RIMS Workshop on

Mathematical Analysis of Viscous Incompressible Fluid

Date: December 7 (Mon) – 9 (Wed), 2020
Venue: Online via Zoom
Organizers: Yasunori Maekawa (Kyoto University)
Yoshihiro Shibata (Waseda University)

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Program

<u>Monday, December 7</u>

14:00 - 14:50 Masahiro Suzuki (Nagoya Institute of Technology)
Stationary solutions to the Euler–Poisson equations in a perturbed half-space

15:10 - 16:00 Kai Koike (Kyoto University)
Refined pointwise estimates for the solutions to the one-dimensional barotropic compressible Navier-Stokes equations: An application to the analysis of the long-time behavior of a moving point mass

16:20 - 16:50 Yusuke Ishigaki (Tokyo Institute of Technology)
Diffusion wave phenomena and L^p decay estimates of solutions of compressible viscoelastic system

Tuesday, December 8

10:00 - 10:50　Yasushi Taniuchi (Shinshu University)
On uniqueness of mild solutions on the whole time axis to the Boussinesq equations in unbounded domains

11:10 - 12:00　Takayuki Kubo (Ochanomizu University)
Analysis of non-stationary Navier-Stokes equations approximated by the pressure stabilization method

14:00 - 14:50　Xin Zhang（Tongji University, Shanghai）
The decay property of the multidimensional compressible flow in the exterior domain

15:10 - 16:00　Masashi Aiki (Tokyo University of Science)
On the head-on collision of coaxial vortex rings

16:20 - 16:50　Yuuki Shimizu (Kyoto University)
Current-valued solutions of the Euler-Arnold equation on surfaces and its applications

Wednesday, December 9

10:00 - 10:50　Hideyuki Miura (Tokyo Institute of Technology)
Estimates of the regular set for Navier-Stokes flows in terms of initial data

11:10 - 12:00　Itsuko Hashimoto (Kanazawa University / OCAMI)
Existence of radially symmetric stationary solutions for the compressible Navier-Stokes equation

RIMS

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Abstracts

Stationary solutions to the Euler–Poisson equations in a perturbed half-space

Masahiro Suzuki

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The purpose of this talk is to mathematically investigate the formation of a plasma sheath near the surface of walls immersed in a plasma. The motion of plasma is governed by the Euler–Poisson equations:

$$\rho_t + \nabla\cdot(\rho\boldsymbol{u}) = 0, \quad \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + K\nabla(\log\rho) = \nabla\phi, \quad \Delta\phi = \rho - e^{-\phi}, \tag{1a}$$

where unknown functions ρ, $\boldsymbol{u} = (u_1, u_2, u_3)$, and $-\phi$ represent the density and velocity of the positive ions and the electrostatic potential, respectively. Furthermore, K is a positive constant. We study an initial–boundary value problem of (1a) in a perturbed half-space $\Omega := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 > M(x_2, x_3)\}$ with $M \in \cap_{k=1}^{\infty} H^k(\mathbb{R}^2)$. The initial and boundary data are prescribed as

$$(\rho, \boldsymbol{u})(0, x) = (\rho_0, \boldsymbol{u}_0)(x), \tag{1b}$$

$$\lim_{x_1\to\infty} (\rho, u_1, u_2, u_3, \phi)(t, x_1, x_2, x_3) = (1, u_+, 0, 0, 0), \tag{1c}$$

$$\phi(t, M(x_2, x_3), x_2, x_3) = \phi_b \quad \text{for } (x_2, x_3) \in \mathbb{R}^2, \tag{1d}$$

where $u_+ < 0$ and $\phi_b \in \mathbb{R}$ are constants. The initial data $(\rho_0, \boldsymbol{u}_0)$ are supposed to satisfy

$$\inf_{x\in\Omega} \rho_0(x) > 0, \quad \inf_{x\in\partial\Omega} \frac{\boldsymbol{u}_0(x)\cdot\nabla(M(x_2, x_3) - x_1)}{\sqrt{1 + |\nabla M(x_2, x_3)|^2}} - \sqrt{K} > 0. \tag{2}$$

For the end state u_+, we assume the Bohm criterion and the supersonic outflow condition:

$$u_+^2 > K + 1, \quad u_+ < 0, \tag{3}$$

$$\inf_{x\in\partial\Omega} \frac{-u_+}{\sqrt{1 + |\nabla M(x_2, x_3)|^2}} - \sqrt{K} > 0. \tag{4}$$

The second condition in (2) is necessary for the well-posedness of the problem (1). We remark that (4) is required if solutions to problem (1) are established in a neighborhood of the constant state $(\rho, u_1, u_2, u_3, \phi) = (1, u_+, 0, 0, 0)$.

In the case of planar wall $M = 0$, Bohm proposed a criterion on the velocity of the positive ion for the formation of sheath [1], and several mathematical results validated the Bohm criterion (3) and defined the fact that the sheath corresponds to the stationary solution of (1a). It is of greater interest to analyze the criterion for nonplanar walls. In this talk, we study the existence and stability of stationary solutions of (1) for $M \neq 0$.

To state our main results, let us introduce the existence theorem of stationary solutions $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1)$ in a one-dimensional half-space. The stationary solutions solve

$$(\tilde{\rho}\tilde{u})' = 0, \quad \tilde{u}\tilde{u}' + K(\log\tilde{\rho})' = \tilde{\phi}', \quad \tilde{\phi}'' = \tilde{\rho} - e^{-\tilde{\phi}}, \quad x_1 > 0, \tag{5a}$$

$$\tilde{\phi}(0) = \phi_b, \quad \lim_{x_1\to\infty} (\tilde{\rho}, \tilde{u}, \tilde{\phi})(x_1) = (1, u_+, 0), \quad \inf_{x_1\in\mathbb{R}_+} \tilde{\rho}(x_1) > 0. \tag{5b}$$

Theorem 1 ([2])**.** *Let u_+ satisfy* (3). *There exists a constant $\delta > 0$ such that if $|\phi_b| < \delta$, a unique monotone solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi}) \in C^\infty(\overline{\mathbb{R}_+})$ to* (5) *exists.*

We have constructed stationary solutions $(\rho^s, \boldsymbol{u}^s, \phi^s)$ in the domain Ω by regarding it as a perturbation of $(\tilde{\rho}, \tilde{u}, 0, 0, \tilde{\phi})(\tilde{M}(x))$, where $\tilde{M}(x) := x_1 - M(x_2, x_3)$. Furthermore, we use the weighted Sobolev space $H^k_\alpha(\Omega)$ for $k = 1, 2, 3, \ldots$ and $\alpha > 0$:

$$H^k_\alpha(\Omega) := \left\{ f \in H^k(\Omega) \mid \|f\|^2_{k,\alpha} < \infty \right\}, \quad \|f\|^2_{k,\alpha} := \sum_{j=0}^{k} \int_\Omega e^{\alpha x_1} |\nabla^j f|^2 \, dx.$$

The existence and stability of stationary solutions are summarized in the following theorems. It is worth pointing out that we do not require any smallness assumptions for the function M representing the boundary $\partial\Omega$.

