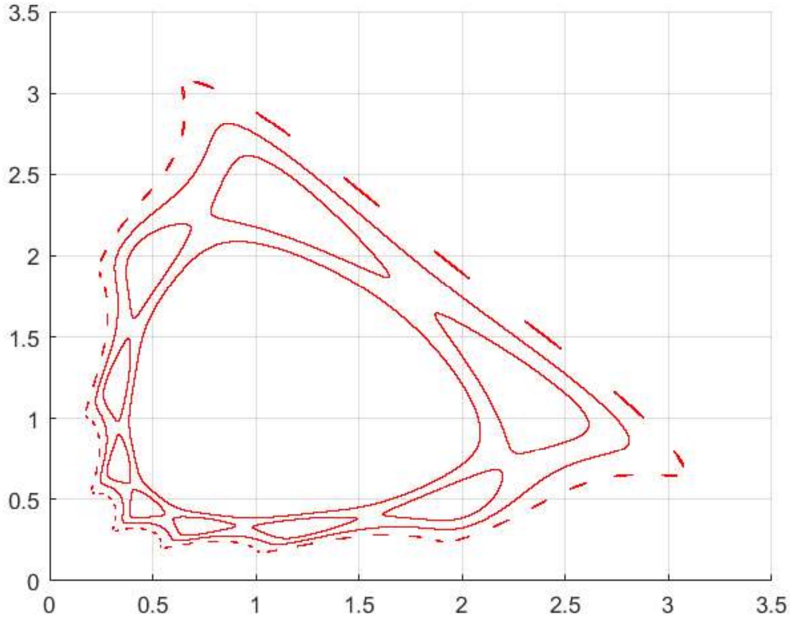

FIGURE 6. h=0.38, ovals obtained with 4 different initial conditions (0.55,0.5), (0.5,0.5), (0.5,0.35), (0.5,0.3). (0.5,0.3) gives rise to an order 55 cyclic oval and (0.5,0.5) gives rise to an order 9 cyclic oval.

The straight lines D and D_1 given in (11) and some of their preimages are plotted in Figures 8-9 as well as the point B (intersection of D and D_1). Let us remark that D_2 is not involved in the evolution of the basin. Indeed, the point C (intersection of D and D_2) is located outside the first quadrant for $h \in [0, 1]$. Numerical simulations show that the ovals are located inside the first quadrant and cannot cross D, D_1 and their preimages of any order. Moreover, we can remark that the size of the basin of ovals decreases when h increases from 0 to 1.

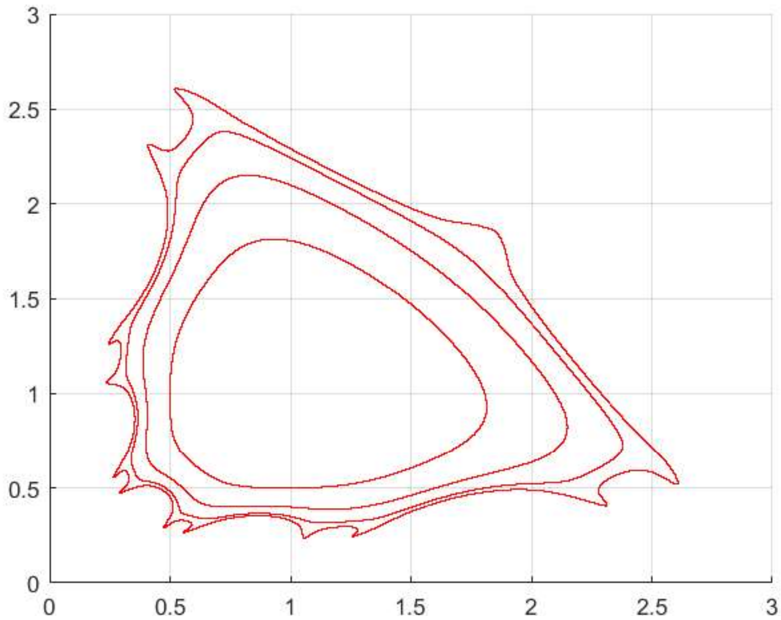

FIGURE 7. h=0.5234, ovals obtained with 4 different initial conditions (1,0.5), (1,0.4), (1,0.35), (1,0.328). The oval obtained with the initial condition (1,0.328) seems to be very close to the boundary of the basin (cf. Figure 8).

