WKB and microlocal approach to various Bohr-Sommerfeld quantization rules

Setsuro Fujiié (Ritsumeikan University)

The well-known Bohr-Sommerfeld quantization rule is the condition for an energy λ to be an eigenvalue of a 1D Schrödinger operator $P = -h^2\Delta + V(x)$ with a simple well potential $V(x)$. It is an equation with respect to the spectral parameter z of the form

$$-e^{iS(z)/h} = 1,$$

where $S(z) = \int_{\gamma(z)} \xi dx$ is the action along the periodic classical trajectory $\gamma(z)$ in the phase space $T^*\mathbb{R} \sim \mathbb{R}_x \times \mathbb{R}_\xi$ associated with the simple well. A root of this equation gives an 'approximation' of an eigenvalue when the semiclassical parameter h is small.

This fact can be justified rigorously in various mathematical methods using WKB method. In this report, we propose a method based on the microlocal analysis. The advantages of this method are the followings:

- We construct solutions only along the periodic curve $\gamma(\lambda)$ in the phase space.
- We do not need to care about the divergence of the microlocal WKB expansion.
- This approach is in line of the intuitive interpretation of the rule that the quantum wave should coincide after a tour around $\gamma(\lambda)$ with the original wave.
- This method is naturally adapted to the quantization of quantum resonances in multi-dimension or for systems.

We explain this microlocal method with emphasis on the above features and with some recent applications.

1 Review of the Bohr-Sommerfeld rule

Let us consider the eigenvalue problem

$$Pu = zu,$$

for the 1D Schrödinger operator

$$P = P_h = -h^2 \frac{d^2}{dx^2} + V(x), \tag{1}$$

where $V(x)$ is a multiplication operator by a function called *potential*. Here, by convention, the mass m of the particle is $1/2$ and the Planck constant $\hbar$ is replaced with h that we regard as small parameter, called *semiclassical parameter*. We want to study the asymptotic distribution of eigenvalues near a prescribed real energy level z_0 as $h \to +0$.

On the potential V and the fixed energy $z_0 \in \mathbb{R}$, we assume a simple well condition : V is a smooth real-valued function on $\mathbb{R}$ and there exist $\alpha < \beta$ such that the condition

$$\frac{V(x) - z_0}{(x-\alpha)(x-\beta)} > 0 \tag{2}$$

holds for all $x \in \mathbb{R}$. Thus condition means that V has two turning points (zeros of $V(x) - z_0$) α, β, which are both simple and that the interval $[\alpha, \beta]$ is classically allowed (i.e. $V(x) \geq z_0$) while the complement is classically forbidden ($V(x) \leq z_0$).

Let ξ denote the momentum variable. The phase space is the product space $\mathbb{R}_x \times \mathbb{R}_\xi$. The underlying classical mechanics is described by the classical Hamiltonian

$$p(x, \xi) = \xi^2 + V(x)$$

corresponding to P defined by (1) and its Hamiltonian vector field

$$H_p := \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi} = 2\xi \frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial \xi}.$$

The integral curves of H_p are called *classical trajectories*. The value of p is invariant along the classical trajectories (energy conservation).

The simple well condition (2) implies that H_p has one and only one periodic classical trajectory $\gamma(z)$ on the hypersurface $p^{-1}(z_0)$. This property holds for $z \in \mathbb{R}$ close enough to z_0. We fix such an interval I. For z in I, we define the *action integral* by the following line integral:

$$S(z) := \int_{\gamma(z)} \xi dx.$$

This is the volume of the domain bounded by $\gamma(z)$.

In addition to the simple-well condition, we assume

$$\lim \inf_{|x|\to\infty} V(x) > \sup I. \tag{3}$$

Then the spectrum of the self-adjoint operator P consists of eigenvalues in I for all $h > 0$.

It is known as Bohr-Sommerfeld quantization rule (we will write simply BS rule below) that the eigenvalues near z_0 are 'approximated' by the roots of the equation

$$e^{iS(z)/h} + 1 = 0, \tag{4}$$

or equivalently,

$$S(z) = (2k+1)\pi h, \quad k \in \mathbb{Z}, \tag{5}$$

when h is small. Since $S(z)$ is smooth near $z = z_0$ and $S'(z_0)$, the period of the classical trajectory $\gamma(z_0)$, is positive, the roots $\{z_k\}$ of (5) make an increasing sequence of real numbers with $z_k - z_{k-1} \sim 2\pi h/S'(z_0)$ near z_0.