Theorem 2 ([3])**.** *Let $m \geq 3$, and u_+ satisfy* (3) *and* (4). *There exists a positive constant δ such that if $\beta + |\phi_b| \leq \delta$, the problem* (1) *has a unique stationary solution $(\rho^s, \boldsymbol{u}^s, \phi^s)$ as*

$$(\rho^s, u_1^s, u_2^s, u_3^s, \phi^s) - (\tilde{\rho} \circ \tilde{M}, \tilde{u} \circ \tilde{M}, 0, 0, \tilde{\phi} \circ \tilde{M}) \in [H^m_\beta(\Omega)]^4 \times H^{m+1}_\beta(\Omega),$$
$$\|(\rho^s - \tilde{\rho} \circ \tilde{M}, u_1^s - \tilde{u} \circ \tilde{M}, u_2^s, u_3^s)\|^2_{m,\beta} + \|\phi^s - \tilde{\phi} \circ \tilde{M}\|^2_{m+1,\beta} \leq C|\phi_b|,$$

where C is a positive constant independent of ϕ_b.

Theorem 3 ([3])**.** *Let u_+ satisfy* (3) *and* (4). *There exists a positive constant δ such that if $\beta + \|(\rho_0 - \rho^s, \boldsymbol{u}_0 - \boldsymbol{u}^s)\|_{3,\beta} + |\phi_b| \leq \delta$ the problem* (1) *has a unique time-global solution $(\rho, \boldsymbol{u}, \phi)$ in the following space:*

$$(\rho - \rho^s, \boldsymbol{u} - \boldsymbol{u}^s, \phi - \phi^s) \in \left[\bigcap_{i=0}^{1} C^i([0, \infty); H^{3-i}_\beta(\Omega)) \right]^4 \times C([0, \infty); H^4_\beta(\Omega)).$$

Moreover, there holds for certain positive constants C and γ independent of ϕ_b and t,

$$\sup_{x \in \Omega} |(\rho - \rho^s, \boldsymbol{u} - \boldsymbol{u}^s, \phi - \phi^s)(t, x)| \leq Ce^{-\gamma t}, \quad t \in [0, \infty).$$

We can conclude from Theorems 2 and 3 that (3) and (4) guarantee the sheath formation as long as the shape of walls is drawn by a graph.

Acknowledgments. This talk is based on a joint work with Prof. Masahiro Takayama (Keio Univ.).

Reference

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Refined pointwise estimates for the solutions to the one-dimensional barotropic compressible Navier–Stokes equations: An application to the analysis of the long-time behavior of a moving point mass

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The objective of this study is to understand the long-time behavior of a point mass moving inside a one-dimensional viscous compressible fluid. In a previous work [2], we showed that the velocity of the point mass $V(t)$ satisfies a decay estimate $V(t) = O(t^{-3/2})$; in this work, we give a necessary and sufficient condition (under some regularity and smallness assumptions) for the corresponding lower bound $C^{-1}(t+1)^{-3/2} \leq |V(t)|$ to hold for large enough C and t.

The system of equations we consider is, in the Lagrangian mass coordinate, the following:

$$\begin{cases} v_t - u_x = 0, & x \in \mathbb{R}_*,\ t > 0, \\ u_t + p(v)_x = \nu \left(\dfrac{u_x}{v}\right)_x, & x \in \mathbb{R}_*,\ t > 0, \\ u(0_\pm, t) = V(t), & t > 0, \\ mV'(t) = [\![-p(v) + \nu u_x/v]\!](t), & t > 0, \\ V(0) = V_0;\ v(x,0) = v_0(x),\ u(x,0) = u_0(x), & x \in \mathbb{R}_*. \end{cases} \tag{1}$$

Here, $v = v(x,t)$ is the specific volume, $u = u(x,t)$ is the velocity, $p(v)$ is the pressure, and the constant $\nu > 0$ is the viscosity of the fluid. We assume that the fluid is barotropic, so that p is a known function; we assume that p is smooth and satisfies $p'(v) < 0$ and $p''(v) \neq 0$ for $v > 0$. In the Lagrangian mass coordinate, the location of the point mass — whose mass and velocity are denoted by m and $V(t)$ — is always $x = 0$, and $\mathbb{R}_* := \mathbb{R}\backslash\{0\}$ is the domain in which the fluid flows. The double brackets $[\![f]\!](t)$ denote the jump of a function $f = f(x,t)$ at $x = 0$, that is, $[\![f]\!](t) := f(0_+,t) - f(0_-,t)$, where $f(0_\pm,t) = \lim_{x\to\pm 0} f(x,t)$. The first two equations in (1) are the barotropic compressible Navier–Stokes equations written in the Lagrangian mass coordinate, the so-called p-system; the third one is the Dirichlet boundary condition for the first two equations, which just says that the fluid does not penetrate through the point mass; the fourth equation is Newton's second law; the final set of equations are initial conditions. In what follows, we set $m = 1$ for simplicity.

We consider small solutions around a steady state $(v,u,V) = (1,0,0)$. We denote by $||\cdot||_k$ the $H^k(\mathbb{R}_*)$-norm. The following theorem is obtained as a corollary to a theorem on the pointwise estimates of the fluid variables [1, Theorem 1.2]; due to the page limitation, we only state a corollary, which reads as follows [1, Corollary 1.2].[2]

Theorem 1. *Let $v_0 - 1, u_0 \in H^6(\mathbb{R}_*)$, and $V_0 \in \mathbb{R}$. Assume that they satisfy suitable compatibility conditions. Then there exist $\delta_0 > 0$ and $C > 1$ such that if*

$$\delta := ||v_0 - 1||_6 + ||u_0||_6 + \sup_{x\in\mathbb{R}^*} \left[(|x|+1)^{9/4} \{ |(v_0 - 1)(x)| + |u_0(x)| \} \right] \leq \delta_0 \tag{2}$$

[1]This work was supported by Grant-in-Aid for JSPS Research Fellow (Grant Number 20J00882).

