Birkhoff normalization for a family of superintegrable symplectic maps and its application

Hidekazu Ito

Kanagawa University

Abstract

Birkhoff normalization is considered for a family of analytic symplectic maps near a fixed point. It is proved that, if the family of those maps have an appropriate number of analytic integrals, there exists an analytic system of symplectic (Birkhoff) coordinates in which this family of maps can be solved explicitly. It is applied for a real analytic superintegrable system to show the existence of special coordinates near singular orbits in which the system can be solved explicitly.

1. Introduction

We consider a Hamiltonian system with d degrees of freedom

$$\dot{x}_k = \frac{\partial H}{\partial y_k}, \qquad \dot{y}_k = -\frac{\partial H}{\partial x_k} \qquad (k = 1, \ldots, d), \tag{1}$$

where H is a smooth function of $(x, y) \in \Omega$, Ω being a domain of $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$. The function H is called *Hamiltonian* and the vector field (1) is denoted by X_H. The Hamiltonian H is an integral of X_H, that is, invariant under the flow of X_H, and the vector field X_H is said to be (*Liouville*) *integrable* if it has d smooth and functionally independent integrals $F_1 = H, F_2, \ldots, F_d$ such that

$$\{F_i, F_j\} = 0 \qquad (i, j = 1, , \ldots, d).$$

Here $\{\cdot, \cdot\}$ denotes the Poisson bracket defined by $\{F, G\} = X_G F$ and the functions $F_1, \ldots, F_d$ are said to be *functionally independent* if the gradient vectors $\nabla F_1, \ldots, \nabla F_d$ are linearly independent on an open dense subset of the phase space Ω.

For such an integrable system X_H, we define the map

$$F \colon \Omega \ni (x, y) \longmapsto F(x, y) = \big(F_1(x, y), \ldots, F_d(x, y)\big) \in \mathbf{R}^d.$$

This is called the *momentum map* of the integrable system. Since $F_1, \ldots, F_d$ are invariant under the flow of X_H, each orbit $(x(t), y(t))$ of X_H is confined on the level set

$$F^{-1}(c) := \{(x, y) \in \Omega \,|\, F_i(x, y) = c_i \ \ (i = 1, \ldots, d)\}, \qquad c_i = F_i(x(0), y(0)).$$

The level set $F^{-1}(c)$ is said to be *regular* (*nonsingular*) if c is the regular value of the map F. The following theorem plays the fundamental role in the study of integrable systems and their perturbation theory.

Theorem 1. (Liouville-Mineur-Arnold) [1] *If $F^{-1}(c)$ is regular, compact and connected, then it is a d-dimensional torus and there exists a neighbourhood D of the origin of $\mathbf{R}^d$ and a local diffeomorphism $\varphi\colon \mathbf{T}^d \times D \ni (\theta, I) \longmapsto (x, y) \in \Omega$ such that the following three conditions hold:*

$$\text{(i)} \quad \varphi^*\Big(\sum_{k=1}^{d} dy_k \wedge dx_k\Big) = \sum_{k=1}^{d} dI_k \wedge d\theta_k, \qquad \text{(ii)} \quad \varphi(\mathbf{T}^d \times \{0\}) = F^{-1}(0),$$

$$\text{(iii)} \quad F_i \circ \varphi(\theta, I) \ \text{does not depend on } \theta \qquad (i = 1, \dots, d).$$

A diffeomorphism (transformation) φ satisfying condition (i) is said to be *symplectic* and the coordinates of $\theta \in \mathbf{T}^d$ and $I \in D$ are called *angle coordinates* and *action coordinates* respectively. In these coordinates, each vector field X_{F_i} is a Hamiltonian system with Hamiltonian $f_i := F_i \circ \varphi$ and is written as

$$\dot{\theta}_k = \frac{\partial f_i}{\partial I_k}, \qquad \dot{I}_k = -\frac{\partial f_i}{\partial \theta_k} = 0 \qquad (k = 1, \dots, d).$$

This can be solved explicitly as

$$\theta_k(t) = \theta_k(0) + \omega_k\, t \quad \Big(\omega_k = \frac{\partial f_i}{\partial I_k}\big(I(0)\big)\Big), \qquad I_k(t) = I_k(0).$$

Namely, the vector fields X_{F_i} are linearized on each regular level set $F^{-1}(c)$ and give periodic or quasi-periodic motions.

The above theorem gives rather perfect description of behaviour of solutions for integrable systems near regular points of the map $F\colon \Omega \longrightarrow \mathbf{R}^d$. However in general, there are singularities of the map F. It is not clear whether an integrable system admits special coordinates near its singularities so that the system can be solved explicitly. This article is devoted to the study of this problem.

Birkhoff normal form theory plays a key role for this purpose. Let $H(x, y)$ be a real analytic function near the origin $(x, y) = (0, 0) \in \mathbf{R}^{2d}$. We assume, for simplicity, that H has the form

$$H(x, y) = H_2(x, y) + O(|x|^3 + |y|^3), \quad H_2(x, y) = \sum_{k=1}^{d} \frac{\alpha_k}{2}(x_k^2 + y_k^2) \qquad (\alpha_k \in \mathbf{R}).$$

The origin is an elliptic equilibrium point of the vector field X_H. We say that the function H is in *Birkhoff normal form* (*BNF*) if the identity $\{H, H_2\} = 0$ holds, that is, H is invariant under the flow of X_{H_2}, or equivalently H_2 is an integral of X_H. The equilibrium point (the origin) is said to be *non-resonant* if the following condition holds.

$$\sum_{j=1}^{d} k_j \alpha_j \neq 0 \quad \text{for any} \quad (k_1, \dots, k_d) \in \mathbf{Z}^d \setminus \{0\}. \tag{2}$$

It is well known that there exists a formal symplectic transformation $\varphi = \mathrm{id} + \cdots$ such that $H \circ \varphi$ is in formal BNF. Here "formal" means that any object (transformation, function,

[1] It is also called "Liouville-Arnold theorem." We added the name of Mineur according to Zung [9, 19-20].

...) is considered as formal power series, and under the non-resonance condition (2) the BNF is a formal power series in d variables $\omega_k = \frac{1}{2}(x_k^2 + y_k^2)$ $(k = 1, \dots, d)$. The transformation φ is called *Birkhoff transformation* and the new coordinates induced by φ are called *Birkhoff coordinates*. The Birkhoff normal form $H \circ \varphi$ is uniquely determined although φ is not.

As Siegel [14] showed, it is exceptional that there exists a convergent Birkhoff transformation. If, however, the Birkhoff transformation φ is convergent, the vector field X_h with $h = H \circ \varphi$ is written as

$$\dot{x}_k = \frac{\partial h}{\partial y_k} = \frac{\partial h}{\partial \omega_k} y_k, \qquad \dot{y}_k = -\frac{\partial h}{\partial x_k} = -\frac{\partial h}{\partial \omega_k} x_k \qquad (k = 1, \dots, d)$$

Since h is a function of $\omega_1, \dots, \omega_d$ alone and $d\omega_k/dt = x_k\dot{x}_k + y_k\dot{y}_k = 0$, this system is linear along solutions and can be solved explicitly. In this case, ω_k are Poisson commuting integrals of X_h and hence the original system X_H has to be Liouville integrable. This gives rise to a natural question: does there exist a convergent Birkhoff transformation φ for analytically Liouville integrable system X_H near a non-resonant equilibrium point? This question was answered affirmatively by the author [4] after preceding works by Rüssman [12] and Vey [17]. It was extended to general vector fields case by Stolovitch [15, 16]. On the other hand, Eliasson [2] proved the same result for smooth Liouville integrable system under the restrictive condition that the quadratic part of the momentum map F at the equilibrium point is nondegenerate (see [9] for its generalization). This type of nondegeneracy condition is needed in smooth case to guarantee the existence of a smooth Birkhoff transformation (see the appendix of [9]), however, it is an open problem to generalize the nondengeneracy condition.

