Degenerate Elliptic Boundary Value Problems with Non-smooth Coefficients and Applications

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Abstract. We review an approach to boundary value problems for strongly elliptic second order operators with a degenerate boundary condition on smooth manifolds with boundary of bounded geometry via (a slight extension of) Boutet de Monvel's calculus. We establish solvability and the existence of a bounded H^∞-calculus. As an application, we derive the short time solvability of the porous medium equation. Moreover, we sketch how this circle of ideas might be applied to a free boundary value problem modeling the melting of ice with a boundary condition between the Gibb-Thomson condition and kinetic undercooling.

Key words: Boundary value problem, degenerate boundary condition, H^∞-calculus, porous medium equation, Stefan problem
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1 Introduction and Presentation of the Results

In the sequel, (X, g) will denote a smooth n-dimensional Riemannian manifold with boundary of bounded geometry, smoothly embedded in a manifold $(\tilde{X}, \tilde{g})$ without boundary of the same dimension and bounded geometry with $\tilde{g}_{|X} = g$. See Section 2 for more details. For example, we could choose $\tilde{X} = \mathbb{R}^n$ with the euclidean metric and $X = \overline{\mathbb{R}}^n_+$, the upper half space $\{x_n \geq 0\}$, we could take for X the closure of a bounded domain with smooth boundary in $\tilde{X} = \mathbb{R}^n$ or (X, g) a smooth compact Riemannian manifold with boundary in its double $(\tilde{X}, \tilde{g})$.

On $\tilde{X}$ we consider an elliptic second order differential operator A that, in local coordinates, is of the form

$$A = A(x, D) = \sum_{j,k=1}^{n} a_{jk}(x) D_j D_k + \sum_{j=1}^{n} b_j(x) D_j + c(x).$$

Here, $D_j = -i\partial_{x_j}$, the coefficients a_{jk} belong to the Hölder space C^τ for some $\tau > 0$ and the b_j and c are L^∞ functions. We assume that there exists a sector Σ_{θ_0}, $0 \leq \theta_0 < \pi$, in the complex plane,

$$\Sigma_{\theta_0} = \{z \in \mathbb{C} : z = re^{i\varphi}, r \geq 0, |\varphi| \leq \theta_0\}, \tag{1.1}$$

and a constant $c_A > 0$ such that, for all $(x, \xi) \in T^*X$,

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \in \Sigma_{\theta_0} \text{ with } \Big| \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \Big| \geq c_A|\xi|^2 > 0. \tag{1.2}$$

In the first part of this article we will endow the operator A with the boundary operator T given by

$$T = \varphi_0\gamma_0 + \varphi_1\gamma_1.$$

Here, γ_0 denotes the evaluation of a function at the boundary and γ_1 that of its exterior normal derivative. Moreover, φ_0 and φ_1 are two non-negative functions satisfying

$$\varphi_0 + \varphi_1 \geq c_\varphi > 0.$$

We will assume that either

$$\varphi_0, \varphi_1 \in C_b^\infty(\partial X)$$

($C_b^\infty(\partial X)$ is the space of all smooth functions on ∂X that are bounded together with all their derivatives) or that

$$\varphi_0 = 1 \text{ and } \varphi_1 = \varphi^2 \text{ for some } \varphi \in C^{2+\tau}(\partial X), \tau > 0.$$

For the choice $\varphi_0 = 1$ and $\varphi_1 = 0$ we obtain the Dirichlet problem, for $\varphi_0 = 0$ and $\varphi_1 = 1$ the Neumann problems. For φ_1 bounded away from zero, we have a Robin problem. All these are elliptic boundary problems and well-studied, at least for the case when X is compact. The Laplacian with Dirichlet boundary conditions on manifolds with boundary and bounded geometry has been treated only some years ago by Ammann, Große and Nistor [3].

However, when φ_1 is not zero but vanishes on some part of ∂X, then the order of the boundary condition varies between zero and one, and thus A with the boundary condition T is not elliptic in the sense of Lopatinski and Shapiro and cannot be treated by standard methods. This is the case we are mainly interested in.

Under the stronger assumption that X is compact and that the coefficients of A and the functions φ_0 and φ_1 are smooth, this problem has been studied - among others - by Egorov and Kondratev [9], Kannai [14] (also in the context of the degenerate oblique derivative problem), and, quite extensively, by Taira starting with [26]; see e.g. [27] and the references therein for subsequent developments. An important point here is to show that the C^∞ regularity assumptions can be relaxed significantly.

The boundary condition T can be viewed as a 'smoothed' version of the Zaremba boundary condition. Indeed, suppose that

$$\partial X = Y_0 \sqcup Z \sqcup Y_1$$

consists of two open subsets Y_0 and Y_1 and their common boundary Z, which is assumed to be a smooth submanifold of ∂X. For the Zaremba problem one imposes Dirichlet conditions on Y_0 and Neumann conditions on Y_1. This corresponds to taking for φ_0 and φ_1 the characteristic functions of Y_0 and Y_1, respectively. The two problems are, however, quite different in spirit. As analyzed by Seeley in [23], the Zaremba problem is basically an edge problem, where Z is the edge and therefore has to be treated by corresponding methods, see e.g. the approach by Dines, Harutyunyan and Schulze in [7].

Let us now look at the results.

Bounded H^∞-calculus and Applications: We will start by studying the L^p-realization A_T of the operator A with the boundary condition T, that is the operator acting like A on the domain

$$\mathscr{D}(A_T) = \{u \in H^2_p(X) : Tu = 0\},$$

and its resolvent. For the definition of $H^2_p(X)$ we refer to Große and Schneider [11].

We recall two important notions: sectoriality and bounded H^∞-calculus. For more information on these subjects see e.g. Denk, Hieber and Prüss' monograph [6].