[2]The assumption on the spatial decay can be slightly weakened as is stated in [1].

and

$$\left[\int_{-\infty}^{\infty}(v_0-1)(x)\,dx\right]\cdot\left[\int_{-\infty}^{\infty}u_0(x)\,dx+V_0\right]\neq 0, \tag{3}$$

then the unique global-in-time solution (v,u,V) *to* (1) *exists and satisfies*

$$C^{-1}\delta^2(t+1)^{-3/2}\leq |V(t)| \quad (t\geq T(\delta)) \tag{4}$$

for some $T(\delta)>0$

Remark 1. (i) The upper bound $|V(t)|=O(t^{-3/2})$ was obtained with less stringent assumptions [2], but we need to make these stronger to prove the lower bound.

(ii) By (4), we see that $V(t)$ does not change its sign after sufficiently long time has elapsed. We can also predict the final sign of $V(t)$ in terms of the initial data: it is the opposite sign of the left-hand side of (3).

(iii) When the left-hand side of (3) is zero, an improved decay estimate $V(t)=O(t^{-7/4})$ can be proved [1, Corollary 1.3]. From some numerical simulations for the corresponding Cauchy problem, we conjecture that the rate $-7/4$ is optimal under the condition that the left-hand side of (3) is zero, but to prove this would require much more work.

(iv) The presence of the point mass does not introduce additional technically difficulties compared to [2]; the main technical advancement lies in the analysis of the corresponding Cauchy problem.

The idea of the proof is to improve the previously known pointwise estimates for the fluid variables, see [3, Theorem 2.6] for the Cauchy problem and [2, Theorem 1.2] for our system, by a refined choice of leading order terms of the solution; we also need to analyze nonlinear interactions more precisely than in the previous works. We also note that we need to make use of finer space-time structure of fundamental solution, obtained in [4], compared to those presented in [3]. These allow us to understand the behavior of the solution around the origin $x=0$ more precisely and lead us to prove lower bound (4) for $V(t)=u(0_{\pm},t)$.

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Diffusion wave phenomena and L^p decay estimates of solutions of compressible viscoelastic system[1]

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1 Introduction

This talk is concerned with the following compressible viscoelastic system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho v) = 0, \\ \rho(\partial_t v + v\cdot\nabla v) - \nu\Delta v - (\nu+\nu')\nabla\mathrm{div}v + \nabla P(\rho) = \beta^2\mathrm{div}(\rho F^\top F), \\ \partial_t F + v\cdot\nabla F = \nabla v F, \\ \mathrm{div}(\rho^\top F) = 0, \\ (\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0),\ \mathrm{div}(\rho_0{}^\top F_0) = 0. \end{cases} \tag{1.1}$$

Here $\rho = \rho(x,t)$, $v = {}^\top(v^1(x,t), v^2(x,t), v^3(x,t))$ and $F = (F^{jk}(x,t))_{1\le j,k\le 3}$ denote the unknown density, velocity field and deformation tensor, respectivity, at time $t \ge 0$ and position $x \in \mathbb{R}^3$. $P(\rho)$ is the pressure that is a smooth function of ρ satisfying $P'(1) > 0$. ν, ν' and β are constants satisfying $\nu > 0,\ 2\nu + 3\nu' \ge 0, \beta > 0$. Here ν and ν' are the viscosity coefficients; β is the strength of elasticity. If we set $\beta = 0$ formally, we obtain the compressible Navier-Stokes equations. Here and in what follows ${}^\top\cdot$ stands for the transposition.

The aim of this talk is to investigate the large time behavior of solutions of the problem (1.1) around a motionless state $(\rho, v, F) = (1, 0, I)$. Here I is the 3×3 identity matrix.

In the case $\beta = 0$, Hoff-Zumbrun[1] derived the following L^p $(1 \le p \le \infty)$ decay estimates and asymptotic properties:

$$\|(\phi(t), m(t))\|_{L^p} \le \begin{cases} C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)}, & 1 \le p < 2, \\ C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}, & 2 \le p \le \infty. \end{cases}$$

$$\left\|\left((\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t}\hat{\mathcal{P}}(\xi)\hat{m}_0\right)\right)\right)\right\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}\left(1-\frac{2}{p}\right)},\ 2 \le p \le \infty,$$

where $(\phi(t), m(t)) = (\rho(t) - 1, \rho(t)v(t))$ and $\hat{\mathcal{P}}(\xi) = I - \frac{\xi^\top\xi}{|\xi|^2}$, $\xi \in \mathbb{R}^3$. The authors of [1] showed that the hyperbolic aspect of sound wave plays a role of the spreading effect of the wave equation, and the decay rate of the solution becomes slower than the heat kernel when $1 \le p < 2$. On the other hand, if $2 < p \le \infty$, the compressible part of the solution $(\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t}\hat{\mathcal{P}}(\xi)\hat{m}_0\right)\right)$ converges to 0 faster than the heat kernel.

In the case $\beta > 0$, Hu-Wu[2] and Li-Wei-Wao[5] established the following L^p $(2 \le p \le \infty)$ decay estimates:

$$\|u(t)\|_{L^p} \le C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)},$$

where $u(t) = (\phi(t), w(t), G(t)) = (\rho(t), w(t), F(t)) - (1, 0, I)$. However the hyperbolic aspects of elastic shear wave and sound wave does not appear. We will clarify the diffusion wave phenomena caused by interaction of three properties; sound wave, viscous diffusion and elastic shear wave and improve the results obtained in [2, 5].

[1] This work was partially supported by JSPS KAKENHI Grant Number 19J10056.

We consider the nonlinear problem for $u(t) = (\phi(t), w(t), G(t))$:

$$
(1.2) \qquad \begin{cases} \partial_t \phi + \mathrm{div} w = g_1, \\ \partial_t w - \nu \Delta w - \tilde{\nu} \nabla \mathrm{div} w + \gamma^2 \nabla \phi - \beta^2 \mathrm{div} G = g_2, \\ \partial_t G - \nabla w = g_3, \\ \nabla \phi + \mathrm{div}^\top G = g_4, \\ u|_{t=0} = u_0 = (\phi_0, w_0, G_0). \end{cases}
$$

Here $\tilde{\nu} = \nu + \nu'$; $g_j,\ j = 1, 2, 3, 4$ are nonlinear terms.

2 Main Result

We have the following result.

Theorem 2.1. ([3]) *Let $1 < p \le \infty$. Assume that ϕ_0, G_0, and F_0^{-1} satisfy $\nabla \phi_0 - \mathrm{div}^\top (I + G_0)^{-1} = 0$ and $F_0^{-1} = \nabla X_0$ for some vector field X_0. If $u_0 = (\phi_0, w_0, G_0)$ satisfies $\|u_0\|_{H^3} \ll 1$ and $u_0 \in L^1$, then there exists a unique solution $u(t) \in C([0, \infty); H^3)$ of the problem (1.2), and $u(t) = (\phi(t), w(t), G(t))$ satisfies*

$$
\|u(t)\|_{L^p} \le C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p}) - \frac{1}{2}(1 - \frac{2}{p})} (\|u_0\|_{L^1} + \|u_0\|_{H^3}),\ t \ge 0.
$$

Here $C(p)$ is a positive constant depending only on p.