Also, it was proved by Zung [18] that, including any resonance cases, an analytically Liouville integrable Hamiltonian system has a convergent Birkhoff transformation near an equilibrium point. Zung's approach is geometric and based on the study of "torus action" associated with integrable systems. It has a wide range of applications (see e.g. the survey articles [19, 20]). However, in resonance cases, it does not necessarily imply the existence of special coordinates in which the corresponding system is solved explicitly. Furthermore, the Birkhoff normal form is not determined uniquely. For this problem, we considered in [6] "superintegrable" situation with more than d (= the degree of freedom) integrals, and proved in analytic category that if the Hamiltonian system X_H has $d + q$ integrals near an equilibrium point of resonance degree q, there exists a convergent Birkhoff transformation such that the BNF becomes a function of d variables $\omega_1, \dots, \omega_d$ alone and can be solved explicitly. Furthermore, the BNF is uniquely determined and actually can be written as function of $d - q$ variables which are linear combinations of $\omega_1, \dots, \omega_d$. There are many examples of superintegrable systems: the Kepler problem, Euler rigid-body motion with symmetry, Toda lattice, $\cdots$. For general information about superintegrable systems, we refer to [3].

In the above mentioned result, the equilibrium point corresponds to rank 0 singularity of the map consisting of $d + q$ integrals. In this article, we generalize this result to situations with more general singularities. For this purpose, we note that, if there exists a nondegenerate periodic orbit for a Hamiltonian system (1), it forms a family of periodic orbits depending on the energy parameter (the value of H). Here "nondegenerate" means that the linear part of the reduced Poincaré map does not have the eigenvalue 1. In the case of integrable systems, it corresponds to the rank 1 singularity of the map F consisting of integrals of X_H. To study the orbit structure near this family of periodic orbits, it is natural to consider Birkhoff normalization in a neighbourhood of the fixed points of the reduced Poincaré maps.

Motivated by this observation, we will formulate theorems on the existence of a convergent Birkhoff normalization for a family of symplectic maps near a fixed point such that the orbits of given maps are obtained explicitly. It is to be noted that the torus action approach does not work for normalization of maps without additional assumptions (see [11]). We will prove them by elementary analysis of Birkhoff normal forms and normalization. It will turn out that the BNF of the parametrized map does not become so complicated although the resonance structure of the fixed points varies with the parameter changes. It shows the rigidity of (super)integrable systems. The Birkhoff transformation will be extended to a tubular neighbourhood of the singular orbits and the desired special coordinates will be obtained.

We conclude this introduction by referring to some recent progress on the study of Birkhoff normal forms. First, we note that there exists a convergent Birkhoff normalization near a non-resonant equilibrium point satisfying a Diophantine condition if the Birkhoff normal form $H \circ \varphi$ becomes a special form, more precisely a power series of the quadratic part of H. This was shown by Rüssmann [13]. This type of phenomenon was found recently in different settings [1]. These results suggest that the Birkhoff normal form itself plays a role in the existence problem of a convergent Birkhoff normalization, more generally in the behaviour of solutions of the original system. In this respect, it is a problem whether the Birkhoff normal form in non-resonance case can be convergent even though there is no convergent normalization.[2] This natural question was raised by Eliasson and its study was initiated by Péres-Marco [10] and recently Krikorian [7] proved that it is generally divergent. See [7] for the precise statement and for discussions on relevant problems and results.

The article is organized as follows. In §2, we review the definition of Birkhoff normal form of a symplectic map. In §3, we state main theorems (Theorem 3 and Theorem 4) about existence of an analytic Birkhoff normalization for an analytic family of symplectic maps near resonant fixed points. In §4, we give an application of these theorems to an analytic superintegrable system near a family of elliptic lower-dimensional invariant tori. In §5, we discuss the idea of their proofs.

Throughout this article, we use the following notation.

- $\exp X_H$: the time-1 map of the flow of the Hamiltonian vector field X_H

- $\mathrm{Diff}(\mathbf{R}^{2n},0)$ $\big(\mathrm{Diff}(\mathbf{C}^{2n},0)\big)$: the group of germs of real (complex) analytic symplectic diffeomorphisms $f\colon (\mathbf{R}^{2n},0) \to (\mathbf{R}^{2n},0)$ $\big((\mathbf{C}^{2n},0) \to (\mathbf{C}^{2n},0)\big)$.

- $\mathrm{Diff}_V(\mathbf{R}^{2n},0)$ $\big(\mathrm{Diff}_V(\mathbf{C}^{2n},0)\big)$: the group of germs of real (complex) analytic symplectic diffeomorphisms $f(v,\cdot)\colon (\mathbf{R}^{2n},0) \to (\mathbf{R}^{2n},0)$ $\big((\mathbf{C}^{2n},0) \to (\mathbf{C}^{2n},0)\big)$ depending real (complex) analytically on a parameter $v \in V$, where V is a domain of $\mathbf{R}^k$ $(\mathbf{C}^k)$ for some integer $k > 0$

[2] In this sense, the title of the paper [4] is misleading. It should have been "Convergence of Birkhoff normalization for analytic integrable systems"

- $\mathcal{A}_V(\mathbf{R}^{2n},0)$ $\big(\mathcal{A}_V(\mathbf{C}^{2n},0)\big)$: the ring of germs of real (complex) analytic functions $F(v,\cdot)$ near the origin of $\mathbf{R}^{2n}$ $(\mathbf{C}^{2n})$ depending real (complex) analytically on a parameter $v \in V$, where V is a domain of $\mathbf{R}^k$ $(\mathbf{C}^k)$ for some integer $k > 0$.

Since any element of $\mathrm{Diff}(\mathbf{R}^{2n},0)$ is analytic, we have natural inclusions

$$\mathrm{Diff}(\mathbf{R}^{2n},0) \subset \mathrm{Diff}(\mathbf{C}^{2n},0), \qquad \mathrm{Diff}_V(\mathbf{R}^{2n},0) \subset \mathrm{Diff}_{\widetilde{V}}(\mathbf{C}^{2n},0),$$

where $\widetilde{V}$ is a complex domain containing the real domain V. For any $f \in \mathrm{Diff}_V(\mathbf{C}^{2n},0)$, we denote it by $f = f(v,z)$, where $v \in V$ is a parameter. We use the notation $Df(v,0)$ to denote the linear part (the Jacobian matrix) of $f \in \mathrm{Diff}_V(\mathbf{C}^{2n},0)$ with respect to the variable $z \in \mathbf{C}^{2n}$. Also, we denote by $G(v,z)$ a power series of z with coefficients being holomorphic functions of $v \in V$.

2. Birkhoff normal forms for symplectic maps

In the previous section, we introduced Birkhoff normal form (BNF) in real category near an elliptic equilibrium point. The BNF is available also for any type of equilibrium point as well as for symplectic maps near a fixed point. However, we need to divide cases to state real BNF in details. Instead, in what follows, we work with complex BNF which can be stated in a unified manner. The BNF in real category will be considered in §4 by imposing appropriate reality condition.