Definition 1.1. *A closed, densely defined linear operator $B : \mathscr{D}(B) = E_1 \subseteq E_0 \to E_0$ in a Banach space E_0 that is injective with dense range is called sectorial of type $\omega < \pi$, if for every $\omega < \theta < \pi$*

$$\sigma(B) \subset \Sigma_\theta \text{ and } \|\lambda(B-\lambda)^{-1}\|_{\mathscr{L}(E_0)} \le C_\theta \text{ for all } \lambda \in \mathbb{C}\backslash\Sigma_\theta.$$

Here $\Sigma_\theta = \{\lambda = re^{i\varphi} \in \mathbb{C} : r \ge 0, |\varphi| \le \theta\}$ is the sector of angle θ about the positive real axis.

To the sector Σ_θ in Definition 1.1 we associate the space $H^\infty(\Sigma_\theta)$ of all bounded holomorphic functions in the interior of the sector Σ_θ and the subspace $H^\infty_*(\Sigma_\theta)$ of all functions f with $|f(\lambda)| \le C(|\lambda|^\epsilon + |\lambda|^{-\varepsilon})^{-1}$ for suitable $C, \varepsilon > 0$. This is a dense subspace with respect to the topology of uniform convergence on compact sets.

Given a sectorial operator B of type ω, $\theta' \in \,]\omega, \theta[$ and $f \in H^\infty_*(\Lambda_\theta)$ let

$$f(B) = \frac{i}{2\pi}\int_{\partial\Sigma_{\theta'}} f(\lambda)(B-\lambda)^{-1}\,d\lambda \in \mathscr{L}(E_0). \tag{1.3}$$

The integral exists in view of the sectoriality of B and the decay of the functions in $H^\infty_*(\Sigma_\theta)$. By Cauchy's integral theorem it is independent of the choice of θ'.

Given $f \in H^\infty(\Sigma_\theta)$, we can approximate f by a sequence $(f_n) \subset H^\infty_*(\Sigma_\theta)$ and define

$$f(B)x := \lim f_n(B)x \text{ for } x \in \mathscr{D}(B) \cap \operatorname{im}(B).$$

It can be shown that $\mathscr{D}(B) \cap \operatorname{im}(B)$ is dense in E_0 and that the above equation defines a closable operator. The closure is again denoted by $f(B)$.

Definition 1.2. *We say that a sectorial operator B of type ω admits a bounded H^∞ calculus of angle ω, if for any $\omega < \theta < \pi$ there exists a constant $C_\theta > 0$, such that*

$$\|f(B)\|_{\mathcal{L}(E)} \le C_\theta \|f\|_\infty, \quad f \in H^\infty(\Sigma_\theta). \tag{1.4}$$

The principle of uniform boundedness implies that it is sufficient to verify estimate (1.4) for all $f \in H^\infty_*(\Sigma_\theta)$.

The notion of bounded H^∞-calculus goes back to McIntosh [20]. It has become an indispensable tool in the modern theory of evolution equations.

Theorem 1.4 in [18] states:

Theorem 1.1. *For every $0 \le \theta_0 < \theta < \pi$ a constant $\nu \ge 0$ exists such that $A_T + \nu$ has a bounded H^∞-calculus in $L_p(X)$.*

Here θ_0 is opening angle in (1.1). The constant ν depends on the norms of the coefficients of A in the respective spaces and the seminorms of φ_0 and φ_1 in $C_b^\infty(\partial X)$ or, in case $\varphi_0 = 1$, on the norm of φ_1 in $C^{2+\tau}(\partial X)$ as well as the lower bounds c_A and c_φ.

Corollary 1.2. *Theorem 1.1 implies the invertibility of $A_T + \lambda$ for $\lambda \in \mathbb{C} \setminus \Sigma_\theta$, $\theta > \theta_0$, when $|\lambda|$ is large and hence the unique solvability of the semihomogeneous boundary value problem*

$$\begin{aligned} (A-\lambda)u &= f \text{ in } X \\ Tu &= 0 \text{ on } \partial X \end{aligned}$$

for $f \in L^p(\partial X)$ by a suitable element $u \in H_p^2(X)$.

One is also interested in the fully inhomogeneous problem. We shall see that its unique solvability can be derived rather easily from Theorem 1.1. Since the problem is not elliptic, we will need a special space for the boundary values. The L^p-Besov space $B_p^s(\partial X) := B_{p,p}^s(\partial X)$, $s \in \mathbb{R}$, $1 < p < \infty$ has been introduced in [11]. Moreover, [11, Theorem 4.10] states that, as in the case of compact manifolds with boundary, $B_p^{s-1/p}(\partial X)$ is the space of restrictions of functions in $H_p^s(X)$, $s > 1/p$.

Definition 1.3. *With the boundary condition T (and hence the functions φ_0 and φ_1) we associate the space*

$$B_{p,T}^{s-1-1/p}(\partial X) = \{v = \varphi_0 v_0 + \varphi_1 v_1 : v_0 \in B_p^{s-1/p}(\partial X), v_1 \in B_p^{s-1-1/p}(\partial X)\}$$

for $s > 1 - 1/p$ and $1 < p < \infty$.

The following theorem is well known for compact manifolds with boundary. In the case of manifolds with boundary of bounded geometry, it can be shown by modifying the proof of [11, Theorem 4.10] in the spirit of the proof of Theorem 2.9.2 in Triebel [28].

Theorem 1.3. *Given $s > 1 + 1/p$, $v_0 \in B_p^{s-1/p}(\partial X)$ and $v_1 \in B_p^{s-1-1/p}(\partial X)$, there exists $u \in H_p^s(\tilde{X})$ such that $\gamma_0 u = v_0$ and $\gamma_1 u = v_1$.*

From this we infer:

Theorem 1.4. *For $\varphi_0, \varphi_1 \in C_b^\infty(\partial X)$, the boundary map*

$$T : H_p^s(X) \to B_{p,T}^{s-1-1/p}(\partial X)$$

is continuous and surjective for all $s > 1 + 1/p$ and $1 < p < \infty$. For $\varphi_0 = 1$ and $\varphi_1 \in C^{2+\tau}(\partial X)$, $\tau > 0$, the same holds provided $1 + 1/p < s < 2 + \tau$.