In the particular case where $V(x) = x^2$ (harmonic oscillator), the simple well condition is satisfied for all positive z. The periodic classical trajectory and the action integral are given by

$$\gamma(z) = \{x^2 + \xi^2 = z\}, \quad S(z) = \pi z.$$

Hence the BS rule gives

$$z_k = (2k+1)h.$$

This exactly coincides with the eigenvalues of the harmonic oscillator.

In the general case, the BS rule does not give the exact eigenvalues, but give the eigenvalues modulo a small error in h. This fact has been proven in various ways in the history. In this report, we present a recently developed approach using the microlocal analysis.

Let

$$\mathrm{BS}_h(I) := \{z \in I; e^{iS(z)/h} + 1 = 0\},$$

be the set of energies satisfying the BS rule and

$$\mathrm{EV}_h(I) := \sigma_{\mathrm{disc}}(P_h) \cap I$$

the set of eigenvalues of $P = P_h$ in I.

Theorem 1.1. *It holds that*

$$\mathrm{dist}(\mathrm{BS}_h(I), \mathrm{EV}_h(I)) = o(h) \quad \textit{as } h \to 0.$$

More precisely, for any $z_h \in \mathrm{EV}_h(I)$, there exists $\tilde{z}_h \in \mathrm{BS}_h(I)$ such that $|z_h - \tilde{z}_h| = o(h)$, and for any $\tilde{z}_h \in \mathrm{BS}_h(I)$, there exists $z_h \in \mathrm{EV}_h(I)$ such that $|z_h - \tilde{z}_h| = o(h)$.

2 Variations in resonance quantization

Let us modify our simple well potential V outside a large enough compact set so that it decays at infinity, keeping the simple well condition for $z \in I$. If the energies z in the interval I are positive, the Hamiltonian flow consists of a periodic trajectory $\gamma(z)$ and 'non-trapping' ones. In such a setting, we have the following spectral property of the operator P:

- The spectrum of P on I is essential: $I \subset \sigma_{\mathrm{ess}}(P)$. There exists no eigenvalue in I.
- The cutoff resolvent $\chi(x)(P-z)^{-1}\chi(x)$ has a meromorphic extension from $\mathbb{C}_+$ to $\mathbb{C}_-$ across $\mathbb{R}_+$, and its poles in $\mathbb{C}_-$ appear near the roots of the BS rule.
- The poles are characterized as eigenvalues of $P_\theta := U_\theta P U_\theta^{-1}$, where U_θ is a complex dilation $(U_\theta f)(x) := f(e^{i\theta}x)$.

These poles are called *resonances*. The imaginary part of resonances (sometimes called *width* of resonance) describes the exponential decay rate of the quantum state. In fact, if we formally replace the operator P with a resonance $z = z_R - iz_I$ in the time evolution operator $e^{-itP/h}$, we have

$$|e^{-itz/h}| = e^{-z_I t/h}.$$

2.1 General facts on the semiclassical resonances in relation with the classical trapped set

Here we consider the general Schrödinger operator

$$P = -h^2\Delta + V(x)$$

in the multidimensional Euclidean space $\mathbb{R}^n$. The real-valued smooth potential $V(x)$ is supposed to decay at infinity, and analytic in sectorial domains near infinity. This last assumption is used to define resonances by a complex dilation. For the purpose of this report, it is enough to consider compactly supported smooth potentials.

The underlying classical Hamiltonian is

$$p(x,\xi) = |\xi|^2 + V(x) = \sum_{j=1}^{n} \xi_j^2 + V(x).$$

For each $z \in \mathbb{R}$, we define the trapped set on the energy hypersurface:

$$K(z) := \{(x,\xi) \in p^{-1}(z); t \mapsto \exp tH_p(x,\xi) \text{ is bounded}\}.$$

A fundamental fact in the semiclassical theory of resonances is the following theorems which suggest the close relation between the trapped set $K(z_0)$ and the asymptotic distribution of resonances in the semiclassical limit near a real energy z_0.