Let us consider the BNF for symplectic maps without parameters. Let $f \in \mathrm{Diff}(\mathbf{C}^{2n},0)$ and let Λ denote the semi-simple part of $Df(0)$. We assume that

$$\Lambda = \mathrm{diag}\,(\lambda_1,\dots,\lambda_n,\lambda_1^{-1},\dots,\lambda_n^{-1}).$$

We call this matrix Λ a *symplectic diagonal matrix.* The map f is said to be in *Birkhoff normal form* (BNF) (*up to order N*) if it commutes with Λ (up to order N), i.e.,

$$f\circ\Lambda = \Lambda\circ f \qquad \big(f\circ\Lambda(z) - \Lambda\circ f(z) = O(|z|^{N+1})\big).$$

Theorem 2. (Birkhoff normal form of a symplectic map) *Let $f \in \mathrm{Diff}\,(\mathbf{C}^{2n},0)$. Then, for any positive integer N, there exists a transformation $\varphi \in \mathrm{Diff}(\mathbf{C}^{2n},0)$ such that $\varphi^{-1}\circ f\circ\varphi$ is in BNF up to order N, that is,*

$$\varphi^{-1}\circ f\circ\varphi(z) = f_N\circ(z + O(|z|^{N+1})), \quad f_N\circ\Lambda = \Lambda\circ f_N, \quad f_N \in \mathrm{Diff}(\mathbf{C}^{2n},0).$$

In particular, if $N \geq 2$ and the eigenvalues of $Df(0)$ satisfy the condition

$$\prod_{i=1}^{n}\lambda_i^{k_i} \neq 1 \quad \textit{for} \quad 0 < |k| \leq N \qquad (|k| = |k_1| + \cdots + |k_n|, \quad k_i \in \mathbf{Z}), \tag{3}$$

then the map $\varphi^{-1}\circ f\circ\varphi$ is written as

$$\varphi^{-1}\circ f\circ\varphi(z) = \Lambda\circ\exp X_h\circ\big(z + O(|z|^{N+1})\big), \quad h = h(\omega_1,\dots,\omega_n), \quad \omega_i = x_iy_i\,,$$

where $\big(D\varphi(0)\big)^{-1}Df(0)D\varphi(0) = \Lambda$ and h is a polynomial of $\omega_1,\dots,\omega_n$ of degree $[N/2]$, the maximum integer that does not exceed $N/2$.

In the above, the transformation φ is not unique, however, the polynomial h is uniquely determined independently of the higher order (i.e., order ≥ 2) terms of φ. The map $f_N := \Lambda\circ \exp X_h$ is called *Birkhoff normal form of f in non-resonance of degree N*. The dynamical system defined by the map f_N is completely understood. In fact, the vector field X_h is written as

$$\dot{x}_i = \frac{\partial h}{\partial \omega_i} x_i, \quad \dot{y}_i = -\frac{\partial h}{\partial \omega_i} y_i \qquad (i = 1, \ldots, n).$$

The products ω_i are invariant under the flow of X_h, and hence the map f_N is written as

$$f_N \colon (x_i, y_i) \longmapsto \left(\lambda_i e^{\partial h/\partial \omega_i} x_i, \lambda_i^{-1} e^{-\partial h/\partial \omega_i} y_i\right) \qquad (i = 1, \ldots, n).$$

Hence m-th iteration of f_N is given explicitly as

$$f_N^m \colon (x_i, y_i) \longmapsto \left(\lambda_i^m e^{m\,(\partial h/\partial \omega_i)} x_i, \lambda_i^{-m} e^{-m\,(\partial h/\partial \omega_i)} y_i\right) \qquad (i = 1, \ldots, n).$$

In this case, we say that the dynamical system of f_N is *solved explicitly*. However, in general case without assuming the condition (3) the map f_N may not be solved in this manner, namely the Birkhoff normal form of f cannot be solved explicitly even formally.

Theorem 2 can be proved by using the following fact.

Proposition 1. (a) *A map $f \in \mathrm{Diff}(\mathbf{C}^{2n}, 0)$ can be written as*

$$f = Df(0)\circ \widehat{f}, \quad \widehat{f} = \left(\exp X_{H_1}\circ \exp X_{H_2}\circ \cdots \circ \exp X_{H_\nu}\right)\circ \psi \tag{4}$$

with

$$\begin{cases} H_k = H_k^{d+2} + \cdots + H_k^{2d+1} & (d = 2^{k-1},\ k = 1, \ldots, \nu), \\ \psi(z) = z + O(|z|^{2^\nu + 1}), \end{cases}$$

where H_k^l are homogeneous polynomials of degree l in z.

(b) *Let Λ be a symplectic diagonal matrix (which is not necessarily equal to the semi-simple part of $Df(0)$). A map $f \in \mathrm{Diff}(\mathbf{C}^{2n}, 0)$ commutes with Λ up to order $N = 2^\nu$ if and only if the following two conditions hold for expression* (4)*.*

(i) *$Df(0)$ commutes with Λ,* (ii) $H_k\circ \Lambda = H_k \quad (1 \leq k \leq \nu)$.

3. Convergent Birkhoff normalization theorems for a family of superintegrable symplectic maps

In this section, we state main theorems about the existence of a convergent Birkhoff transformation for superintegrable symplectic maps.

First we consider a map $f \in \mathrm{Diff}(\mathbf{C}^{2n}, 0)$ without parameters. For the matrix

$$\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}),$$

the set

$$\mathcal{R} := \left\{ k = (k_1, \ldots, k_n) \in \mathbf{Z}^n \;\middle|\; \prod_{i=1}^n \lambda_i^{k_i} = 1 \right\}$$

is called *the resonance lattice of* Λ. This is a discrete subgroup of $\mathbf{Z}^n$. If $\mathcal{R}$ is generated by q $(0 \leq q \leq n)$ elements, we write $\operatorname{rank} \mathcal{R} = q$ and say that the fixed point $z = 0$ of f (or the matrix Λ) is of *resonance degree* q. Here $q = 0$ means that $\mathcal{R} = \{\mathbf{0}\}$, and in this case the fixed point (or the matrix Λ) is also said to be *non-resonant*.

Next, we consider symplectic maps $f \in \mathrm{Diff}_V(\mathbf{C}^{2n}, 0)$ depending on a parameter $v \in V$, where V is a domain of $\mathbf{C}^k$. Let $\Lambda(v)$ be the semi-simple part of the linear part $Df(v, 0)$ and assume that

$$\Lambda(v) = \operatorname{diag}\left(\lambda_1(v), \ldots, \lambda_n(v), \lambda_1^{-1}(v), \ldots, \lambda_n^{-1}(v)\right)$$

with eigenvalues $\lambda_i(v)$ being holomorphic functions of $v \in V$. We set

$$\mathcal{R}(v) := \left\{k = (k_1, \ldots, k_n) \in \mathbf{Z}^n \;\middle|\; \prod_{i=1}^{n} \lambda_i^{k_i}(v) = 1\right\}$$

and call it *the resonance lattice of* $\Lambda(v)$, which depends on the parameter $v \in V$. Since each $\mathcal{R}(v)$ contains the zero vector $0 \in \mathbf{Z}^n$, the intersection

$$\mathcal{R}_0 := \bigcap_{v \in V} \mathcal{R}(v)$$

is nonempty and is a discrete subgroup of $\mathbf{Z}^n$. We introduce the following definition.

Definition 1. We say that the fixed point $z = 0$ of the parametrized map $f \in \mathrm{Diff}_V(\mathbf{C}^{2n}, 0)$ (or the parametrized matrix $\Lambda(v)$) is *of resonance degree* q if $\operatorname{rank} \mathcal{R}_0 = q$.