Proof. Continuity follows from [11, Theorem 4.10]. Let $v = \varphi_0 v_0 + \varphi_1 v_1 \in B_{p,T}^{s-1-1/p_p}(\partial X)$ with $v_0 \in B_p^{s-1/p}(\partial X), v_1 \in B_p^{s-1-1/p}(\partial X)$. According to Theorem 1.4 we find $u_0, u_1 \in H_p^s(X)$ with $\gamma_0 u_0 = v_0$, $\gamma_1 u_0 = 0$ and $\gamma_0 u_1 = 0$, $\gamma_1 u_1 = v_1$. Then $T(u_0 + u_1) = v$. □

With this at hand, it is easy to prove the unique solvability of the fully inhomogeneous problem.

Theorem 1.5. *For every $\theta_0 < \theta < \pi$ the operator*

$$\begin{pmatrix} A-\lambda \\ T \end{pmatrix} : H^2_p(X) \longrightarrow \begin{matrix} L_p(X) \\ \oplus \\ B^{1-1/p}_{p,T}(\partial X) \end{matrix} \tag{1.5}$$

is a topological isomorphism for $\lambda \in \Sigma_\theta$, $|\lambda|$ sufficiently large.

Proof. Given $f \in L_p(X)$ and $v \in B^{1-1/p}_{p,T}(\partial X)$, we first fix $w_0 \in H^2_p(X)$ with $Tw_0 = v$. By Theorem 1.1, the problem $(A-\lambda)w = f - (A-\lambda)w_0$, $Tu = 0$ has a unique solution $w \in H^2_p(X)$. Then $u = w + w_0$ is the (unique) solution to $(A-\lambda)u = f$, $Tu = v$. Hence (1.5) is a bijection. As it is continuous, it is a topological isomorphism in view of the closed graph theorem. □

Theorem 1.1 is a very useful result: According to a theorem by Dore and Venni [8], the existence of a bounded H^∞-calculus for an operator $B : \mathscr{D}(B) = E_1 \subseteq E_0 \to E_0$ in a sector Σ_θ with $0 < \theta < \pi/2$ implies maximal L^q-regularity, $1 < q < \infty$, of B in the associated evolution equation, i.e., for $f \in L^q([0,T]), E_0)$ and u_0 in the real interpolation space $(E_1, E_0)_{1/q,q}$, the initial value problem

$$\partial_t u + Bu = f, \quad u(0) = u_0$$

has a unique solution

$$u \in L^q((0,T), E_1) \cap W^1_q((0,T), E_0)$$

depending continuously on f and u_0.

This can be used to deduce the short time existence of quasilinear parabolic equations of the form

$$\dot{u} + A(u)u = g(t,u) + h(t),\ 0 < t < t_0, \quad u|_{t=0} = u_0, t \in \tag{1.6}$$

as a consequence of the following theorem of Clément and Li [5]:

Theorem 1.6. *Let $E_1 \hookrightarrow E_0$ be Banach spaces, $1 < q < \infty$. (1.6) has a short-time solution on an interval $(0,t^*)$ with $0 < t^* \le t_0$:*

$$u \in L^q((0,t^*), E_1) \cap W^1_q((0,t^*), E_0),$$

provided $A(u_0)$ has maximal regularity and there exists a neighborhood U of u_0 in $E_{1-1/q} = (E_1, E_0)_{1/q,q}$ with

(H1) $u \mapsto A(u) \in \mathrm{Lip}(U; \mathscr{L}(X_1, X_0))$,

(H2) $(t,u) \mapsto g(t,u) \in \mathrm{Lip}([0,t_0] \times U, E_0)$, *and*

(H3) $h(t) \in L^q((0,t_0), X_0)$

In that case, $u \in C([0,t^*], E_{1-1/q})$.

In fact, the following variant of this theorem is more useful, since it covers additional time dependence of A. It goes back to Prüss; in the present, slightly extended variant, it is Theorem 6.1 in [18]. With the above notation consider

$$\dot{u}(t) + A(t,u(t))u(t) = F(t,u(t)), \ \ t \in \,]0,t_0[,\ u(0) = 0. \tag{1.7}$$

Theorem 1.7. *Assume $A : [0, t_0] \times E_q \to \mathcal{L}(E_1, E_0)$ is continuous, $A_0 = A(0, u_0)$ has maximal L^q-regularity, $F(\cdot, u)$ is measurable for each $u \in E_q$, $F(t, \cdot)$ is continuous for a.a. $t \in [0, t_0]$, and $f(\cdot) := F(\cdot, 0) \in L^q([0, t_0]; E_0)$. Moreover, suppose that there is an $R^* > 0$ such that*

(A) *For each $R \in (0, R^*)$ there is a constant $C = C(R)$ such that*

$$\|A(t, u)v - A(t, \bar{u})v\|_{E_0} \leq C\|u - \bar{u}\|_{E_q}\|v\|_{E_1}, \quad t \in [0, t_0], \ u, \bar{u} \in B(0, R), \ v \in E_1.$$

(F) *For each $R \in (0, R^*)$ there is a function $\psi_R \in L^q(J_0)$ such that*

$$\|F(t, u) - F(t, \bar{u})\|_{E_0} \leq \psi_R(t)\|u - \bar{u}\|_{E_q}, \quad \text{a.a. } t \in [0, t_0], \ u, \bar{u} \in B(0, R).$$

Then there exists a $t^* > 0$ such that (1.7) has a unique solution u in $W_q^1((0, t^*); E_0) \cap L^q((0, t^*); E_1)$.