Theorem 2.1. *([7], [12]) There is no resonance 'near' non-trapping energies z (i.e. $K(z) = \emptyset$).*

More precisely, Helffer and Sjöstrand proved the non-existence of resonance in a neighborhood of size $\mathcal{O}(1)$ of a non-trapping energy under the global analyticity condition on the potential, whereas Martinez proved the same statement in a neighborhood of size $\mathcal{O}(h|\log h|)$ under the analyticity condition only near infinity (which is our setting).

Another fundamental fact is that the resonant states are "concentrated on the trapped set". In order to state this fact, we should introduce semiclassical and microlocal terminologies.

The *semiclassical Fourier transform* and its inverse are defined respectively with a small parameter h by

$$(\mathcal{F}_h f)(\xi) := \frac{1}{(2\pi h)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi/h} f(x)dx,$$
$$(\mathcal{F}_h^{-1} g)(x) := \frac{1}{(2\pi h)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi/h} g(\xi)d\xi.$$

Definition 1. Let $u(x,h)$ be an L^2 function in $\mathbb{R}^n$ depending on h with $\|u\| \le 1$ for all small $h > 0$. We write

$$u \equiv 0 \quad \text{at } (x_0, \xi_0) \in \mathbb{R}^{2n},$$

if there exist $\chi_1(x) \in C_0^\infty(\mathbb{R}^n)$ with $\chi_1(x_0) = 1$ and $\chi_2(\xi) \in C_0^\infty(\mathbb{R}^n)$ with $\chi(\xi_0) = 1$ such that

$$\|\chi_2 \mathcal{F}_h \chi_1 u\|_{L^2} = \mathcal{O}(h^\infty). \tag{6}$$

Remark that the left hand side of (6) is equal to $\|\mathcal{F}_h^{-1}\chi_2\mathcal{F}_h\chi_1 u\|_{L^2}$ thanks to the unitarity of the Fourier transform, and

$$\mathcal{F}_h^{-1}\chi_2\mathcal{F}_h\chi_1 u = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h}\chi_1(y)\chi_2(\xi)u(y)dxd\xi$$

This is a particular form of the so-called *semiclassical pseudo-differential operator*. Let $\chi(x,\xi)$ be a smooth function defined in the phase space. The semiclassical pseudo-differential operator called *Weyl quantization* χ^W of $\chi(x,\xi)$ is the operator defined by the integral

$$\chi^W u := \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h}\chi\left(\frac{x+y}{2}, \xi\right) u(y)dxd\xi.$$

In fact the above notion 'microlocally zero' is usually defined by using the semiclassical pseudo-differential operator and rather called *microlocaly infinitely small* (see [6], [11]).

The second fundamental fact is the following assertion on the resonant state (or quasi-mode):

Theorem 2.2. *([2]) If $u = u(x, h) \in L^2(\mathbb{R}^n)$ satifies*

$$\begin{cases} \|(P_\theta - z)u\| = \mathcal{O}(h^\infty), \\ u \equiv 0 \quad \textit{microlocally on } K(z) \end{cases}$$

then $\|u\| = \mathcal{O}(h^\infty)$.

In the case of the 1D eigenvalue problem, this theorem can be replaced by a simpler fact that the eigenfunction (or quasi-mode) cannot be microlocally zero along the characteristic set $\gamma(z)$. This fact will be used in the proof of Theorem 1.1 given in section 3.3.

2.2 Semiclassical resonance distribution created by a trapped trajectory

After the discovery of Theorem 2.1, we have been interested in the resonance asymptotics near a trapping energy. In particular, the precise asymptotic distribution have been studied for trapped sets with simple geometrical structure such as a periodic trajectory, hyperbolic fixed point and homoclinic or heteroclinic trajectory etc. Here we quickly review some of them. Here we are interested in the BS rules and do not specify the rigorous meaning of the 'approximation' for simplicity.

2.2.1 Hyperbolic closed trajectory

Suppose that the trapped set $K(z_0)$ consists of a hyperbolic closed trajectory. This is the case when for example the potential $V(x)$ consists of two suitably shaped bumps like

$$V(x) = f(|x - a|) + g(|x + a|), \tag{7}$$

where $f(r)$ and $g(r)$ are strictly decreasing functions on $\mathbb{R}_+$ with compact support on $[0, 1)$ such that their extensions to $\mathbb{R}$ as even functions are smooth. If $a = (a_1, 0, \ldots, 0)$ with $a_1 > 1$ and $0 < z_0 < \min(f(0), g(0))$, the trapped set $K(z_0)$ consists of a unique closed trajectory whose x-space projection is an interval on the x_1-axis contained in $(-a, a)$.