In this case, the following holds:

Proposition 2. *Assume that the fixed point of the parametrized map* $f \in \mathrm{Diff}_V(\mathbf{C}^{2n}, 0)$ *is of resonance degree* q. *Then, there exists a dense subset* V_0 *of* V *satisfying the following three conditions*

(i) $\mathcal{R}(v) \supset \mathcal{R}_0$ *for every* $v \in V$, (ii) $\operatorname{rank} \mathcal{R}_0 = q$, (iii) $\mathcal{R}(v) = \mathcal{R}_0$ *for every* $v \in V_0$.

In fact, claims (i) and (ii) are trivial. To see (iii), let

$$V_0 = \bigcap_{k \in \mathbf{Z}^n \setminus \mathcal{R}_0} \{v \in V \mid \prod_{i=1}^{n} \lambda_i^{k_i}(v) \neq 1\}. \tag{5}$$

Here $\lambda_i(v)$ are holomorphic functions in the complex domain V, and hence for each $k \in \mathbf{Z}^n \setminus \mathcal{R}_0$ fixed, the set $\{v \in V \mid \prod_{i=1}^{n} \lambda_i^{k_i}(v) \neq 1\}$ is dense in V. Otherwise, we have contradiction by the identity theorem. Therefore, V_0 is the intersection of countable numbers of open dense subsets of V. This implies that V_0 is dense in V by the Baire property. For $v \in V_0$, it holds that $\mathcal{R}(v) = \mathcal{R}_0$ and claim (iii) follows. ■

In the case $f \in \mathrm{Diff}_V(\mathbf{R}^{2n}, 0)$, we need to extend $V \subset \mathbf{R}^k$ to a complex domain $\widetilde{V} \subset \mathbf{C}^k$ in order to apply the identity theorem when $k \geq 2$. In other words, the set V_0 defined by (5) is not necessarily dense in $V \subset \mathbf{R}^k$. By this fact, we introduce the following

Definition 2. We say that a real parametrized map $f \in \mathrm{Diff}_V(\mathbf{R}^{2n}, 0)$ with $V \subset \mathbf{R}^k$ is *of resonance degree* q if there exists a dense subset V_0 of V satisfying the conditions (i)-(iii) of Proposition 2.

Remark 1. In the case $k = 1$, Proposition 2 holds true for $f \in \mathrm{Diff}_V(\mathbf{R}^{2n}, 0)$ without any modification. Therefore, in the case $k = 1$ it is possible to define $f \in \mathrm{Diff}_V(\mathbf{R}^{2n}, 0)$ to be of resonance degree q by Definition 1.

Example. Let $f \in \mathrm{Diff}_V(\mathbf{R}^2, 0)$ and assume that the eigenvalues of $Df(0)$ are $e^{\pm 2\pi i \alpha(v)}$ ($i = \sqrt{-1},\ \alpha \in \mathbf{R}$). If $\alpha(v)$ varies with v, then the fixed point is resonant or non-resonant according to $\alpha(v) \in \mathbf{Q}$ or $\alpha(v) \in \mathbf{R} \setminus \mathbf{Q}$. In this case, the origin is a non-resonant fixed point of the family of maps $f(v, \cdot)$ with $v \in V$.

We now consider the Birkhoff normalization of $f \in \mathrm{Diff}_V(\mathbf{C}^{2n}, 0)$ near a fixed point of resonance degree q and want to find a special coordinate system in which the dynamical system of f is solved explicitly. In order to state the results, let $\mathcal{R}_0$ be the minimal resonance lattice defined above of the parametrized matrix $\Lambda(v)$ and introduce

$$\begin{cases} \rho^{(1)}, \ldots, \rho^{(q)} \in \mathbf{Z}^n & : \text{ generators of } \mathcal{R}_0 \\ \rho^{(q+1)}, \ldots, \rho^{(n)} \in \mathbf{R}^n & : \text{ linearly independent vectors which are orthogonal to} \\ & \quad\ \rho^{(1)}, \ldots, \rho^{(q)} \text{ with respect to the Euclidean inner product.} \end{cases}$$

Furthermore, we set

$$\rho^{(i)} = (\rho_1^{(i)}, \ldots, \rho_n^{(i)}), \qquad \rho^{(i)} = \rho_+^{(i)} - \rho_-^{(i)} \quad (\rho_+^{(i)},\ \rho_-^{(i)} \in \mathbf{Z}_+^n)$$

and define $n + q$ variables ω_i $(i = 1, \ldots, n + q)$ as well as n variables τ_i $(i = 1, \ldots, n)$ as follows:

$$\begin{cases} \omega_i = x_i y_i, \qquad \tau_i = \displaystyle\sum_{j=1}^{n} \rho_j^{(i)} \omega_j \qquad (i = 1, \ldots, n), \\ \omega_{n+i} = x^{\rho_+^{(i)}} y^{\rho_-^{(i)}} \qquad (i = 1, \ldots, q). \end{cases}$$

Let V_0 be the dense subset of V described in Proposition 2 and $\Lambda := \Lambda(v)$ for any $v \in V_0$ fixed. Then, the variables $\omega_1, \ldots, \omega_{n+q}$ are monomials of z which are invariant under Λ.

Since the fixed point of f is resonant, one cannot expect in general that the Birkhoff normal form of f is solved explicitly, nor can expect that there exists a convergent Birkhoff normalization. Moreover, since we consider a parametrized map, one may wonder that resonance terms of its Birkhoff normal form appear or disappear as the parameter v varies. This phenomenon happens in general cases, however, our first result below shows that it does not occur in integrable systems, more precisely provided that f has $n+q$ functionally independent integrals. It turns out that the Birkhoff normal form of f becomes a map which commutes with $\Lambda(v_0)$ for any $v_0 \in V_0$ fixed (see Lemma 2 in §5). In this case, the Birkhoff normal form will be called Λ-*normal form*. The result is stated as follows.

Theorem 3. *Let $f(v, \cdot) \in \mathrm{Diff}_V(\mathbf{C}^{2n}, 0)$ and assume that the following conditions hold:*

[A1] *all eigenvalues of $Df(v, 0)$ are holomorphic functions of $v \in V$.*

[A2] *the fixed point $z = 0$ of the parametrized map $f(v, \cdot)$ is of resonance degree q.*

[A3] *f has $n+q$ analytic integrals $G_i(v,z) \in \mathcal{A}_V(\mathbf{C}^{2n},0)$ $(i=1,\dots,n+q)$ which are functionally independent functions of z for each $v \in V$ fixed.*

Then there exists an open dense subset $\widehat{V}$ of V with the following property: For any $v_0 \in \widehat{V}$, there exists its neighbourhood $\widehat{V}(v_0) \subset \widehat{V}$ and a symplectic transformation $\varphi \in \mathrm{Diff}_{\widehat{V}(v_0)}(\mathbf{C}^{2n},0)$ such that

$$\varphi^{-1}\circ f\circ\varphi = \begin{cases} \Lambda(v)\circ \exp X_{h(v,\cdot)} & (0 \le q \le n-1), \\ \Lambda(v) & (q=n). \end{cases} \tag{6}$$

Here $\Lambda(v)$ is the symplectic diagonal matrix and $h(v,\cdot)$ is a convergent power series of $\omega_1,\dots,\omega_n$ with coefficients being holomorphic functions of $v \in \widehat{V}(v_0)$ such that it depends actually on $n-q$ variables $\tau_{q+1},\dots,\tau_n$ only.