Outline of the proof. We note that the solenoidal part of the solution of linearized system $(w_s, \tilde{G}_s) = (\mathcal{F}^{-1}(\hat{\mathcal{P}}(\xi)\hat{w}), \beta \mathcal{F}^{-1}(\hat{\mathcal{P}}(\xi)\hat{G}))$ satisfies the following linear symmetric parabolic-hyperbolic system:

$$
\begin{cases} \partial_t w_s - \nu \Delta w_s - \beta \mathrm{div} \tilde{G}_s = 0, \\ \partial_t \tilde{G}_s - \beta \nabla w_s = 0. \end{cases}
$$

It follows from [4, 6] that if $p > 2$, then the L^p norm of the solution of the linearized problem decays faster than the case $\beta = 0$. In the case of the nonlinear problem, we use a nonlinear transform $\tilde{\psi}$ defined by $\psi = \tilde{\psi} - (-\Delta)^{-1} \mathrm{div}^\top (\phi \nabla \tilde{\psi} + (1 + \phi) h(\nabla \tilde{\psi}))$ instead of G. Here $\tilde{\psi}$ is a displacement vector and $(-\Delta)^{-1} = \mathcal{F}^{-1} |\xi|^{-2} \mathcal{F}$. We then see that the nonlinear constraint $\mathrm{div}(\rho^\top F) = 0$ becomes the linear condition $\phi + \mathrm{tr}(\nabla \psi) = \phi + \mathrm{div} \psi = 0$ and straightforward application of the semigroup theory works well. Here $h(\nabla \tilde{\psi})$ is a nonlinear term satisfying $h(\nabla \tilde{\psi}) = O(|\nabla \tilde{\psi}|^2), |\nabla \tilde{\psi}| \ll 1$.

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On uniqueness of mild solutions on the whole time axis to the Boussinesq equations in unbounded domains

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In this talk, we consider the Boussinesq equations in 3-dimensional unbounded domains Ω. The Boussinesq equations describe the heat convection in a viscous incompressible fluid.

$$\text{(B)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = g\theta, \qquad t \in \mathbb{R}, x \in \Omega, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = S, \qquad t \in \mathbb{R}, \quad x \in \Omega, \\ \quad \nabla \cdot u = 0, \qquad t \in \mathbb{R}, \quad x \in \Omega, \\ \quad u|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = \xi, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$, $\theta = \theta(x,t)$ and $p = p(x,t)$ denote the velocity vector, the temperature and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times \mathbb{R}$. Here $\xi = \xi(x,t)$ is a given boundary temperature, S is a given external heat source and g denotes the acceleration of gravity. When Ω is some unbounded domain (e.g. the half-space $\mathbb{R}^3_+$), we can show the existence theorem of small mild solutions on whole time axis to (B). Typical examples of solutions on whole time axis are stationary, time-periodic and almost time-periodic solutions. In this talk, we consider the uniqueness of such solutions. Very roughly speaking, we will show that if there are two solutions (u, θ_1) and (v, θ_2) in some function spaces with the same data and if we assume that θ_1 is small in some sense, then $(u, \theta_1) = (v, \theta_2)$. Here, we do not need to assume any smallness condition on u, v, θ_2.

Analysis of non-stationary Navier-Stokes equations approximated by the pressure stabilization method

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The results of this talk are joint works with Dr. R. Matsui (Ushiku high school affiliated with Toyo University) and with Dr. H. Kikuchi (Univ. of Tsukuba).

The mathematical description of fluid flow is given by the Navier-Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = f & t > 0, x \in \Omega, \\ \nabla \cdot u = 0 & t > 0, x \in \Omega, \\ u(0, x) = a & x \in \Omega, \\ u(t, x) = 0 & x \in \partial\Omega, \end{cases} \tag{NS}$$

where the fluid vector fields $u = u(t, x)$ and the pressure $\pi = \pi(t, x)$ are unknown function, the external force $f = f(t, x)$ is a given vector function, the initial data a is a given solenoidal function and Ω is a bounded domain with smooth boundary. It is well-known that one of the difficulty of analysis for Navier Stokes equations (NS) is the pressure term $\nabla\pi$ and incompressible condition $\nabla \cdot u = 0$.

In order to overcome this difficulty, we often use Helmholtz decomposition. The Helmholtz decomposition means that for $1 < p < \infty$, $L^p(\Omega)^n = L_{p,\sigma}(\Omega) \oplus G_p(\Omega)$, where $L_{p,\sigma}(\Omega) = \overline{\{u \mid u_j \in C_0^\infty, \nabla \cdot u = 0\}}^{\|\cdot\|_{L_p}}$ and $G_p(\Omega) = \{\nabla\pi \in L_p(\Omega)^n \mid \pi \in L_{p,\mathrm{loc}}(\Omega)\}$. On the other hand, in numerical analysis, some penalty methods are employed as the method to overcome this difficulty. They are methods that eliminate the pressure by using approximated incompressible condition. For example, setting $\alpha > 0$ as a perturbation parameter, we use $\nabla \cdot u = -\pi/\alpha$ in the penalty method,

$$(u, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi, \nabla\varphi)_\Omega \qquad (\varphi \in \widehat{W}^1_{q'}(\Omega)) \tag{wi}$$

in the pressure stabilization method and $\nabla \cdot u = -\partial_t\pi/\alpha$ in the pseudocompressible method.

In this talk, we consider (wi) instead of incompressible conditions $\nabla \cdot u = 0$ in (NS). Namely we consider the following equations:

$$\begin{cases} \partial_t u_\alpha - \Delta u_\alpha + (u_\alpha \cdot \nabla)u_\alpha + \nabla\pi_\alpha = f & t > 0, x \in \Omega, \\ u_\alpha(0, x) = a_\alpha & x \in \Omega, \\ u_\alpha(t, x) = 0, \qquad \partial_n\pi_\alpha(t, x) = 0 & x \in \partial\Omega. \end{cases} \tag{NSa}$$

under the approximated weak incompressible condition (wi) in L^q-framework ($n/2 < q < \infty$). We shall use the maximal regularity theorem for linearized problem for (NSa) in order to prove the local in time existence theorem and the error estimate in the L_p in time and the L_q in space framework with $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$.

Main result in this talk is concerned with the local in time existence theorem for (NSa) with approximated weak incompressible condition (wi).