This periodic trajectory is hyperbolic. Let $\{\theta_j(z)\}_{j=1}^{n-1}, \{\theta_j(z)^{-1}\}_{j=1}^{n-1}$ be the eigenvalues with $|\theta_j(z)| > 1$ of the linearized Poincaré map associated with this hyperbolic closed trajectory. Then we have the following result about the asymptotic distribution of resonances. This is a potential scattering version of the famous paper by Ikawa [10] on the resonance distribution for the obstacle scattering.

Theorem 2.3. *([5]) There exists an analytic function $\rho(z)$ satisfying $|\rho(z)| = |\theta_1(z)\cdots\theta_{n-1}(z)|^{-\frac{1}{2}}$ such that the resonances closest to the real axis 'near' z_0 are 'approximated' by the roots of the BS rule*

$$\rho(z)e^{iS(z)/h} = 1.$$

This is a natural extension of the BS rule (4) for eigenvalues to the higher dimension. We have $\rho(z)$ instead of -1. We easily see by taking the modulus of this rule that any real z can no longer be a root, since $|\rho(z)| = |\theta_1(z)\cdots\theta_{n-1}(z)|^{-\frac{1}{2}} < 1$. The above BS rule is equivalent to

$$S(z) = 2k\pi h + ih\log\rho(z), \quad k \in \mathbb{Z},$$

and the width of these resonances is approximated by $-\frac{h}{2}\sum_j \log\theta_j(z_0)$. The resonance width is smaller when θ_j's are smaller. In other words, the life time is longer when the trap is stronger.

In the paper [5], the authors give not only the closest resonances to the real axis but also all the resonances which make a lattice structure.

2.2.2 Homoclinic trajectory

In the above example (7), suppose that $f(0) < g(0)$. Then the trapped set $K(z_0)$ for $z_0 = f(0)$ consists of a fixed point $(a, 0)$ and a homoclinic trajectory which tends to $(a, 0)$ as t tends to $+\infty$ and $-\infty$.

We may expect that the asymptotic distribution of resonances strongly depends on the geometry near the fixed point since the classical particles on the trapped set spend infinite time at these points. In the case where the fixed points are hyperbolic, we have the following results.

Suppose that $(a, 0)$ is a hyperbolic fixed point with exponents $\pm\lambda_1, \ldots, \pm\lambda_n$. This is the case when the potential has a non-degenerate maximum at $x = a$ and the Hessian $V''(a)$ at this point has negative eigenvalues $-\lambda_1^2/2, \ldots, -\lambda_n^2/2$. Let λ_1 is the smallest among all λ_j's.

We assume that $K(z_0)$ consists of this hyperbolic fixed point and an associated homoclinic orbit γ. Let $\zeta(z)$ be a linear function of z defined by

$$\zeta(z) = \frac{1}{\lambda_1}\left(\frac{1}{2}\sum_{j=2}^{n}\lambda_j - i\frac{z - z_0}{h}\right).$$

Theorem 2.4. *([2]) There exists an h-independent function $q(\zeta)$ such that the resonances closest to the real axis near z_0 are 'approximated' by the roots of the BS rule*

$$h^{\zeta(z)}e^{iS(z_0)/h}q(\zeta(z)) = 1.$$

The above BS rule is equivalent to

$$z = z_0 + \lambda_1 \frac{2k\pi h - S(z_0)}{|\log h|} - i\left(\frac{h}{2}\sum_{j=2}^{n}\lambda_j - i\lambda_1 \log q(\zeta(z))\frac{h}{|\log h|}\right).$$

We see from this formula that the width of resonances closest to the real axis is of order h, or more precisely $\sim \frac{h}{2}\sum_{j=2}^{n}\lambda_j$, if $n \geq 2$, but it is of order $h/|\log h|$, infinitely smaller than h if $n = 1$. We also see that the distance of neighboring resonances is of order $h/|\log h|$. Compared with the usual eigenvalue or resonance distribution, we observe a densification at a homoclinic level.