Remark 2. (i) The function h is uniquely determined as power series in $\omega_1,\dots,\omega_n$. More precisely, it is independent of the higher order (i.e., order ≥ 2) terms of φ.

(ii) The monomials $\omega_1,\dots,\omega_{n+q}$ are invariant under $\varphi^{-1}\circ f\circ\varphi$. Moreover, if $G(v,z)$ is an analytic function which is invariant under f, then $G\circ\varphi$ is a function (Laurent series) of $n+q$ variables $\omega_1,\dots,\omega_{n+q}$.

(iii) By the implicit function theorem, the condition [A1] holds if all eigenvalues of $Df(v,0)$ are simple. Also, when the parameter space V is one-dimensional, a multiple eigenvalue can be expressed as Puiseux series of $v \in V$. Therefore, condition [A1] holds for this case under slight modification so that the same conclusion of the theorem holds.

(iv) The theorem holds also when the parameter is fixed. It is obvious how to modify assumptions and consequences. The condition [A1] is deleted and there is no need to take the subset $\widehat{V}$ of V.

(v) Since $\omega_1,\dots,\omega_n$ are Poisson commuting integrals of $\varphi^{-1}\circ f\circ\varphi$, it turns out that the original map f is Liouville integrable near the origin although it is not assumed.

The normal form (6) will be obtained through normalization of f into Λ-normal form and will turn out to have the same form as in non-resonance case, i.e., h is a function of $\omega_1,\dots,\omega_n$ (actually of smaller number of variables stated above). This implies that the normal form (6) can be solved explicitly.

As mentioned in §1, the symplectic map in Theorem 3 typically appears as the reduced Poincaré map associated to a family of periodic orbits. In the case where several Poisson commuting functions are given, the corresponding flows give rise to commuting Poincaré maps. For such situations, we have the following

Theorem 4. *Let $f_i(v,\cdot) \in \mathrm{Diff}_V(\mathbf{C}^{2n},0)$ $(i=1,\dots,k)$ satisfy the following conditions:*

[B1] $f_i\circ f_j = f_j\circ f_i \qquad (i,j=1,\dots,k)$,

[B2] *the fixed point $z=0$ of f_i $(i=1,\dots,k)$ is of resonance degree q_i and $q_i \ge q_1 (=q)$,*

[B3] *$f_1,\dots,f_k$ have common $n+q$ analytic integrals $G_i(v,z)$ $(i=1,\dots,n+q)$ which are functionally independent functions of z for each $v \in V$ fixed,*

[B4] *all eigenvalues of $Df_i(v,0)$ $(i=1,\dots,k)$ are holomorphic functions of $v \in V$ and all $Df_i(v,0)$ are diagonalized by a symplectic matrix whose elements are holomorphic functions of $v \in V$.*

Then the same conclusion of Theorem 3 holds with $f(v,\cdot)$, $\Lambda(v)$, $h(v,\cdot)$ replaced by $f_i(v,\cdot), \Lambda_i(v)$, $h_i(v,\cdot)$ respectively.

Remark 3. The condition [B4] holds if there is some map $f \in \mathrm{Diff}_V(\mathbf{C}^{2n}, 0)$ such that it commutes with all of $f_1, \ldots, f_k$ and all eigenvalues of $Df(v,0)$ are simple.

In fact, all eigenvalues of $Df(v,0)$ are holomorphic functions of $v \in V$ under this assumption. If $P = P(v)$ is a symplectic matrix whose elements are holomorphic functions of $v \in V$ such that $P^{-1}Df(v,0)P$ is diagonal, then the commuting relation $Df(v,0)Df_i(v,0) = Df_i(v,0)Df(v,0)$, together with simplicity of the eigenvalues of $Df(v,0)$, implies that $Df_i(v,0)$ is also diagonalized by P and its eigenvalues are holomorphic functions of $v \in V$.

Remark 4. If $f \in \mathrm{Diff}_V(\mathbf{R}^{2n}, 0)$ or $f_i \in \mathrm{Diff}_V(\mathbf{R}^{2n}, 0)$, it is important to consider Birkhoff normalization in real category. The real Birkhoff normalization is obtained by imposing reality condition on the diagonalization matrix. The real BNF is obtained by $\omega_j = x_j y_j$ replaced with the real variables such as $\widehat{\omega}_j = \frac{1}{2}(x_j^2 + y_j^2)$ mentioned in §1.

4. Action-angle with Birkhoff coordinates near singular orbits for superintegrable systems

Let (M,σ) be a real analytic symplectic manifold of dimension $2d$. For a real analytic function H on M, the Hamiltonian vector field X_H is defined and is written in the form (1) by introducing the standard symplectic structure $\sigma = \sum_{k=1}^{d} dy_k \wedge dx_k$.

Let $F_1, \ldots, F_k, F_{k+1}, \ldots, F_{d+q}$ be $d+q$ real analytic functions on M satisfying the condition

$$\{F_i, F_j\} = 0 \quad (i = 1, \ldots, k,\ j = 1, \ldots, d+q) \tag{7}$$

and $dF_1, \ldots, dF_{d+q}$ are linearly independent on an open dense subset of M (i.e., functionally independent). Namely, $F_1, \ldots, F_k$ are Poisson commuting functions which have $d+q$ common integrals $F_1, \ldots, F_{d+q}$. We note that $k \le d$ and set

$$F = (F_1, \ldots, F_k).$$

Then, near any regular point of the map F, there exists a system of local symplectic coordinates $(u, v, x, y) \in \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}^{d-k} \times \mathbf{R}^{d-k}$ such that

$$\begin{cases} \sigma = \displaystyle\sum_{i=1}^{k} dv_i \wedge du_i + \sum_{j=1}^{d-k} dy_j \wedge dx_j \\ F_i = v_i \quad \text{and hence} \quad X_{F_i} = \dfrac{\partial}{\partial u_i} \qquad (i = 1, \ldots, k). \end{cases}$$

Then condition (7) implies that $F_{k+1}, \ldots, F_{d+q}$ are functions of $2d-k$ variables v, x, y alone. Since $F_1, \ldots, F_{d+q}$ are functionally independent, we have $d+q \le 2d-k$ and hence

$$q \le d - k (\equiv n)$$

Let $p_0 \in M$ and assume that

$$\operatorname{rank}(dF_1, \dots, dF_{d+q})(p_0) = \operatorname{rank}(dF_1, \dots, dF_k)(p_0) = k, \qquad F(p_0) = 0. \tag{8}$$

We consider orbits of commuting vector fields $X_{F_1}, \dots, X_{F_k}$. Let $\phi^t_{F_i}$ be the flow of X_{F_i}, and for a point p in a neighbourhood of p_0 we define

$$\phi^t_F(p) := \phi^{t_1}_{F_1} \circ \cdots \circ \phi^{t_k}_{F_k}(p) \quad \text{for} \quad t = (t_1, \dots, t_k) \in \mathbf{R}^k.$$

We assume that the level set of $F_1, \dots, F_{d+q}$ through p_0 is compact. Then $\phi^t_F(p_0)$ is defined for all $t \in \mathbf{R}^k$. We call the set

$$\Phi_F(p_0) := \{\phi^t_F(p_0) \in M \mid t \in \mathbf{R}^k\}$$

the *orbit of* $X_{F_1}, \dots, X_{F_k}$ *through* $p_0 \in M$. The set

$$\operatorname{Per}_F(p_0) := \{t \in \mathbf{R}^k \mid \phi^t_F(p_0) = p_0\}$$

is called the *period lattice of* ϕ^t_F. It is a discrete subgroup of $\mathbf{R}^k$ and the action $t \mapsto \phi^t(p_0)$ gives rise to a diffeomorphism $\mathbf{R}^k/\operatorname{Per}_F(p_0) \cong \Phi_F(p_0)$. Since $\Phi_F(p_0)$ is compact, $\operatorname{Per}_F(p_0)$ is a k-dimensional lattice and hence $\Phi_F(p_0)$ is diffeomorphic to a k-dimensional torus $\mathbf{T}^k = \mathbf{R}^k/2\pi\mathbf{Z}^k$ and is called *invariant torus of* $X_{F_1}, \dots, X_{F_k}$.