Theorem 1. *Let $n \geq 2$, $n/2 < q < \infty$ and $\max\{1, n/q\} < p < \infty$. Let $\alpha > 0$ and $T_0 \in (0, \infty)$. For any $M > 0$, assume that the initial data $a_\alpha \in B^{2(1-1/p)}_{q,p}(\Omega) = (L_q(\Omega), W^2_q(\Omega))_{1-1/p,p}$ and the external force $f \in L_p((0, T_0), L_q(\Omega)^n)$ satisfy*

$$\|a_\alpha\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|f\|_{L_p((0,T_0),L_q(\Omega)^n)} \leq M.$$

Then, there exists T^* depending on only M such that (NSa) *under* (wi) *has a unique solution (u_α, π_α) of the following class:*

$$\begin{aligned} &u_\alpha \in W^1_p((0, T^*), L_q(\Omega)^n) \cap L_p((0, T^*), W^2_q(\Omega)^n),\\ &\pi_\alpha \in L_p((0, T^*), \widehat{W}^1_q(\Omega)). \end{aligned}$$

Moreover the following estimate holds:

$$\|u_\alpha\|_{L_\infty((0,T^*),L_q(\Omega))} + \|(\partial_t u_\alpha, \nabla^2 u_\alpha, \nabla \pi_\alpha)\|_{L_p((0,T^*),L_q(\Omega))} + \|\nabla u_\alpha\|_{L_r((0,T^*),L_q(\Omega))} \leq C$$

for $1/p - 1/r \leq 1/2$, where C is the positive constant depend on n, p, q and T^*.

Next we consider the error estimate between the solution (u, π) to (NS) under the weak incompressible condition $(u, \nabla\varphi)_\Omega = 0$ for $\varphi \in \widehat{W}^1_{q'}(\Omega)$ and solution (u_α, π_α) to (NSa) under (wi). To this end, setting $u_e = u - u_\alpha$ and $\pi_e = \pi - \pi_\alpha$, we see that (u_e, π_e) enjoys that

$$\begin{cases} \partial_t u_e - \Delta u_e + \nabla \pi_e + N(u_e, u_\alpha) = 0, & t > 0, x \in \Omega,\\ u_e(0, x) = a_e, & x \in \Omega,\\ u_e(t, x) = 0, & x \in \partial\Omega, \end{cases} \tag{PE}$$

where $N(u_e, u_\alpha) = (u_e \cdot \nabla)u_e + (u_e \cdot \nabla)u_\alpha + (u_\alpha \cdot \nabla)u_e$ and $a_e = a - a_\alpha$ under the approximated weak incompressible condition

$$(u_e, \nabla\varphi)_\Omega = \alpha^{-1}(\nabla\pi_e, \nabla\varphi)_\Omega + \alpha^{-1}(\nabla\pi, \nabla\varphi)_\Omega \qquad \varphi \in \widehat{W}^1_{q'}(\Omega) \tag{wie}$$

for $1 < q < \infty$. In a similar way to Theorem 1, we obtain the following theorems:

Theorem 2. *Let $n \geq 2$, $n/2 < q < \infty$, $\max\{1, n/q\} < p < \infty$ and $\alpha > 0$. Let T^* be a positive constant obtained in Theorem 1 and (u_α, π_α) be a solution obtained in Theorem 1. For any $M > 0$, assume that $a_e \in B^{2(1-1/p)}_{q,p}(\Omega)$ satisfies*

$$\|a_e\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq M\alpha^{-1}.$$

Then there exists $T^\flat$ such that (PE) *has a unique solution (u_e, π_e) which satisfies*

$$\|u_e\|_{L_\infty((0,T^\flat),L_q(\Omega))} + \|\nabla u_e\|_{L_r((0,T^\flat),L_q(\Omega))} + \|(\nabla^2 u_e, \partial_t u_e, \nabla\pi_e)\|_{L_p((0,T^\flat),L_q(\Omega))} \leq C\alpha^{-1}$$

for $1/p - 1/r \leq 1/2$.

Furthermore, in this talk, we will introduce the estimates for error (u_e, π_e) derived from the maximal regularity theorem for the linearized problem for (NSa) under the approximated weak incompressible condition (wi).

The decay property of the multidimensional compressible flow in the exterior domain

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This talk discusses about the decay property of the compressible flow in the exterior domain in the general L_p framework. Here, we consider the motion of the gases with some free surface Γ_t, which can be described by the following (barotropic) compressible Navier-Stokes system in the exterior domain $\Omega_t \subset \mathbb{R}^N (N \geq 3)$:

$$\begin{cases} \partial_t \rho + \operatorname{div}\big((\rho_e + \rho)\mathbf{v}\big) = 0 & \text{in} \quad \Omega_t, \\ (\rho_e + \rho)(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \operatorname{Div}\big(\mathbf{S}(\mathbf{v}) - P(\rho_e+\rho)\mathbf{I}\big) = 0 & \text{in} \quad \Omega_t, \\ \big(\mathbf{S}(\mathbf{v}) - P(\rho_e+\rho)\mathbf{I}\big)\mathbf{n}_{\Gamma_t} = P(\rho_e)\mathbf{n}_{\Gamma_t}, \;\; V_{\Gamma_t} = \mathbf{v}\cdot\mathbf{n}_{\Gamma_t} & \text{on} \quad \Gamma_t, \\ (\rho, \mathbf{v}, \Omega_t)|_{t=0} = (\rho_0, \mathbf{v}_0, \Omega). \end{cases} \tag{1}$$

Given the initial data and the reference density $\rho_e > 0$, we seek for the velocity field $\mathbf{v}$, the mass density $\rho + \rho_e$ and the pattern of Ω_t. In (1), the Cauchy stress tensor

$$\mathbf{S}(\mathbf{v}) = \mu\mathbf{D}(\mathbf{v}) + (\nu - \mu)\operatorname{div}\mathbf{v}\mathbf{I} \;\text{ for constants } \mu, \nu > 0,$$

and the doubled deformation tensor $\mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^\top$. Moreover, the (i,j)th entry of the matrix $\nabla\mathbf{v}$ is $\partial_i v_j$, $\mathbf{I}$ is the $N\times N$ identity matrix, and $\mathbf{M}^\top$ is the transposed the matrix $\mathbf{M} = [M_{ij}]$. In addition, $\operatorname{Div}\mathbf{M}$ denotes an N-vector of functions whose i-th component is $\sum_{j=1}^N \partial_j M_{ij}$, $\operatorname{div}\mathbf{v} = \sum_{j=1}^N \partial_j v_j$, and $\mathbf{v}\cdot\nabla = \sum_{j=1}^N v_j\partial_j$ with $\partial_j = \partial/\partial x_j$. On the moving boundary Γ_t of Ω_t, $\mathbf{n}_{\Gamma_t}$ is the outer unit normal vector to the boundary Γ_t of Ω_t, and V_{Γ_t} stands for the normal velocity of the moving surface Γ_t.