In addition to the above assumptions on A suppose that the matrix $(a_{jk})_{j,k=1,\ldots,n}$ formed by the coefficients of the principal part of A is positive definite – this is the case e.g. when A is the Laplace-Beltrami operator $\Delta_{\tilde{g}}$ with respect to the metric $\tilde{g}$ on $\tilde{X}$. Then we can choose $\theta_0 = 0$ in (1.1). Combining Theorem 1.1 and the cited theorem of Dore and Venni, we see that $\Delta_{\tilde{g},T}$, the realization of $\Delta_{\tilde{g}}$ with respect to T has maximal regularity.

We will consider the porous medium equation with the boundary condition T. Recall that the porous medium equation

$$\partial_t u - \Delta_{\tilde{g}} u^m = 0$$

with $m > 0$, is a nonlinear generalization of the heat equation. We obtain the following result:

Theorem 1.8. *Let $1 < p, q < \infty$, $n/p + 2/q < 1$, $m > 0$, $v_0 \in H_p^2(X)$ with $v_0 \geq c > 0$, and $\phi \in C^1(J_0; B_{p,T}^{1-1/p}(\partial X))$ satisfying the compatibility condition $\phi(0) = Tv_0$. Here $J_0 = [0, t_0]$ with $t_0 > 0$. Then the system*

$$\begin{cases} \dot{v} - \Delta_{\tilde{g}} v^m = 0 \\ Tv = \phi \\ v|_{t=0} = v_0 \end{cases} \tag{1.8}$$

has a unique short time solution of maximal regularity, i.e. there exists an interval $J = [0, t^*]$ with $t^* > 0$ and a unique solution

$$v \in L^q(J; H_p^2(X)) \cap W_q^1(J; L_p(X)).$$

A Quasi-stationary Stefan Problem. As it turns out, the boundary value problem (Δ, T) also appears in a different context. The following project arose in discussions with Joachim Escher (Hannover). It is a quasi-stationary Stefan problem modeling the melting/solidification process of ice in water of constant temperature zero (one therefore often speaks of a one phase problem, even though there are actually two phases).

Considering t as a time parameter, denote by $X(t)$ the volume filled by the ice at time t. Here we can assume $X(t)$ to be the closure of a bounded domain in $\mathbb{R}^n$. The evolution of $X(t)$ in time is determined by the temperature $u(x,t)$ in x at time t, the exterior normal derivative $\partial_\nu u(x,t)$ at the boundary, the normal velocity $V(x,t)$ of $\partial X(t)$ and the mean curvature $\kappa(x,t)$ of $\partial X(t)$ in x. The process is modeled by the system of equations

$$\Delta u = 0 \quad \text{in } X(t) \tag{1.9}$$
$$V + \partial_\nu u = 0 \quad \text{on } \partial X(t) \tag{1.10}$$
$$\mu V + \kappa = u \quad \text{on } \partial X(t) \tag{1.11}$$
$$X(0) = X_0 \quad \text{at } t = 0. \tag{1.12}$$

The first equation is derived from the heat equation, assuming stationarity, i.e. $\partial_t u = 0$. The second states that the velocity with which the ice expands in the normal direction is given by $-\partial_\nu u$, as one would expect, and the final equation fixes the initial form of the volume covered by the ice.

The interesting equation is the third. From a naive point of view, one might ask that $u_{|\partial X} = 0$, as the surrounding temperature is zero. However, this immediately implies that $u \equiv 0$; hence this boundary condition does not model the observed natural phenomenon. One standard assumption, motivated by physical considerations at the interface, is the Gibbs-Thomson condition

$$u = \sigma\kappa \text{ on } \partial X(t),$$

where $\sigma > 0$ is the surface tension coefficient. Another is the condition of kinetic undercooling:

$$u = \alpha V$$

with a positive constant α.

Here we will consider a mixture of the two cases with a function $\mu \geq 0$ that may vanish on some parts of the boundary. The boundary condition $V + \kappa = u$ was treated by Kneisel [15]. For the case where $\mu V + \kappa = u$ with μ smooth and bounded away from zero an attempt was made by Lukarevski [19]. For simplicity we assume that $\mu \in C_b^\infty(\mathbb{R}^n)$ is a nonnegative function with a (possibly) non-empty zero set. We next partly follow a strategy used by Escher and Simonett [10] and Kneisel in [15].

We choose a smoothly bounded domain D and suppose that there exists a smooth vector field $\mathscr{V}$ that is nowhere tangent to ∂D and outward pointing with respect to D. The flow F of $\mathscr{V}$ therefore provides a diffeomorphism from $]-a, a[\, \times \partial D$, $a > 0$, to a neighborhood $\mathcal{R}$ of ∂D which we assume to contain ∂X_0. We write $F(s; y)$ for the point x reached by the flow line starting in $y \in \partial D$ after 'time' $s \in\,]-a, a[$. Conversely, we call (say) the flow coordinates of x.

Let

$$\mathcal{M} = \{\rho \in C^\tau(\partial D) : \|\rho\|_\infty < b\} \tag{1.13}$$

for some $b < a$ and $\tau > 2$. We assume that ∂X_0 is a graph over ∂D; i.e. there exists a function ρ_0 in $\mathcal{M}$ such that $\partial X_0 = \{F(\rho_0(x), x) : x \in \partial D\}$.

More generally, for ρ in $\mathcal{M}$, the image of the map $\theta_\rho : \partial D \to \mathcal{R}$, $x \mapsto F(\rho(x), x)$ is a hypersurface in $\mathbb{R}^n$. Using the so-called Hanzawa transform we will now extend θ_ρ to a C^τ-diffeomorphism $\Theta_\rho : \mathbb{R}^n \to \mathbb{R}^n$. To this end we note that we may assume without loss of generality that $0 < b < a/4$. We next choose $\phi \in C^\infty(\mathbb{R}, [0, 1])$ with

$$\begin{aligned} \phi(r) &= 1;\ |r| < b \\ \phi(r) &= 0; |r| \geq 3b \\ |\phi'(r)| &\leq 1/b. \end{aligned}$$

We then define, using the flow coordinates (y, s) of $x \in \mathcal{R}$,

$$\Theta_\rho(x) = \begin{cases} F(s + \phi(s)\rho(y); y); & x \in \mathcal{R} \\ x; & x \notin \mathcal{R}. \end{cases}$$

Going even further, consider functions $\rho(t)$ depending on a variable t in an interval $J \subset \mathbb{R}$, say $\rho \in C^1(J, C(\partial D)) \cap C(J, \mathcal{M})$. For every $\rho(t)$ we obtain a diffeomorphism $\Theta_{\rho(t)}$. We let $X_{\rho(t)} = \Theta_{\rho(t)}(D)$. Our aim now is to find $X(t)$ as $X_{\rho(t)}$ for a suitable function ρ on ∂D.