Let us consider orbits of $X_{F_1}, \dots, X_{F_k}$ in a neighbourhood of this torus. Let U be a neighbourhood of $p_0 \in M$ and V a neighbourhood of $v = 0 \in \mathbf{R}^k$. For any $v \in V$, we consider the level set of $F = (F_1, \dots, F_k)$, denoted by $F^{-1}(v) := \{p \in M \mid F(p) = v\}$. The flows of $X_{F_1}, \dots, X_{F_k}$ give rise to a local action of $\mathbf{R}^k$ on $F^{-1}(v) \cap U$, and the quotient space

$$\Sigma_v := F^{-1}(v) \cap U/\mathbf{R}^k$$

is obtained by identifying the orbit of this action. It is a symplectic manifold of dimension $2n\big(= 2(d-k)\big)$ and is called the *reduced phase space*. Using the coordinates (u, v, x, y), the point p_0 corresponds to $(0,0,0,0)$ and Σ_v is identified as the set $\{(u,v,x,y) \in U \mid u = 0,\ v = \text{const.}\}$.

Let

$$\pi_v \colon F^{-1}(v) \cap U \to \Sigma_v$$

be the projection map. Let $T = (T_1, \dots, T_k) \in \operatorname{Per}_F(p_0)$ and consider the parametrized map

$$\widehat{\phi}^T_F(v, \cdot) := \pi_v \circ \phi^T_F \circ \pi_v^{-1} \colon \Sigma_v \longrightarrow \Sigma_v\,.$$

We call this map the *reduced period-T map of the vector fields* $X_{F_1}, \dots, X_{F_k}$. Here v is the value of F and $(x, y) = (0, 0)$ is the fixed point of the map $\widehat{\phi}^T_F(0, \cdot)$

Definition 3. An invariant torus $\Phi_F(p_0)$ of $X_{F_1}, \dots, X_{F_k}$ is said to be *nondegenerate* if there exists a period $T \in \operatorname{Per}_F(p_0)$ such that $\pi_0(p_0)$ is a nondegenerate fixed point of the reduced period-T map $\widehat{\phi}^T_F(0, \cdot)$, i.e., the linear map $D\widehat{\phi}^T_F(0, 0)$ does not have eigenvalue 1.

Then we have

Proposition 3. *If $\Phi_F(p_0)$ is a nondegenerate invariant k-torus of $X_{F_1}, \dots, X_{F_k}$, then there exists a unique k-parameter family of k-dimensional invariant tori $\Phi_F(p_v) \subset F^{-1}(v)$ of $X_{F_1}, \dots, X_{F_k}$, where p_v is a (vector-valued) real analytic function of $v \in V$, V being a neighbourhood of $F(p_0) = 0$ in $\mathbf{R}^k$.*

We call this family of invariant k-tori simply an *analytic family of invariant k-tori* and denote it by $\{\Phi_F(p_v)\}_V$. By this proposition, the reduced period-T map $\widehat{\phi}_F^T(v, \cdot)$ can be considered as $\widehat{\phi}_F^T(v, \cdot) \in \mathrm{Diff}_V(\mathbf{R}^{2n}, 0)$ after parallel translation of taking p_v to the origin.

We now state a result establishing the existence of action-angle with Birkhoff coordinates for superintegrable systems with singularities. For simplicity, we restrict ourselves to the case where those invariant k-tori are elliptic. We introduce the following

Definition 4. An analytic family of invariant k-tori $\{\Phi_F(p_v)\}_V$ is said to be (i) *elliptic*, (ii) *simple*, (iii) *of resonance degree* q if there exists a period $T = T(v) \in \mathrm{Per}_F(p_v)$ such that the associated reduced period-T map $\widehat{\phi}_F^T(v, \cdot)$ satisfies the following conditions respectively:

(i) p_v is an elliptic fixed point of $\widehat{\phi}_F^T(v, \cdot)$,

(ii) all eigenvalues of $D\widehat{\phi}_F^T(v, 0)$ are simple,

(iii) p_v is a fixed point of the parametrized map $\widehat{\phi}_F^T(v, \cdot)$ of resonance degree q.

Then, by applying Theorem 4 we can prove the following

Theorem 5. *Let $F_1, \dots, F_k, F_{k+1}, \dots, F_{d+q}$ $(0 \le q \le d-k)$ be real analytic functions satisfying (7) and (8) on a real analytic symplectic manifold (M, σ) of dimension $2d$. Let $\{\Phi_F(p_v)\}_V$ be a real analytic family of invariant k-tori of the vector fields $X_{F_1}, \dots, X_{F_k}$ and assume that it is elliptic, simple and of resonance degree q. Then, there exists an open dense subset $\widehat{V}$ of V with the following property: For any $v \in \widehat{V}$, there exists a neighbourhood of $\Phi_F(p_v)$ in which one can introduce a system of real analytic symplectic coordinates*

$$(\theta, I, x, y) \in \mathbf{T}^k \times \mathbf{R}^k \times \mathbf{R}^{d-k} \times \mathbf{R}^{d-k}; \quad \sigma = \sum_{j=1}^{k} dI_j \wedge d\theta_j + \sum_{j=1}^{d-k} dy_j \wedge dx_j$$

such that $F_1, \dots, F_k$ are functions of $d-q$ variables

$$I_1, \dots, I_k, \widehat{\tau}_{q+1}, \dots, \widehat{\tau}_{d-k} \quad (0 \le q \le d-k-1), \qquad I_1, \dots, I_k \quad (q = d-k),$$

where $\widehat{\tau}_i = \sum_{j=1}^{n} \rho_j^{(i)} \widehat{\omega}_j$ and $\widehat{\omega}_i = \dfrac{1}{2}(x_i^2 + y_i^2)$. The vector fields X_{F_i} $(i = 1, \dots, k)$ have $d+q$ integrals $I_1, \dots, I_k$ and $\widehat{\omega}_1, \dots, \widehat{\omega}_{n+q}$ $(n = d-k)$.

5. Sketch of proofs

We give a sketch of proofs of Theorems 3–5. The details will be published elsewhere.

(i) Preliminary facts

First we note the the following fact.

Lemma 1. *Let $\Lambda \in \mathrm{sp}\,(2n, \mathbf{C})$ be a symplectic diagonal matrix of resonance degree q. If a function $G(v,z) \in \mathcal{A}_V\,(\mathbf{C}^{2n},0)$ is Λ-invariant, then $G(v,z)$ can be written as Laurent series of $n+q$ variables $\omega_1,\ldots,\omega_{n+q}$ whose coefficients are holomorphic functions in $v \in V$.*

In fact, if we write G in the form $G = \sum_{\alpha,\beta\in\mathbf{Z}_+^n} c_{\alpha\beta}(v)x^\alpha y^\beta$, we can easily see that the term $x^\alpha y^\beta$ in Λ-normal form can be expressed as a quotient of two monomials in $\omega_1,\ldots,\omega_{n+q}$ (see the proof of [6, Prop 1]).