To study (1), we transfer (1) to some system in the fixed (or initial) domain Ω. Assume that $\Omega \subset \mathbb{R}^N$ with the boundary Γ is an exterior domain such that $\mathcal{O} = \mathbb{R}^N \setminus \Omega$ is a subset of the ball B_R, centred at origin with radius $R > 1$. Let κ be a C^∞ functions which equals to one for $x \in B_R$ and vanishes outside of B_{2R}. Then we define the partial Lagrangian coordinates

$$x = X_{\mathbf{w}}(y, T) = y + \int_0^T \kappa(y)\mathbf{w}(y,s)\,ds \quad (\forall\, y \in \Omega), \tag{2}$$

for some vector field $\mathbf{w} = \mathbf{w}(\cdot, s)$ defined in Ω. By using (2), denoting $(\gamma_1, \gamma_2) = \big(\rho_e, P'(\rho_e)\big)$ and neglecting the nonlinear forms, we obtain the following linearized equations *

$$\begin{cases} \partial_t\rho + \gamma_1\operatorname{div}\mathbf{v} = 0 & \text{in} \quad \Omega\times\mathbb{R}_+, \\ \gamma_1\partial_t\mathbf{v} - \operatorname{Div}\big(\mathbf{S}(\mathbf{v}) - \gamma_2\rho\mathbf{I}\big) = 0 & \text{in} \quad \Omega\times\mathbb{R}_+, \\ \big(\mathbf{S}(\mathbf{v}) - \gamma_2\rho\mathbf{I}\big)\mathbf{n}_\Gamma = 0 & \text{on} \quad \Gamma\times\mathbb{R}_+, \\ (\rho, \mathbf{v})|_{t=0} = (\rho_0, \mathbf{v}_0) & \text{in} \quad \Omega. \end{cases} \tag{3}$$

*In fact, our result can be extended to more general linear equations with variable coefficients.

Now, our main result of (3) reads as follows:

Theorem 1 (L_p-L_q type decay estimate)**.** *Let Ω be a C^3 exterior domain in $\mathbb{R}^N$ with $N \geq 3$. Assume that $(\rho_0, \mathbf{v}_0) \in L_q(\Omega)^{1+N} \cap H_p^{1,0}(\Omega)$ with $H_p^{1,0}(\Omega) = H_p^1(\Omega) \times L_p(\Omega)^N$ for $1 \leq q \leq 2 \leq p < \infty$, and $\{T(t)\}_{t \geq 0}$ is the semigroup associated to (3) in $H_p^{1,0}(\Omega)$. For convenience, we set $\mathcal{P}_v(\rho, \mathbf{v}) = \mathbf{v}$ and*

$$\|\!|(\rho_0, \mathbf{v}_0)|\!\|_{p,q} = \|(\rho_0, \mathbf{v}_0)\|_{L_q(\Omega)} + \|(\rho_0, \mathbf{v}_0)\|_{H_p^{1,0}(\Omega)}.$$

Then for $t \geq 1$, there exists a positive constant C such that

$$\begin{aligned}
\|T(t)(\rho_0, \mathbf{v}_0)\|_{L_p(\Omega)} &\leq C t^{-(N/q - N/p)/2} \|\!|(\rho_0, \mathbf{v}_0)|\!\|_{p,q}, \\
\|\nabla T(t)(\rho_0, \mathbf{v}_0)\|_{L_p(\Omega)} &\leq C t^{-\sigma_1(p,q,N)} \|\!|(\rho_0, \mathbf{v}_0)|\!\|_{p,q}, \\
\|\nabla^2 \mathcal{P}_v T(t)(\rho_0, \mathbf{v}_0)\|_{L_p(\Omega)} &\leq C t^{-\sigma_2(p,q,N)} \|\!|(\rho_0, \mathbf{v}_0)|\!\|_{p,q},
\end{aligned}$$

where the indices $\sigma_1(p,q,N)$ and $\sigma_2(p,q,N)$ are given by

$$\sigma_1(p,q,N) = \begin{cases} (N/q - N/p)/2 + 1/2 & \text{for } 2 \leq p \leq N, \\ N/(2q) & \text{for } N < p < \infty, \end{cases}$$

$$\sigma_2(p,q,N) = \begin{cases} 3/(2q) & \text{for } N = 3, \\ (N/q - N/p)/2 + 1 & \text{for } N \geq 4 \text{ and } 2 \leq p \leq N/2, \\ N/(2q) & \text{for } N \geq 4 \text{ and } N/2 < p < \infty. \end{cases}$$

The proof of Theorem 1 relies on the spectral analysis and the local energy method. This is a joint work with Yoshihiro Shibata from Waseda University.

On the Head-on Collision of Coaxial Vortex Rings

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1 Introduction

The study of the interaction of coaxial vortex rings dates back to the pioneering paper by Helmholtz [2]. In [2], Helmholtz considered vortex motion in a incompressible and inviscid fluid based on the Euler equations. His study includes the motion of circular vortex filaments, and he observed that motion patterns such as head-on collision may occur. Since then, many researches have been done on head-on collision of coaxial vortex rings, and interaction of coaxial vortex rings in general. Most of these research are either experiments conducted in a laboratory or numerical simulations of the Navier–Stokes equations, and the rigorous mathematical treatment of head-on collision of vortex rings are very scarce.

In light of this, we consider the head-on collision of two coaxial vortex rings, which have circulations of opposite sign, described as the motion of two coaxial circular vortex filaments under the localized induction approximation. A vortex filament is a space curve on which the vorticity of the fluid is concentrated. In our present work, we approximated thin vortex structures, such as vortex rings, by vortex filaments, and described the motion as the motion of a curve in the three-dimensional Euclidean space. In this formulation, a vortex ring is a space curve in the shape of a circle. We prove the existence of solutions to a system of nonlinear partial differential equations modelling the interaction of two vortex filaments proposed by the speaker [1] which exhibit head–on collision. We also give a necessary and sufficient condition for the initial configuration and parameters of the filaments for head-on collision to occur. Our results suggest that there exists a critical value $\gamma_* > 1$ for the ratio γ of the magnitude of the circulations satisfying the following. When $\gamma \in [1, \gamma_*]$, two approaching rings will collide, and when $\gamma \in (\gamma_*, \infty)$, the ring with the larger circulation passes through the other and then separate indefinitely. As far as the speaker knows, the existence of such threshold γ_* is only indirectly suggested via numerical investigations of the head-on collision of coaxial vortex rings, for example by Inoue, Hattori, and Sasaki [3]. Hence, our result is the first to obtain the threshold in a way that is possible to numerically calculate γ_*, as well as prove that the threshold exists in a framework of a mathematical model.