2.2.3 Energy-level crossing

Let $\mathcal{P}$ be a 1D matrix Schrödinger operator

$$\mathcal{P} = \begin{pmatrix} P_1 & hW \\ hW & P_2 \end{pmatrix},$$

where P_1 and P_2 are semiclassical Schrödinger operators with real valued smooth potentials V_1 and V_2:

$$P_j = -h^2\frac{d^2}{dx^2} + V_j(x),$$

and W is a first order differential operator

$$W = r_0(x) + r_1(x)h\frac{d}{dx},$$

with real-valued smooth bounded coefficients $r_0(x)$ and $r_1(x)$. Such an operator appears as a model of the Born-Oppenheimer approximation in the quantum chemistry, where the semiclassical parameter h comes from the ratio of the mass of electrons and the nuclei.

We assume that $V_1(x)$ is strictly increasing and $V_2(x)$ is strictly decreasing. Then any energy z_0 is a non-trapping for both p_1 and p_2, the classical Hamiltonian associated with P_1 and P_2 respectively. Theorem 2.1 implies that the scalar operators P_1 and P_2 have no resonance near z_0.

On the other hand, we also assume that the graphs of V_1 and V_2 cross at one point, say at the origin $x = 0$. Then, for energies z above $V_1(0) = V_2(0) =: V_0$, the classical trajectories $\gamma_1(t)$ for p_1 and $\gamma_2(t)$ for p_2 on the energy surfaces $p_1^{-1}(z)$ and $p_2^{-1}(z)$ respectively cross at two points $(0, \sqrt{z - V_0})$ and $(0, -\sqrt{z - V_0})$. More precisely, $\gamma_1(t)$ starting from $(0, \sqrt{z - V_0})$ at time $t = 0$ goes to the other crossing point $(0, -\sqrt{z - V_0})$ at a positive time t_1 and $\gamma_2(t)$ starting from $(0, -\sqrt{z - V_0})$ at time t_1 goes to $(0, \sqrt{z - V_0})$ at time $t_1 + t_2$ with a positive t_2. The union of these trajectories

$$\left(\bigcup_{0 \le t \le t_1} \gamma_1(t) \right) \cup \left(\bigcup_{t_1 \le t \le t_1 + t_2} \gamma_2(t) \right)$$

make a closed curve in the phase space. Let $S(z)$ be the area of the domain bounded by this closed curve. Then we have the following theorem:

Theorem 2.5. *([8]) There exists a non-zero constant q_0 such that the resonances near z_0 are 'approximated' by the roots of the BS rule*

$$q_0 h e^{iS(z)/h} = 1.$$

The above BS rule is equivalent to

$$S(z) = 2k\pi h - ih \log \frac{1}{h} + ih \log q_0.$$

We see that the imaginary part the roots is of order $h|\log h|$. This model supplies us with an interesting feature of matrix Schrödinger operators. Such a matrix valued operator has multiple classical dynamics. Even though each dynamical system is non-trapping at an energy, this example says that there may be resonances near this energy created by a *generalized closed trajectory* made by a combination of different systems.

Remark that, even in the case where both V_1 and V_2 are strictly increasing (or decreasing), the classical trajectories may make a bounded domain but such a domain does not create resonances with width of order $h|\log h|$. In fact the two classical trajectories do not make a closed curve if the orientation is taken into account.

3 Microlocal approach

There are various methods to derive the Bohr-Sommerfeld quantization condition of eigenvalues or resonances but they are usually based on the construction of WKB solutions, which are the power series solutions to the semiclassical Schrödinger equation with respect to the small parameter h:

$$u(x,h) = e^{i\phi(x)/h} \sum_{k=0}^{\infty} a_k(x) h^k. \tag{8}$$

The main difficulty consists in the divergence of this infinite series. This is due to the fact that the small parameter is multiplied to the principal term (i.e. the Laplacian) of the differential equation. This results not only in the divergence of the series but also in the singularities of the phase $\phi(x)$ and the symbols $a_k(x)$ at the turning points or at the caustics. Thus, we need to give a meaning (asymptotic expansion, resummation, etc.) to such formal solutions away from the singularities, and to study the Stokes phenomena, namely the discontinuous change of asymptotic form, which occur at the singularities.

The microlocal approach to the eigenvalue or resonance asymptotics created by a closed trajectory γ consists in the study of the eigenfunctions or resonant state along γ in the phase space. More precisely, we construct a microlocal solution at a point $\rho_0 \in \gamma$, and continue it along γ. Then the matching condition at ρ_0 of the initial microlocal solution and the final one after the tour along $\gamma(z)$ gives the Bohr-Sommerfeld quantization rule. We justify this formal procedure by contradiction arguments.