Figures 9-10 show the largest oval obtained numerically respectively when $h = 0.7$ and $h = 0.9$. This oval is surrounded by the points of a saddle cycle and we can conjecture that the boundary of the basin is connected with the invariant manifolds of this saddle cycle. Moreover, when $h = 0.9$, a cycle of order 4 appears. We note that this is the same order than those of the cycles described in the case $h = 1$.

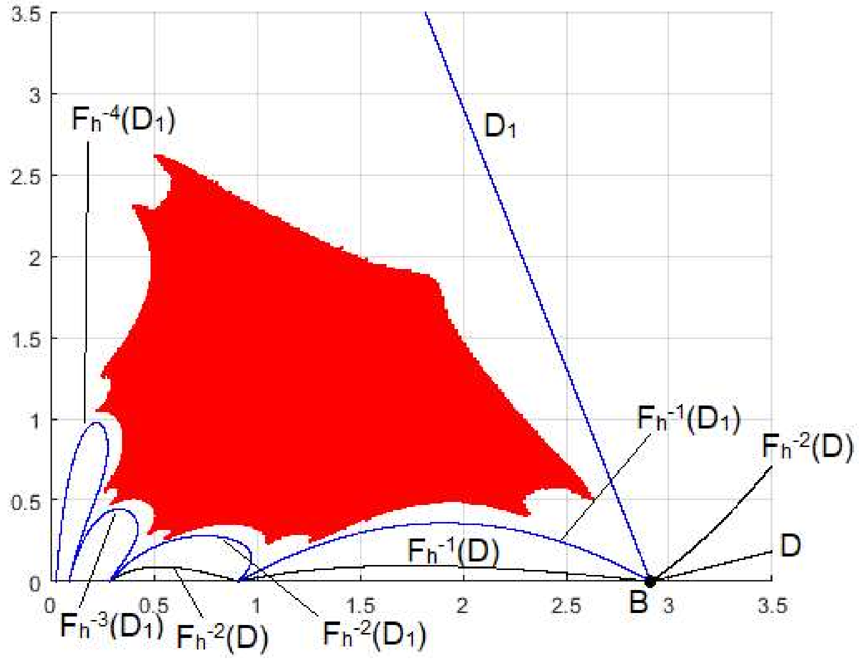

FIGURE 8. h=0.5234, the basin of ovals has "the shape of a bat". The lines D and D_1 and the point B are plotted as well as some of their preimages. The basin is located inside the area limited by these curves.