To this end we transform the system under the diffeomorphism $\Theta_{\rho(t))}$. A computation leads to the equations

$$\begin{aligned} A(\rho)v &= 0 \quad \text{in } J \times D && (1.14) \\ \partial_t \rho + L_\rho D_\partial v &= 0 \quad \text{on } J \times \partial D && (1.15) \\ \gamma_0 v + \gamma_0(\mu D_\partial v) &= H(\rho) \quad \text{on } J \times \partial D && (1.16) \\ \rho(0) &= \rho_0 \quad \text{on } \partial D. && (1.17) \end{aligned}$$

Here, $A(\rho)$ is a second order strongly elliptic differential operator. If $\rho(t)$ is of regularity C^τ, then the top order coefficients of A are $C^{\tau-1}$, the lower oder coefficients are $C^{\tau-2}$. Moreover, D_∂ and $H(\rho)$ are the expressions for the normal derivative and the mean curvature, respectively, in the new coordinates, and L_ρ is a strictly positive function.

We observe that equations (1.14) and (1.16) furnish a boundary value problem with a degenerate boundary condition similar to that studied above. The difference is that instead of the normal derivative we here have the non-tangential derivative D_∂.

The strategy now is to first solve this boundary value problem in dependence of ρ. Inserting the result into (1.15) we obtain an evolution equation for ρ with initial value given by (1.17). In fact, this evolution equation is quasilinear. To see this, we decompose

$$H(\rho) = P(\rho) + Q(\rho) \qquad (1.18)$$

into a second order strongly elliptic differential operator $P(\rho)$ and a remainder $Q(\rho)$, see e.g. [10, Lemma 3.1] for details. The goal then is to establish the existence of a short time solution to this quasilinear equation using continuous maximal regularity and a result of Simonett [25, Theorem 3.1]. For more details see below.

2 Background and Ideas for the Proofs

2.1 Manifolds with Boundary and Bounded Geometry

The notion of manifolds with boundary and bounded geometry goes back to Schick [24]; it was taken up by Ammann, Große and Nistor [3]. We recall the important ideas.

Definition 2.1. (a) *A Riemannian manifold* $(\tilde{X}, \tilde{g})$ *is said to have bounded geometry, if it has positive injectivity radius and all covariant derivatives of the curvature* R *are bounded: Denoting by* ∇ *the Levi-Civita connection, one asks that*

$$\|\nabla^k R\|_{L^\infty(\tilde{X})} < \infty.$$

(b) *Let* $(\tilde{X}, \tilde{g})$ *be of bounded geometry and let* Y *be a hypersurface with unit normal vector field* ν.

One identifies the normal bundle of Y *in* $\tilde{X}$ *with* $Y \times \mathbb{R}$ *using* ν. *Hence the second fundamental form of* Y *is simply a smooth family of symmetric bilinear maps* $\mathrm{II}_y : T_yY \times T_yY \to \mathbb{R}$, $y \in Y$. *In particular,* II *defines a smooth tensor.*

Y *is said to be a bounded geometry hypersurface, if the following conditions are fulfilled:*

(i) Y *is a closed subset of* $\tilde{X}$

(ii) $(Y, \tilde{g}_{|Y})$ *is a manifold of bounded geometry*

(iii) *The second fundamental form* II *of* Y *in* $\tilde{X}$ *and all its covariant derivatives along* Y *are bounded, i.e.,*

$$\|(\nabla^Y)^k \mathrm{II}\|_\infty \le C_k \text{ for all } k \in \mathbb{N}_0$$

(iv) *There is a* $\delta > 0$ *such that* $\exp^\perp : Y \times]-\delta, \delta[\to \tilde{X}$ *is injective.*

(c) *A Riemannian manifold* (X, g) *with (smooth) boundary has bounded geometry, if there is a Riemannian manifold* $(\tilde{X}, \tilde{g})$ *with bounded geometry satisfying*

(i) $\dim \tilde{X} = \dim X$

(ii) X *is contained in* $\tilde{X}$, *in the sense that there is an isometric embedding* $(X, g) \to (\tilde{X}, \tilde{g})$

(iii) ∂X *is a bounded geometry hypersurface in* $\tilde{X}$.

2.2 The Parametrix Construction

We first consider the model case, where $X = \overline{\mathbb{R}}^n_+$ and the coefficients of A and the functions φ_0 and φ_1 belong to $C_b^\infty(\overline{\mathbb{R}}^n_+)$. In order to cover the parameter $\lambda \in \Sigma_\theta$ we use Agmon's trick [2] of introducing an additional variable. We write $\lambda = |\lambda| e^{i\phi}$ and consider, instead of the operator $A - \lambda$, the operator

$$A_\phi = A + e^{i\phi} D_z^2,$$

where $z \in \mathbb{R}$ is an artificial additional variable. The symbol of A_ϕ is

$$\sigma(A_\phi)((x, z), (\xi, \zeta)) = \sigma(A)(x, \xi) + e^{i\phi} \zeta^2.$$

Since $|\phi| \le \theta < \theta_0$, the estimates in (1.2) also hold for A_ϕ, uniformly in ϕ. The basic observation is that solving the problem for A_ϕ is equivalent to solving the parameter-dependent problem for A. Note, however, that we have replaced $|\lambda|$ by ζ^2; this has to be taken into account when considering e.g. sectoriality. (In case X is a compact manifold with boundary, one may apply the same trick with $z \in \mathbb{S}^1$, thus avoiding non-compactness of the underlying domain.)