The advantages of this method are the followings:

1. We have only to construct solutions along the periodic curve $\gamma(z)$. This is due to the fact that the normalized eigenfunctions are microlocally concentrated on $\gamma(z)$ in the semiclassical limit. More precisely, an eigenfunction 'microlocally zero' on the trapped set $\gamma(z)$ is globally identically zero (see Theorem 2.2 and the remark after this theorem).

2. We need not to care about the divergence of the microlocal WKB expansion. Microlocal solutions are defined modulo $\mathcal{O}(h^\infty)$, and hence a Borel resummation of a divergent WKB series gives a microlocal solution and it is a generator of the one-dimensional microlocal solution space associated with a Lagrangian submanifold in the energy surface.

3. This method is naturally adapted to the quantization of quantum resonances in multi-dimension or for systems. In fact, this fact has first been established for a problem of resonances in higher dimension created by a homoclinic trajectory ([2]). It is also used in the quantization of eigenvalues and resonances for matrix Schrödinger operators ([8]).

3.1 Microlocal WKB construction

Let Λ be a *Lagrangian submanifold* in $p^{-1}(z)$ (recall $p(x,\xi) = |\xi|^2 + V(x)$) and $\rho_0 = (x_0, \xi_0)$ a point on Λ near which the projection $\Lambda \to \mathbb{R}^n_x$ is diffeomorphic. Then one can construct a formal solution of the form (8), where ϕ is a *generating function* of Λ:

$$\Lambda = \{(x,\xi); \xi = \partial_x \phi(x)\},$$

and $a_k(x)$ are determined inductively solving the *transport equations* along the classical trajectories:

$$2\partial_x\phi \cdot \partial_x a_k + (\Delta\phi)\, a_k = i\Delta a_{k-1}.$$

The power series $\sum_{k=0}^{\infty} a_k(x) h^k$ is divergent, but it suffices to take a Borel's resummation $a(x,h)$, which satisfies, locally near x_0,

$$(P-z)\left(a(x,h)e^{i\phi(x)/h}\right) = \mathcal{O}(h^\infty).$$

This means that u is a microlocal solution near ρ_0.

3.2 Propagation of microlocal solutions

Let ρ_0 and ρ_1 be two points connected by a classical trajectory γ of principal type. Let $u(x,h)$ be a microlocal solution in a neighborhood of γ:

$$(P-z)u \equiv 0 \quad \text{on } \gamma.$$

We consider a *microlocal Cauchy problem*: Does a microlocal datum in a neighborhood of ρ_0 determines the microlocal solution in a neigborhood of ρ_1? The answer is yes. The following theorem implies the *uniqueness* of the microlocal Cauchy problem.

Theorem 3.1. *If* $u \equiv 0$ *near* ρ_0, *then* $u \equiv 0$ *near* ρ_1.

This is nothing but the semiclassical version of the well known theorem of propagation of singularities due to Hörmander (see [9]).

Suppose that the projection of Λ to $\mathbb{R}^n_x$ is diffeomorphic at both ρ_0 and ρ_1. Then the microlocal solution at ρ_1 is explicitly described modulo $\mathcal{O}(h)$ in terms of the initial datum at ρ_0.

Theorem 3.2. *If u is of WKB form microlocally near* ρ_0:

$$u \equiv a(x,h)e^{i\phi(x)/h} \quad \textit{near } \rho_0,$$

then it is also of WKB form microlocally near ρ_1*:*

$$u \equiv b(x,h)e^{i\psi(x)/h} \quad \text{near } \rho_1.$$

Moreover, the phase ψ *is the generating function of the evolution of* Λ *near* ρ_1*, and there exists an integer* ν *such that*

$$\psi(x_1) = \phi(x_0) + \int_{\rho_0}^{\rho_1} \xi dx - \frac{\pi}{2}\nu h.$$

The integer ν *is called Maslov index.*

The symbol $b(x,h)$ is expressed in terms of $a(z,h)$, but we omit this.

3.3 Justification of the Bohr-Sommerfeld quantization condition

Here we prove Theorem 1.1 on the simplest 1D eigenvalue problem using the contradiction arguments established in [2] for the justification of the BS rule for resonances (see Theorem 2.4).