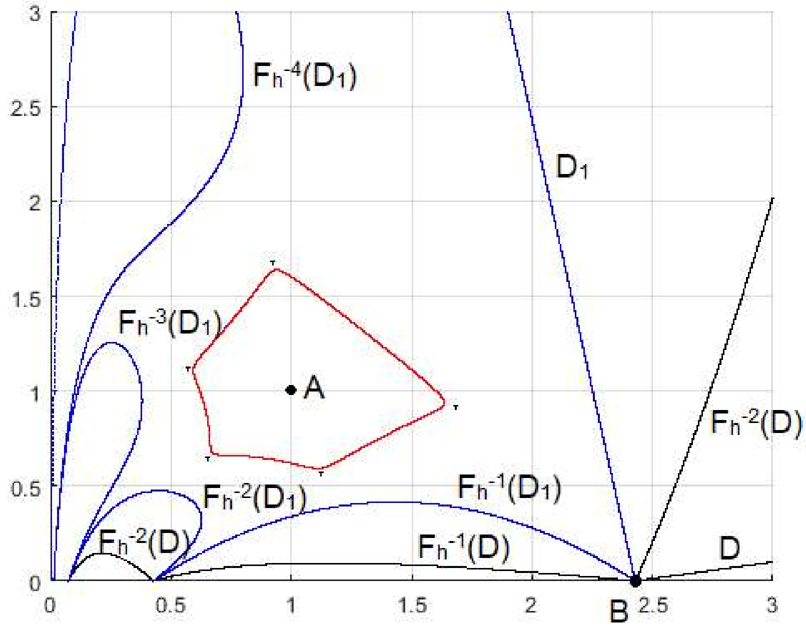

FIGURE 9. h=0.7, an oval obtained with the initial condition (1,0.62) close to the boundary of the basin of ovals. An order 5 saddle cycle (points in black) is located close to this boundary. The lines D and D_1 and the point B are plotted as well as some of their preimages. The basin is located inside the area limited by these curves. We can conjecture that the invariant manifolds of the saddle also play a role in the structure of the boundary of the basin.

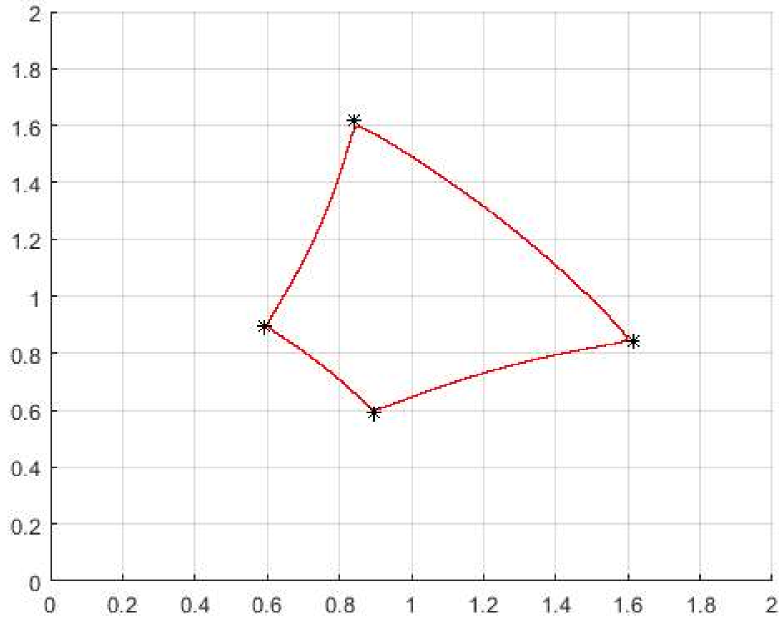

FIGURE 10. h=0.9, an oval obtained with the initial condition (0.755,0.755) close to the boundary of the basin of ovals. An order 4 saddle cycle (points in black) is located close to this boundary. We can conjecture that the invariant manifolds of the saddle also play a role in the structure of the boundary of the basin.

6. Conclusion and perspectives

This article revisits a discretization of quadratic vector fields called the Kahan-Hirota-Kimura discrete dynamical systems (KHK map). The study of this map relates with integrable systems and soliton theory, in particular with QRT-maps ([5, 6, 7, 8, 9, 12, 14, 17, 18, 22]). The proof of the Jacobian identity is included in the article. We focus on the discretization of the classical Lotka-Volterra prey-predator system and we derive a direct proof of the Sanz-Serna theorem from the Jacobian identity. Several numerical simulations are further discussed. They give evidences that for some values of the parameter, the boundary of the domain of Lagrange stability displays an interesting geometric structure and the "shape of a bat".

There are several perspectives to push further this study. In particular, the study of the boundary of the domain of stability in the sense of Lagrange and its relation with saddle cycles and preimages of D and D_1 will be the subject of further work.

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