Next we note the following fact due to Proposition 2.

Lemma 2. *Let $\Lambda(v)$ be a parametrized symplectic diagonal matrix of resonance degree q. Let $\Lambda = \Lambda(v_0)$ for any fixed $v_0 \in V_0$, where V_0 is the dense subset of V described in Proposition 2. Then, for any $f \in \mathrm{Diff}_V\,(\mathbf{C}^{2n},0)$ and for any function $G \in \mathcal{A}_V\,(\mathbf{C}^{2n},0)$ the following hold:*

$$(1)\quad G(v,\Lambda(v)z) = G(v,z) \quad (\forall v\in V) \quad\Longleftrightarrow\quad G(v,\Lambda z) = G(v,z) \quad (\forall v \in V)$$

$$(2)\quad f(v,\Lambda(v)z) = \Lambda(v)f(v,z) \quad (\forall v\in V) \Longleftrightarrow f(v,\Lambda z) = \Lambda f(v,z) \quad (\forall v \in V)$$

This lemma implies that a parametrized map (or function) is in $\Lambda(v)$-normal form for each $v \in V$ if and only if it is in Λ-normal form for all $v \in V$.

(ii) Proof of Theorem 3

Our proof of the existence of a convergent Birkhoff normalizing transformation relies on the existence of $n+q$ integrals of f. We first give a sketch of the proof of Theorem 3 under the condition that $Df(v,0)$ is a diagonal matrix. The following fact is crucial.

Proposition 4. *Let $f \in \mathrm{Diff}_V\,(\mathbf{C}^{2n},0)$ and assume that $Df(v,0)$ is a parametrized symplectic diagonal matrix of resonance degree q. Let $\Lambda = Df(v,0)$ for any $v \in V_0$ fixed. Assume that f has an integral $G \in \mathcal{A}_V\,(\mathbf{C}^{2n},0)$ and that f is in Λ-normal form up to order d for all $v \in V$. Then G is in Λ-normal form up to order $s+d-1$ for all $v \in V$, where s is the degree of the lowest order part of G as power series expansion of z with holomorphic coefficients in $v \in V$.*

Let us write the Taylor expansions of the integrals $G_i(v,z)$ of f in the form

$$G_i(v,z) = G_i^0(v,z) + G_i^1(v,z) + \cdots + G_i^d(v,z) + \cdots, \quad \deg G_i^d = s_i + d$$

where $G_i^d(v,z)$ denotes the homogeneous polynomial of degree s_i+d in z whose coefficients are holomorphic functions of $v \in V$. The functional independence of $G_1,\ldots,G_{n+q}$ does not necessarily imply that of their lowest order parts $G_1^0,\ldots,G_{n+q}^0$. However, we have the following lemma which will play a crucial role in our proof.

Lemma 3. ([5, Lemma 5.8]) *Let $G_1,\ldots,G_m \in \mathcal{A}_V\,(\mathbf{C}^{2n},0)$ be functionally independent functions of z for any $v \in V$ fixed. Then there exists an open dense subset $\widehat{V}$ of V with the following property: For any $v_0 \in \widehat{V}$, there exists its neighbourhood $\widehat{V}(v_0) \subset \widehat{V}$ such that the following holds: there exist m functions $\widehat{G}_1,\ldots,\widehat{G}_m \in \mathcal{A}_{\widehat{V}(v_0)}(\mathbf{C}^{2n},0)$ such that* (i) *they are polynomials of $G_1,\ldots,G_m$ whose coefficients are holomorphic functions of $v \in \widehat{V}(v_0)$, and* (ii) *their lowest order parts $\widehat{G}_i^0$ $(i = 1,\ldots,m)$ are functionally independent functions of z for each $v \in \widehat{V}(v_0)$.*

In view of Proposition 1, we will construct the Birkhoff normalizing transformation $z = \varphi(v, z)$ in the form

$$\varphi(v,z) = \lim_{\nu\to\infty} \Big(\exp X_{W_1} \circ \exp X_{W_2} \circ \cdots \circ \exp X_{W_\nu} \Big), \quad W_k = W_k^{d+2} + \cdots + W_k^{2d+1}, \\ (d = 2^{k-1},\ k = 1, \ldots, \nu).$$

Each iteration step is described as follows:

Proposition 5. *Let $f \in \mathrm{Diff}_V\,(\mathbf{C}^{2n}, 0)$ and assume that $Df(v, 0)$ is a parametrized symplectic diagonal matrix of resonance degree q. Assume that f has $n+q$ integrals $G_i(v, z) \in \mathcal{A}_V\,(\mathbf{C}^{2n}, 0)$ $(i = 1, \ldots, n + q)$ whose lowest order parts $G_i^0(v, \cdot)$ are functionally independent functions of z for each $v \in V$ fixed. Let $\Lambda = Df(v_0, 0)$ for any $v_0 \in V_0$ fixed and suppose that f is in Λ-normal form up to order $d = 2^\nu$ $(\nu = 0, 1, \ldots)$ for any $v \in V$. Then there exists a unique polynomial $W(v, z)$ with coefficients being holomorphic in $v \in V$ of the form*

$$W = W^{d+2} + W^{d+3} + \cdots + W^{2d+1} \quad \textit{with} \quad P_N W = 0, \tag{9}$$

$W^l = W^l(v, \cdot)$ being homogeneous polynomials of degree l, such that $\varphi = \exp X_W$ takes f into Λ-normal form up to order $2d = 2^{\nu+1}$ for any $v \in V$. Here P_N is the projection operator from the space $\mathcal{A}_V\,(\mathbf{C}^{2n}, 0)$ to its subspace consisting of power series in Λ-normal form.

It is standard to prove that, for each $v \in V_0$ fixed, there exists a polynomial W such that $\varphi = \exp X_W$ takes f into $\Lambda(v)$-normal form up to order $2d$. It is done without using the existence of integrals. We can use the existence of $n + q$ integrals of f to show that the polynomial $W(v, z)$ defined only for $v \in V_0$ can be extended to a function $W(v, z) \in \mathcal{A}_V\,(\mathbf{C}^{2n}, 0)$ defined for all $v \in V$. The argument goes as follows:

Let $\Lambda = \Lambda(v_0)$ for some fixed $v_0 \in V_0$ and suppose that $f = f(v, z)$ is in Λ-normal form up to order d for all $v \in V$. Then its integrals G_i are written as

$$G_i(v, z) = g_i(v, z) + \widehat{G}_i(v, z)\,; \quad g_i \circ \Lambda = g_i, \qquad \widehat{G}_i = O(|z|^{s_i+d}). \tag{10}$$

Let $v \in V_0$ be taken arbitrarilly and and let $W = W^{d+2} + \cdots + W^{2d+1}$ be the polynomial such that $\varphi(v, z) = \exp X_W$ takes f into $\Lambda(v)$-normal form up to order $2d$. Then $G_i \circ \varphi$ can be written as

$$G_i \circ \varphi = g_i(v, z) + \{g_i(v, z), W(v, z)\} + \widehat{G}_i(v, z) + O(|z|^{s_i+2d}) \qquad (v \in V_0).$$

Since $G_i \circ \varphi$ are Λ-invariant up to order $s_i + 2d - 1$, the homogeneous parts W^{l+2} of W satisfy the following equations for $l = d, d + 1, \ldots, 2d - 1$:

$$\{g_i^0, W^{l+2}\} = -(id - P_N)\widehat{G}_i^l - \sum_{\nu=1}^{l-d} \{g_i^\nu, W^{l+2-\nu}\} \qquad (i = 1, \ldots, n + q). \tag{11}$$