Let $z = z_h$ be in a complex neighborhood of I of size $\mathcal{O}(h)$. The following proposition asserts the existence of elements of $\mathrm{BS}_h(I)$ near each eigenvalue.

Proposition 3.3. *If there exists a positive h-independent constant* C *such that, for every small* h*,*

$$\textit{dist}\ (z, \mathrm{BS}_h(I)) \geq Ch, \tag{9}$$

then $z \notin \mathrm{EV}_h(I)$ *for small enough* h*, and there exists* $N \in \mathbb{N}$ *such that*

$$(P-z)^{-1} = \mathcal{O}(h^{-N}) \quad \text{as } h \to +0.$$

Proof. If the conclusion were false, then there would exist $u = u(x,h) \in L^2$ and $z = z_h$ satisfying (9) such that

$$\|u\| = 1, \quad \text{and} \quad (P-z)u = \mathcal{O}(h^\infty).$$

Let ρ_0 be a point on the periodic trajectory $\gamma = \gamma(\operatorname{Re} z)$ where the x-space projection is diffeomorphic. Microlocally near this point, u is of WKB form:

$$u(x,h) \equiv a(x,h)e^{i\phi(x)/h} \quad \text{near } \rho_0, \tag{10}$$

where $a(x,h)$ is a resummation of the infinite series $\sum_k a_k(x)h^k$. We apply the theorems in the previous section to continue u along γ from ρ_0 to $\hat{\rho}_0$, the same point as ρ_0 but after a tour along the periodic classical trajectory γ. Then we obtain

$$u \equiv ae^{i(\phi+S(z))/h-i\pi} \quad \text{near } \hat{\rho}_0, \tag{11}$$

because the Maslov index is $\nu = 2$ counted at two turning points on $\gamma(z)$. The two expression (10) and (11) should coincide. This compatibility condition reads

$$ae^{i\phi/h}(1+e^{iS(z)/h}) \equiv 0 \quad \text{near } \rho_0.$$

The condition (9) is equivalent to that $1+e^{iS(z)/h}$ is estimated from below by a positive constant. Hence the above identity implies

$$u \equiv ae^{i\phi/h} \equiv 0 \quad \text{near } \rho_0.$$

It follows from the propagation of singularities (Theorem 3.1) that u is microlocally zero all along γ:

$$u \equiv 0 \quad \text{on } \gamma(z).$$

Finally we apply Theorem 2.2 to conclude that u is globally infinitely small:

$$\|u\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty).$$

This is a contradiction against the assumption that u is of norm 1. ∎

On the contrary, the following proposition asserts the existence of eigenvalues near $\mathrm{BS}_h(I)$. This proposition with the previous one finishes the proof of Theorem 1.1.

Proposition 3.4. *Let $\tilde{z} = \tilde{z}_h \in \mathrm{BS}_h(I)$. For any $\epsilon > 0$, there exists $h_0 > 0$ such that for any $h < h_0$, one has*

$$\mathrm{EV}_h(I) \cap (\tilde{z} - \epsilon h, \tilde{z} + \epsilon h) \neq \emptyset.$$

Proof. If this were false, then there would exist $\epsilon > 0$, a sequence h tending to 0 and $\tilde{z} = \tilde{z}_h \in \mathrm{BS}_h(I)$ such that $\mathrm{EV}_h(I) \cap (\tilde{z} - \epsilon h, \tilde{z} + \epsilon h) = \emptyset$. Since the eigenvalues are real, the interval $(\tilde{z} - \epsilon h, \tilde{z} + \epsilon h)$ in this assertion can be replaced with the complex disk $D(\tilde{z}, \epsilon h)$ centered at $\tilde{z}$ with radius ϵh.

Take $z \in \partial D(\tilde{z}, \epsilon h)$ and a point ρ_0 on γ where the x-space projection is diffeomorphic.

Let $w(x,h)$ be a microlocal WKB solution to the homogenous equation $(P - z)u = 0$ near ρ_0, and set

$$v := \chi_2^W [P, \chi_1^W] w.$$

Here χ_1^W and χ_2^W are the Weyl quantization of the symbols $\chi_1(x,\xi), \chi_2(x,\xi)$ satisfying the following properties: χ_1 is supported in a small neighborhood of ρ_0 and it is identically 1 in a smaller neighborhood of ρ_0. Then $\nabla\chi_1 \cap \gamma$ consists of two small connected curves; one is in the forward side of ρ_0 with respect to the time parametrization of γ, and the other one is in the backward side. χ_2 is identically 1 on the curve in the backward side and supported nearby so that it vanishes on the forward side.