2 Problem Setting and Main Results

We consider the motion of two interacting vortex filaments. In [1], under physically intuitive assumptions, we derived the following system of partial differential equations which describes the interaction of two vortex filaments.

$$
\begin{cases}
\boldsymbol{X}_t = \beta \dfrac{\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi}}{|\boldsymbol{X}_\xi|^3} - \alpha \dfrac{\boldsymbol{Y}_\xi \times (\boldsymbol{X} - \boldsymbol{Y})}{|\boldsymbol{X} - \boldsymbol{Y}|^3}, \\
\boldsymbol{Y}_t = \dfrac{\boldsymbol{Y}_\xi \times \boldsymbol{Y}_{\xi\xi}}{|\boldsymbol{Y}_\xi|^3} - \alpha\beta \dfrac{\boldsymbol{X}_\xi \times (\boldsymbol{Y} - \boldsymbol{X})}{|\boldsymbol{X} - \boldsymbol{Y}|^3},
\end{cases}
\tag{1}
$$

where $\boldsymbol{X}(\xi,t) = {}^t(X_1(\xi,t), X_2(\xi,t), X_3(\xi,t))$ and $\boldsymbol{Y}(\xi,t) = {}^t(Y_1(\xi,t), Y_2(\xi,t), Y_3(\xi,t))$ are the position vectors of the vortex filaments parametrized by ξ at time t, $\times$ is the exterior product in the three-dimensional Euclidean space, subscripts denote partial differentiation with the respective variables, $\beta \in \mathbf{R} \setminus \{0\}$ is the quotient of the vorticity strengths of the filaments, and $\alpha > 0$ is a constant which is introduced in the course of the derivation of the system. We consider the case $\beta < 0$ to describe colliding vortex rings. To make things more simple, we set $\gamma = -\beta$ and consider $\gamma > 0$.

In this talk, we introduce recent results on the existence of solutions to system (1) which correspond to head-on collision. We give necessary and sufficient conditions on the initial data and parameters for head-on collision to occur. This, in particular, gives the threshold for γ mentioned in the introduction.

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Current-valued solutions of the Euler-Arnold equation on surfaces and its applications

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The motion of incompressible and inviscid fluids in the Euclidean plane is governed by the Euler equation and its solution is called an Euler flow. For the Euler flow, the vorticity is a Lagrange invariance. Then, since the fluid velocity and the pressure can be recovered from the vorticity, an Euler flow is determined by a solution of the vorticity equation. Namely, if ω_t is a solution of the vorticity equation,

$$\partial_t \omega_t + (-\mathcal{J}\mathrm{grad}\langle G, \omega_t\rangle \cdot \nabla)\omega_t = 0,$$

(v_t, p_t) is an Euler flow, defined by

$$v_t = -\mathcal{J}\mathrm{grad}\langle G, \omega_t\rangle, \quad p_t = \langle G, \mathrm{div}(v_t \cdot \nabla)v_t\rangle, \tag{1}$$

where $\mathcal{J}$ is the symplectic matrix and G is the Green function for the Laplacian. On the other hand, the formulae (1) still make sense in the sense of distributions when we give a time-dependent distribution Ω_t by a linear combination of delta functions centered at $q_n(t)$ for $n = 1, \ldots, N$ with the linear coefficient $\Gamma_n \in \mathbb{R}$. Then, replacing ω_t by Ω_t in (1), we formally obtain a fluid velocity V_t and a pressure P_t. However, we can not define the dynamics of $q_n(t)$ from the vorticity equation. Instead, to determine the evolution of $q_n(t)$ by V_t, Helmholtz considered the following regularized equation for $q_n(t)$ [6].

$$\dot{q}_n = \lim_{q \to q_n} \left[V_t(q) + \mathcal{J}\mathrm{grad}\langle G, \Gamma_n \delta_{q_n(t)}\rangle(q)\right] = -\mathcal{J}\mathrm{grad} \sum_{\substack{m=1 \\ m \neq n}}^{N} \Gamma_m G(q_n, q_m) \equiv v_n(q_n). \tag{2}$$

It is called the *point vortex equation*, and the solution of (2) is called the *point vortex dynamics*. Then, there arises a natural question; How can we interpret (V_t, P_t) as an Euler flow in an appropriate mathematical sense? In other words, we need to determine a space of solutions of the Euler equation which contains (V_t, P_t). Since L^p space does not contain (V_t, P_t), a more sophisticated space is to be considered. This is one of the central problems in the analysis of 2D Euler equation as discussed in [2, 3, 4]. From the viewpoint of the application, the point vortex dynamics is sometimes considered in the presence of a time-dependent vector field $X_t \in \mathfrak{X}^r(\mathbb{R}^2)$, called the *point vortex dynamics in the background field* X_t. Then, the evolution of $q_n(t)$ is governed by the following equation.

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \ldots N \tag{3}$$

for a given $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. Some experimental studies confirm the importance of background fields in two-dimensional turbulence [1, 7].

The purpose of this research is justifying the point vortex dynamics in background fields as an Euler flow mathematically. To this end, we establish a weak formulation of the Euler equation in the space of currents, which is developed in the theory of geometric analysis and geometric measure theory. Since the notion of currents is defined not only for the Euclidean plane but also general curved surface, the formulation established here can be naturally generalized for surfaces. From the viewpoint of the application, it is of a great significance to justify the point vortex dynamics in a background field on curved surfaces as an Euler-Arnold flow, which is a generalization of the Euler equation to the case of the surfaces, since the point vortex dynamics in the rotational vector field on the unit sphere is adapted as a mathematical model of a geophysical flow in order to take effect of the Coriolis force on inviscid flows into consideration [5] for instance.

The main results consist of two theorems. For a current-valued solution of the Euler-Arnold equation with a regular-singular decomposition, we first prove that, if the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$ for $n = 1, \ldots, N$, $q_n(t)$ is a solution of (3). Conversely, we next prove that, if $q_n(t)$ is a solution of (3), there exists a current-valued solution of the Euler-Arnold equation with a regular-singular decomposition such that the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$. Therefore, we conclude that the point vortex dynamics in a background field on a surfaces is a current-valued solution of the Euler-Arnold equation with a regular-singular decomposition.