We will next construct a parametrix for A_ϕ relying on (an extension of) Boutet de Monvel's calculus; see [4], the monographs [21] or [12] or the short introduction [22] for details on the (classical version of the) calculus.

The Dirichlet problem $\binom{A_\phi}{\gamma_0}$ for A_ϕ is an elliptic problem in the Boutet de Monvel calculus. Hence there exists a parametrix

$$\begin{pmatrix} A_\phi \\ \gamma_0 \end{pmatrix}^{-\#} = \begin{pmatrix} (A_\phi^{-\#})_+ + G_\phi^D & K_\phi^D \end{pmatrix}.$$

Here, $A_\phi^{-\#}$ is a parametrix to A_ϕ on $\mathbb{R}^n$, and $(A_\phi^{-\#})_+$ denotes its truncation to $\mathbb{R}^n_+$, i.e. the operator

$$(A_\phi^{-\#})_+ = r^+ A_\phi^{-\#} e^+,$$

where e^+ denotes the extension by zero of functions on $\mathbb{R}^n_+$ to functions on $\mathbb{R}^n$ and r^+ denotes the restriction of functions/distributions on $\mathbb{R}^n$ to $\mathbb{R}^n_+$. Moreover, G_ϕ^D is a singular Green operator and K_ϕ^D is a Poisson type operator.

Since $\gamma_0 K_\phi^D \sim I$, where '$\sim$' here and in the following considerations denotes equality up to smoothing operators, we obtain

$$TK_\phi^D = (\varphi_0\gamma_0 + \varphi_1\gamma_1)K_\phi^D = \varphi_0 + \varphi_1\Pi_\phi =: S_\phi,$$

where $\Pi_\phi = \gamma_1 K_\phi^D$ is the Dirichlet-to-Neumann operator for $\binom{A_\phi}{\gamma_0}$. Hence

$$\begin{pmatrix} A_\phi \\ T \end{pmatrix} \begin{pmatrix} (A_\phi^{-\#})_+ + G_\phi^D & K_\phi^D \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ T((A_\phi^{-\#})_+ + G_\phi^D) & S_\phi \end{pmatrix}.$$

Supposing that we can construct a parametrix $S_\phi^{-\#}$ to S_ϕ, we can find a parametrix to the triangular matrix on the right and therefore a parametrix $\binom{A_\phi}{T}^{-\#}$ to $\binom{A_\phi}{T}$, namely

$$\begin{pmatrix} A_\phi \\ T \end{pmatrix}^{-\#} \sim \begin{pmatrix} (A_\phi^{-\#})_+ + G_\phi^D - K_\phi^D S_\phi^{-\#} T((A_\phi^{-\#})_+ + G_\phi^D) & K_\phi^D S_\phi^{-\#} \end{pmatrix}.$$

In the sequel, we will be interested in a parametrix $(A_\phi)_T^{-\#}$ to the realization $(A_\phi)_T$ of A_ϕ with respect to the boundary condition T. That is given by the left entry in the row matrix on the right hand side. Summing up, we have shown the following result:

Proposition 2.1. *Suppose that the operator S_ϕ has a parametrix $S_\phi^{-\#}$. Then the realization $(A_\phi)_T$ has the parametrix*

$$(A_\phi)_T^{-\#} \sim (A_\phi^{-\#})_+ + G_\phi^D - K_\phi^D S_\phi^{-\#} T((A_\phi^{-\#})_+ + G_\phi^D)$$

Unless $\varphi_1 \equiv 0$ or $\varphi_1(x) \neq 0$ for all x, the operator S_ϕ will not be elliptic. However, as has been observed early on in the study of this problem:

Lemma 2.2. *S_ϕ has a parametrix $S_\phi^{-\#}$ with a symbol in the Hörmander class $S^0_{1,1/2}$.*

Proof. S_ϕ is a classical element in the Hörmander class $S^1_{1,0}$. Its symbol is

$$\sigma(S_\phi) = \varphi_0 + \varphi_1\sigma(\Pi_\phi).$$

The symbol of Π_ϕ is known to be elliptic with values in the sector Σ_{θ_0}. In view of the fact that $\varphi_0 + \varphi_1 \geq c_\varphi > 0$, the symbol of S_ϕ is bounded away from zero. A short computation shows that its inverse belongs to $S^0_{1,1/2}$. Then a standard parametrix construction completes the argument. □

While the term $(A_\phi^{-\#})_+ + G_\phi^D$ in the parametrix of $(A_\phi)_T$ belongs to Boutet de Monvel's calculus, the second term, i.e. $K_\phi^D S_\phi^{-\#} T((A_\phi^{-\#})_+ + G_\phi^D)$ does not unless $\varphi_1 \equiv 0$ or $\varphi_1(x) \geq c > 0$ for all x. We study it more closely and observe that, since $\gamma_0((A_\phi^{-\#})_+ + G_\phi^D) \sim 0$,

$$K_\phi^D S_\phi^{-\#} T((A_\phi^{-\#})_+ + G_\phi^D) \sim K_\phi^D S_\phi^{-\#} \varphi_1 \gamma_1((A_\phi^{-\#})_+ + G_\phi^D).$$

Here K_ϕ^D and $\gamma_1((A_\phi^{-\#})_+ + G_\phi^D)$ are elements in the standard Boutet de Monvel calculus. Moreover, the composition $S_\phi^{-\#}\varphi_1$ is better than expected, namely:

Lemma 2.3. *$S_\phi^{-\#}\varphi_1$ has a symbol in $S^{-1}_{1,1/2}$.*

In [18] an extension of Boutet de Monvel's calculus has been worked out with symbols modeled on the Hörmander classes of type $(1, \delta)$, $0 \leq \delta < 1$. (A similar calculus was devised by Krainer in the edge calculus setting in [16].) It was then shown:

Theorem 2.4. *The parametrix*

$$(A_\phi)_T^{-\#} \sim (A_\phi^{-\#})_+ + G_\phi^D - K_\phi^D (S_\phi^{-\#}\varphi_1)\gamma_1((A_\phi^{-\#})_+ + G_\phi^D)$$

is an operator of order -2 and class zero in the extended Boutet de Monvel calculus. As a consequence, the symbol seminorms of $(A_\phi)_T^{-\#}$ as a zero order element are $O(\langle\zeta\rangle^{-2})$ as $\zeta \to +\infty$.