The function v is in $L^2(\mathbb{R})$ with bounded norm with respect to h. Since z is at a distance ϵh from BS_h, Proposition 3.3 permits us to define $u := (P-z)^{-1}v$ and guarantees that the norm of u is at most polynomial in h.

Let us observe this function u microlocally along γ from the initial point ρ_0 to the final point $\hat{\rho}_0$ as in the proof of Proposition 3.3.

Since $v \equiv 0$ microlocally near ρ_0, u is a microlocal solution to the homogeneous equation $(P-z)u = 0$ there, and hence it is of WKB form:

$$u \equiv ae^{i\phi/h} \quad \text{near } \rho_0. \tag{12}$$

We continue this along γ in the direction of the time evolution of this classical trajectory. Let ρ_1 be a point on this way outside the support of χ_1. It is still of WKB form since $v \equiv 0$ is valid from ρ_0 to ρ_1:

$$u \equiv \tilde{a}e^{i\tilde{\phi}/h} \quad \text{near } \rho_1. \tag{13}$$

Now we compute the microlocal data of u at $\hat{\rho}_0$ continuing it from ρ_1. We write $\tilde{\gamma}$ this sub-curve from ρ_1 to $\hat{\rho}_0$. u is the sum $u = u_{\mathrm{hom}} + u_{\mathrm{inhom}}$ of the solutions to the following two 'microlocal Cauchy problem':

$$\begin{cases} (P-z)u_{\mathrm{hom}} & \equiv 0 \quad \text{near } \tilde{\gamma}, \\ u_{\mathrm{hom}} & \equiv \tilde{a}e^{i\tilde{\phi}/h} \quad \text{near } \rho_1, \end{cases}$$

$$\begin{cases} (P-z)u_{\mathrm{inhom}} & \equiv v \quad \text{near } \tilde{\gamma}, \\ u_{\mathrm{inhom}} & \equiv 0 \quad \text{near } \rho_1. \end{cases}$$

Just as in the proof of Proposition 9, we have

$$u_{\mathrm{hom}} \equiv ae^{i(\phi+S(z))/h - i\pi} \quad \text{near } \hat{\rho}_0.$$

On the other hand, u_{inhom} is explicitly given by

$$u_{\mathrm{inhom}} = \chi_1^W w.$$

In fact, we have

$$(P-z)\chi_1^W w \equiv [(P-z), \chi_1^W]w = [P, \chi_1^W]w \equiv \chi_2^W[P, \chi_1^W]w = v,$$

since χ_2 is identically 1 on the support of the symbol of $[P, \chi_1^W]$ restricted on $\tilde{\gamma}$, which is equal to $(\mathrm{supp}\nabla\chi_1) \cap \tilde{\gamma}$. The initial condition is also satisfied since ρ_1 is outside the support of χ_1.

Thus we have obtained

$$u \equiv ae^{i(\phi+S(z))/h-i\pi} + w \quad \text{near } \hat{\rho}_0, \tag{14}$$

because χ_1 is identically 1 in a neighborhood of ρ_0.

Now the compatibility condition between (12) and (14) reads

$$ae^{i\phi/h} \equiv -e^{iS/h}ae^{i\phi/h} + w \quad \text{near } \rho_0,$$

and, since $u \equiv ae^{i\phi/h}$ near ρ_0 and $u = (P-z)^{-1}v$, we get

$$(P-z)^{-1}v \equiv \frac{w}{1+e^{iS/h}} \quad \text{near } \rho_0.$$

We integrate this identity with respect to z along the boundary $\partial D(\tilde{z}, \epsilon h)$

$$\int_{\partial D(\tilde{z},\epsilon h)} (P-z)^{-1}v dz \equiv \int_{\partial D(\tilde{z},\epsilon h)} \frac{w}{1+e^{iS(z)/h}} dz.$$

The left hand side should be zero because there exists no eigenvalue in $D(\tilde{z}, \epsilon h)$ by assumption. On the other hand, the right hand side does not vanish since $\tilde{z}$ is a unique zero of $1 + e^{iS(z)/h}$. This is a contradiction. ∎

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