Suppose that $W^{d+2}, \ldots, W^{l+1}$ are extended to polynomials of z whose coefficients are holomorphic functions of $v \in V$. Since the left-hand side of (11) can be written as

$$\{g_i^0, W^{l+2}\} = \sum_{j=1}^{n+q} \frac{\partial g_i^0}{\partial \omega_j} \{\omega_j, W^{l+2}\},$$

the system (11) can be considered as a system of linear equations for $\{\omega_j, W^{l+2}\}$ $(j = 1,\ldots,n+q)$. Then, by Cramer's formula we have the expression

$$\{\omega_j, W^{l+2}\} = \frac{q_k^l(v,z)}{p(v,z)} \qquad (k = 1,\ldots,n+q\,;\; l = d,\ldots,2d-1), \tag{12}$$

where $q_k^l(v,z), p(v,z)$ are some polynomials of z with holomorphic coefficients in $v \in V$ and q_k^l is divisible by p for each $v \in V_0$ fixed. Then one can see that $q_k^l(v,z)$ is divisible by p for all $v \in V$ (see [5, p.392]). Therefore $\{\omega_j, W^{l+2}\}$ and hence W^{l+2} are extended to polynomials with holomorphic coefficients in $v \in V$. By induction argument, we see that W is a polynomial of z with holomorphic coefficients in $v \in V$. Therefore φ is defined for all $v \in V$ and $\varphi^{-1}\circ f \circ \varphi$ is in Λ-normal form up to order $2d$ for all $v \in V$.

By repeating this iteration procedure, we can find a formal symplectic transformation φ which takes f into Λ-normal form up to infinite order. Convergence of the transformation φ can be proved by the technique of KAM theory. The small divisor difficulty can be also avoided by using the division formula (12).

The existence of $n+q$ integrals of f as above implies also that the Λ-normal form in Proposition 5 has the special form. Suppose that f is in Λ-normal form up to order $d = 2^\nu$. Since Proposition 1 holds also for parametrized map case, $f(v,\dot{})$ can be written in the form

$$\begin{cases} f(v,\cdot) = \Lambda(v)\circ \exp X_{H_1}\circ \exp X_{H_2}\circ \cdots \circ \exp X_{H_\nu}\circ \psi(v,\cdot), \\ H_k = H_k^{d+2} + \cdots + H_k^{2d+1} \quad (d = 2^{k-1}, \;\; k = 1,\ldots,\nu), \qquad \psi(v,z) = z + O(|z|^{2^\nu+1}), \end{cases}$$

where $H_k^l = H_k^l(v,z)$ are homogeneous polynomials of degree l in z with coefficients being holomorphic in $v \in V$ and satisfy $H_k^l(v,\Lambda z) = H_k^l(v,z)$. We note that any integral $G_i(v,\cdot)$ of f is in Λ-normal form (hence also $\Lambda(v)$-invariant) up to oder $s_i + d - 1$. Suppose that $H_1,\ldots,H_{k-1}$ are polynomials of $\tau_{q+1},\ldots,\tau_n$ with holomorphic coefficients in $v \in V$. Then we have

$$\exp X_{H_1}\circ \exp X_{H_2}\circ \cdots \circ \exp X_{H_{k-1}} = \exp X_{h_{k-1}}, \qquad h_{k-1} = H_1 + \cdots + H_{k-1}.$$

Moreover we have $g_i\circ \exp X_{h_{k-1}} = g_i$, where g_i is the normal form part of G_i given in (10). Then we can derive from the identity $G_i\circ f = G_i$ that

$$\{G_i^0, H_k^{d+2}\} = 0 \qquad (i = 1,\ldots,n+q).$$

Since G_i^0 are functions of $\omega_1,\ldots,\omega_{n+q}$, we have

$$\{G_i^0, H_k^{d+2}\} = \sum_{j=1}^{n+q} \frac{\partial G_i^0}{\partial \omega_j}\{\omega_j, H_k^{d+2}\} = 0.$$

Here $G_1^0,\ldots,G_{n+q}^0$ are functionally independent and hence $\det\left(\partial G_i^0/\partial\omega_j\right) \not\equiv 0$ for any $v \in V$ fixed. Therefore the above identities are equivalent to

$$\{\omega_j, H_k^{d+2}\} = 0 \quad (j = 1,\ldots,n+q).$$

This implies that H_k^{d+2} is a polynomial of $n-q$ variables $\tau_{q+1},\ldots,\tau_n$ only (see [6]). We can continue this argument to show that this holds also for $H_k^{d+3},\ldots,H_k^{2d+1}$ and conclude that H_k is a polynomial of $n-q$ variables $\tau_{q+1}, \dots, \tau_n$ only. By induction, this leads to the proof that $\varphi^{-1} \circ f \circ \varphi$ has the special form (6).

It remains to show that the linear part $Df(v,0)$ is semi-simple. For this purpose, we use the fact that the square of a linear symplectic map can be written as the time-1 map of a linear autonomous Hamiltonian system (see cf. [8]). We have its parametrized version as follows.

Lemma 4. *Let $f \in \mathrm{Diff}_V(\mathbf{C}^{2n}, 0)$. Then the following holds:*

(i) *If $Df(v,0)$ does not have a negative eigenvalue, then there exists a symmetric matrix $A(v)$ whose elements are holomorphic functions of $v \in V$ such that*

$$Df(v,0) = e^{JA(v)}.$$

(ii) *There exists a symmetric matrix $A(v)$ whose elements are holomorphic functions of $v \in V$ such that*

$$\big(Df(v,0)\big)^2 = \exp X_{H_2} \Big(= e^{JA(v)} \Big), \qquad H_2 = \frac{1}{2} {}^t z A(v) z\,.$$

Suppose that f has $n+q$ integrals $G_i(v,z)$ $(i = 1, \dots, n+q)$. Their lowest order parts $G_i^0(v,z)$ are invariant under the linear map $Df(v,0)$ and hence invariant also under its square $\big(Df(v,0)\big)^2$. Therefore, the functions

$$\widehat{G}_i^0(v,z,t) := G_i^0\big(v, \exp(-tX_{H_2})z\big) \qquad (i = 1, \dots, n+q)$$

are time-dependent integrals of the vector field X_{H_2}. Using this fact, one can prove that $H_2(v,z)$ is a function of $\omega_1, \dots, \omega_n$ only. This implies particularly that the map $\exp X_{H_2}$ and hence $Df(v,0)$ are semi-simple.

(iii) Proofs of Theorem 4 and Theorem 5

If f_1 is in Λ-normal form, the commuting relation $f_1 \circ f_j = f_j \circ f_1$ implies that f_j is also in Λ-normal form. However, it is not trivial at all that f_j has the special form $f_j = Df_j(v,0) \circ \exp X_{h_j(v,\cdot)}$. To prove it, we carry out further normalization of $f_2, \dots, f_k$ so that f_j are also taken into Λ_j-normal form, where $\Lambda_j = Df_j(v,0)$ for some $v \in V_0$, V_0 being the dense subset of V described in Proposition 2. It concludes the proof of Theorem 4. Finally, we can prove Theorem 5 by the arguments similar to those of [5].

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Faculty of Engineering
Kanagawa University
Rokkakubashi 3-27-1, Kanagawa-ku, Yokohama, 221-8686
JAPAN