2.3 Sectoriality and H^∞-calculus

With this at hand we go back to the original operator A_T in $L^p(\mathbb{R}^n)$. The above parametrix construction implies that

Corollary 2.5.

$$(A_T - \zeta^2 e^{i\phi})(A_\phi)_T^{-\#} = I + R_\phi(\zeta),$$

where $R_\phi(\zeta)$, $\zeta \geq 0$, is a family of smoothing operators for which the operator norms in $\mathscr{L}(L^p(\mathbb{R}^n_+))$ tend to zero rapidly as $\zeta \to \infty$, uniformly in ϕ for $|\phi| \geq \theta$. In particular, there exists a $C \geq 0$ such that $A_T - \zeta^2 e^{i\phi}$ is invertible for $\zeta^2 \geq C$, uniformly in ϕ, and

$$\|A_T - \zeta^2 e^{i\phi}\|_{\mathscr{L}(L^p(\mathbb{R}^n))} = O(\langle\zeta\rangle^{-2}).$$

Hence we obtain the sectoriality of $A_T + \nu$ in Σ_θ for $\nu \geq C$.

Idea for the proof of Theorem 1.1. In order to obtain the existence of a bounded H^∞-calculus, we first consider the case where φ_0 and φ_1 are smooth and A only consists of the leading part and has constant coefficients in $\overline{\mathbb{R}}^n_+$. In order to obtain the required estimate (1.4) for the Dunford integral (1.3), here applied to the case where $B = A_T + \nu$, we note that in (1.3) we can replace the resolvent by a suitable parameter-dependent parametrix. As a consequence of the calculus, all terms in the parametrix - apart from the contribution of the leading symbol - will be $O(\langle\lambda\rangle^{-3/2})$, so that the estimate (1.4) trivially holds. The contribution of the leading symbol, however, can be computed explicitly and the estimate can be checked directly. Hence we obtain the existence of a bounded H^∞-calculus for this situation.

It remains to cover the case, where A only consists of the principal part, but has variable coefficients in C^τ, $\tau > 0$, and finally the general case, where A also has lower order terms and variable coefficients. The existence of a bounded H^∞-calculus then follows from the two perturbation results, below:

Theorem 2.6. *Let A have a bounded H^∞-calculus in the UMD Banach space E and $0 \in \rho(A)$. Suppose that B is a linear operator in E with $\mathscr{D}(B) \supseteq \mathscr{D}(A)$.*

(a) Same order perturbations. *Let $\gamma \in (0,1)$, $C \geq 0$, with*

$$B(\mathscr{D}(A^{1+\gamma})) \subseteq \mathscr{D}(A^\gamma) \text{ and } \|A^\gamma Bu\|_E \leq C\|A^{1+\gamma}u\|_E, \quad u \in \mathscr{D}(A^{1+\gamma}).$$

Then $A + B$ has a bounded H_∞-calculus, provided

$$\|Bu\|_E \leq \varepsilon\|Au\|_E, \quad u \in \mathscr{D}(A)$$

for suitably small $\varepsilon > 0$.

(b) Lower order perturbations. *Assume $\gamma \in (0,1)$, $C \geq 0$ with*

$$\|Bu\|_E \leq C\|A^{1-\gamma}u\|_E, \quad u \in \mathscr{D}(A).$$

Then $\nu + A + B$ has a bounded H^∞-calculus for ν sufficiently large.

Finally, to treat the case where X is a manifold with boundary and bounded geometry, we use a patching procedure, see [18, Section 4.3] for details. This concludes the argument for the case where φ_0 and φ_1 are smooth.

The case where $\varphi_0 = 1$ and $\varphi_1 = \varphi^2$ for some $\varphi \in C^{2+\tau}(X)$ requires a more elaborate parametrix construction for the operator $S_\phi\phi_1$ above, using symbol smoothing and [1]. Details can be found in [18, Section 5]. □

2.4 Application to the Porous Medium Equation

For the proof of Theorem 1.8 we start with the following observation, see [18, Lemma 6.2]:

Lemma 2.7. *Given $\phi \in C^1(J_0; B^{1-1/p}_{p,T}(\partial X))$ and $v_0 \in H^2_p(X)$ there exists a $w \in C^1(J_0; H^2_p(X))$, such that $Tw = \phi$ and $w(0) = v_0$.*

Letting $u := v - w$ we consider the problem:

$$\begin{aligned}\dot{u}(t) - \Delta_g(u(t)+w(t))^m &= -\dot{w}(t)\\ Tu(t) &= 0\\ u(0) &= 0.\end{aligned}$$

Obviously, v solves (1.8) if and only if u solves this system. Now

$$\begin{aligned}\Delta_g(u+w)^m =& m(u+w)^{m-1}\Delta_g u\\ &+ m(m-1)(u+w)^{m-2}|\nabla(u+w)|_g^2 + m((u+w))^{m-1}\Delta_g w.\end{aligned}$$

Let

$$\begin{aligned}A(t,u(t)) &= -m(u(t)+w(t))^{m-1}\Delta_g \text{ and}\\ F(t,u(t)) &= m(m-1)(u(t)+w(t))^{m-2}|\nabla(u(t)+w(t))|_g^2\\ &\quad + m((u(t)+w(t)))^{m-1}\Delta_g w(t) - \dot{w}(t).\end{aligned}$$

Then it can be checked that the conditions of Theorem 1.7 are fulfilled, and we obtain the assertion.

2.5 The Stefan Problem

This is joint work in progress with J. Seiler, Turin.

For the analysis of the Stefan problem, we work in Zygmund and little Hölder spaces. Recall that we now assume X to be compact and that the boundary condition is given on ∂D by

$$T = \gamma_0 + \mu\gamma_0 D_\partial \tag{2.1}$$

with D_∂ expressing the normal derivative in the flow coordinates; in particular, D_∂ is non-tangential.

Definition 2.2. *Denote by $C^s_*(D)$ and $C^s_*(\partial D)$ the Zygmund spaces of order $s \in \mathbb{R}$ on D and ∂D, respectively, and by $C^s_{*,T}(\partial D)$ the space*

$$C^s_{*,T}(\partial D) = \{u_0 + \mu u_1 : u_0 \in C^{s+1}_*(\partial D), u_1 \in C^s_*(\partial D)\}.$$

Recall that for $s > 0$, $s \notin \mathbb{N}$, the Zygmund spaces coincide with the usual Hölder spaces and that $C^s \subseteq C^s_*$ for $s \geq 0$.

We have the following analog of Theorem 1.3:

Theorem 2.8. *Given $s > 1$, $v_0 \in C^s_*(\partial D)$ and $v_1 \in C^{s-1}_*(\partial D)$, there exists $u \in C^s_*(D)$ such that $\gamma_0 u = v_0$ and $\gamma_1 u = v_1$.*

Proof. For a compact manifold D with boundary, it is shown in [12, Lemma 1.6.4] that there exists a potential operator in Boutet de Monvel's calculus which is a right inverse to the restriction operator $(\gamma_0, \ldots, \gamma_k) : H^s_2(D) \to \prod_{j=0}^k H_2^{s-j-1/2}(\partial D)$ for $s - k - 1/2 > 0$. According to [13, Theorem 1.1] this operator moreover acts continuously on the Zygmund spaces and therefore provides a right inverse also there. □

Proposition 2.9. *Let $\tau > 2$ be the regularity index in* (1.13) *The boundary condition T in* (2.1) *provides a surjective map*

$$T : C^\tau_*(D) \to C^{\tau-1}_{*,T}(\partial D).$$

Proof. Obviously, T maps $C^\tau_*(D)$ to $C^{\tau-1}_{*,T}(\partial D)$. So suppose we are given $v_0 \in C^\tau_*(\partial D)$ and $v_1 \in C^{\tau-1}_*(\partial D)$. We decompose $D_\partial = D_{tan} + \sigma\partial_\nu$ with a tangential part D_{tan} and a normal part $\sigma\partial_\nu$ with σ nowhere vanishing so that

$$T = \gamma_0 + \mu\gamma_0(D_{tan} + \sigma\partial_\nu).$$

Since we assume that $\rho \in C^\tau(\partial D)$ for $\tau > 2$, we will have $\sigma \in C^{\tau-1}(\partial D)$. We choose u_0 such that $\gamma_0 u_0 = v_0, \gamma_1 u_0 = 0$ and u_1 such that $\gamma_0 u_1 = 0$ and $\gamma_1 u_1 = (v_1 - \gamma_0 D_{tan} u_0)/\sigma$ and let $u = u_0 + u_1$. Then $u \in C^\tau_*(D)$ and, since $u_1 = 0$ on ∂D,

$$\gamma_0 u + \mu\gamma_0(D_{tan}u + \sigma\partial_\nu u) = v_0 + \mu\gamma_0 D_{tan}u_0 + \mu v_1 - \mu\gamma_0 D_{tan}u_0 = v_0 + \mu v_1.$$

Hence u is a preimage to $v_0 + \mu v_1$ under T. □

Corollary 2.10. *The unique solvability of the semi-homogeneous problem $Au = f$, $Tu = 0$ for given $f \in C^{\tau-2}_*(D)$ with $u \in C^\tau_*(D)$ is equivalent to the unique solvability of $Au = 0$, $Tu = g$ for given $g \in C^{\tau-1}_{*,T}(\partial D)$ with $u \in C^\tau_*(D)$.*

Proposition 2.11. *Assume the Dirichlet problem for $A(\rho)$ in* (1.14) *is uniquely solvable, and K^D_ρ is the associated Poisson operator. Then the solution u to the boundary problem $A(\rho)u = 0$, $Tu = g$ for $A(\rho)$ is given by*

$$u = K^D_\rho(I + \mu D_\partial K^D_\rho)^{-1} g.$$

Proof. This follows from the fact that $A(\rho)K^D_\rho$ is zero and $\gamma_0 K^D_\rho = I$ whenever the Dirichlet problem is uniquely solvable. □

Theorem 2.12. *Assume the Dirichlet problem for $A(\rho)$ in* (1.14) *is uniquely solvable, K^D_ρ is the associated Poisson operator, and $P(\rho), Q(\rho)$ are as in* (1.18)*. Then the solution to the system* (1.14)*,* (1.15)*,*(1.16) *and* (1.17) *is given by the solution to the quasilinear evolution equation*

$$\partial_t\rho + C(\rho)\rho = F(\rho), \quad \rho(0) = \rho_0,$$

where $C(\rho) = L_\rho D_\partial K^D_\rho(I + \mu D_\partial K^D_\rho)^{-1}P(\rho)$ and $F(\rho) = -L_\rho D_\partial K^D_\rho(I + \mu D_\partial K^D_\rho)^{-1}Q(\rho)$.

Remark 2.1. We have not specified the spaces in which this is to be solved as they are not determined yet. In fact, this is a rather delicate issue. The goal is to apply continuous maximal regularity on suitable interpolation spaces of little Hölder spaces, using the results of Simonett [25]. These are topics presently under investigation.

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