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Estimates of the regular set for Navier-Stokes flows in terms of initial data

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This talk is based on a joint work with Kyungkeun Kang (Yonsei university) and Tai-Peng Tsai (University of British Columbia).We consider the regularity of weak solutions for the incompressible Navier-Stokes equations

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \operatorname{div} v = 0 \tag{NS}$$

associated with the initial value $v|_{t=0} = v_0$ with $\operatorname{div} v_0 = 0$. In [2], Caffarelli, Kohn, and Nirenberg established local regularity theory for suitable weak solutions. As an application of their ϵ-regularity theorem, they showed the following result:

Theorem 1 ([2]). *There exists $\epsilon_0 > 0$ such that if $v_0 \in L^2(\mathbb{R}^3)$ satisfies*

$$\|v_0\|_{L^{2,-1}}^2 = \epsilon < \epsilon_0, \tag{1}$$

then there exists a suitable weak solution which is regular in the set $\Pi_{\epsilon_0 - \epsilon}$, where

$$\begin{aligned} \|v_0\|_{L^{2,\alpha}} &:= \||x|^{\frac{\alpha}{2}} v_0\|_{L^2(\mathbb{R}^3)}, \\ \Pi_\delta &:= \left\{ (x,t) \in \mathbb{R}^3 \times (0,\infty) : t > \frac{|x|^2}{\delta} \right\}. \end{aligned}$$

This theorem asserts that smallness of initial data in a weighted space implies regularity of the solution above a paraboloid with vertex at the origin. There are at least two interesting features in this result: No regularity condition (better than L^2) is assumed away from the origin and the regularity around the origin is propagated globally in time. We also note that if the size of v_0 tends to 0, $\Pi_{\epsilon_0 - \epsilon}$ increases and converges to a limit set Π_{ϵ_0}. This observation leads to the following questions: (a) Can the size of regular set Π_δ be enlarged? (b) Can the asssumtions of the initial data be relaxed in terms of regularity, decay, and smallness? The question (a) is addressed by D'Ancona and Luca [3], where it is shown that there exists $\delta_0 > 0$ such that if $v_0 \in L^2(\mathbb{R}^3)$ satisfies

$$\|v_0\|_{L^{2,-1}} < \delta_0 e^{-4L^2} \tag{2}$$

for some $L > 1$, (NS) has a suitable weak solution which is regular in the set $\Pi_{L\delta_0}$ invading the whole half space $\mathbb{R}^3 \times (0,\infty)$ when v_0 tends to zero, though (2) still assumes smallness of the data. The aim of this talk is trying to answer questions (a) and (b). To this end, we recall the notion of the local energy solutions introduced by [4] and later modified by [1]. The local energy solution is a suitable weak solution of (NS) defined in $\mathbb{R}^3$ which satisfies certain uniformly local energy bounds and pressure representation.

In this context, let us also recall the uniformly local L^q spaces for $1 \le q < \infty$. We say $f \in L^q_{\text{uloc}}$ if $f \in L^q_{\text{loc}}(\mathbb{R}^3)$ and

$$\|f\|_{L^q_{\text{uloc}}} = \sup_{x\in\mathbb{R}^3} \|f\|_{L^q(B_1(x))} < \infty.$$

The following result shows estimates of the regular set for the local energy solution for arbitrary initial data in $L^{2,-1}(\mathbb{R}^3)$ and the data with small scaled energy.

Theorem 2. *Let (v,p) be a local energy solution in $\mathbb{R}^3 \times (0,\infty)$ for the initial data $v_0 \in L^2_{\text{uloc}}(\mathbb{R}^3)$.*

(i) For any $v_0 \in L^{2,-1}(\mathbb{R}^3)$ there exist positive constants $T(v_0)$ and $c(v_0)$ such that v is regular in the set

$$\left\{(x,t) \in \mathbb{R}^3 \times (0,\infty) \,:\, c(v_0)|x|^2 \le t < T(v_0)\right\}.$$

(ii) There exist positive absolute constants ϵ_* and c such that if v_0 satisfies

$$\dot{N}_0 := \sup_{r>0} \frac{1}{r} \int_{B_r} |v_0(x)|^2 \, dx \le \epsilon_*$$

and

$$\sup_{x_0\in\mathbb{R}^3} \sup_{r\ge 1} \frac{1}{r} \int_{B_r(x_0)} |v_0(x)|^2 \, dx < \infty,$$

then v is regular in the set

$$\left\{(x,t) \in \mathbb{R}^3 \times (0,\infty) \,:\, c\dot{N}_0^2 \, |x|^2 \le t\right\}.$$

It is not difficult to see that the assumptions of (ii) are slightly weaker than those in Theorem 1. (ii) also refines the convergence rate of the regular set to $\mathbb{R}^3 \times (0,\infty)$ in [3] as $\|v_0\|_{L^{2,-1}}$ tends to zero.

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Existence of radially symmetric stationary solutions for the compressible Navier-Stokes equation

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(joint work with Akitaka Matsumura of Osaka university)

In this talk, we consider the existence of radially symmetric problems in $\mathbb{R}^n (n \geq 2)$ to the compressible Navier-Stokes equation:

$$\begin{cases} (r^{n-1}\rho)_t + (r^{n-1}\rho u)_r = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_r + (n-1)\dfrac{\rho u^2}{r} = \mu\Big(\dfrac{(r^{n-1}u)_r}{r^{n-1}}\Big)_r, \quad t > 0,\ r > r_0, \end{cases} \tag{1}$$

where $\rho = \rho(t, r) > 0$ is the mass density, $u = u(t, r)$ is the fluid velocity, and $p = p(\rho)$ is the pressure given by a smooth function of ρ satisfying $p'(\rho) > 0$ $(\rho > 0)$, and μ is a positive constant. We consider the initial boundary value problems to (1) under the initial condition

$$(\rho, u)(0, r) = (\rho_0, u_0)(r), \quad r > r_0,$$

the far field condition

$$\lim_{r\to\infty} (\rho, u)(t, r) = (\rho_+, u_+), \quad t > 0,$$

and also the following two types of boundary conditions depending on the sign of the velocity on the boundary

$$\begin{cases} (\rho, u)(t, r_0) = (\rho_-, u_-), \quad t > 0, \quad (u_- > 0), \\ u(t, r_0) = u_-, \quad t > 0, \qquad\qquad\quad (u_- \leq 0), \end{cases}$$

where $\rho_\pm > 0, u_\pm$ are given constants. The case $u_- > 0$ is known as "inflow problem", the case $u_- = 0$ as "impermeable wall problem", and the case $u_- < 0$ as "outflow problem". The equation (1) is given by letting

$$\rho(t, x) = \rho(t, r), \quad U(t, x) = \frac{x}{r}u(t, r), \quad r = |x|,$$

for the compressible Navier-Stokes equation which describes a barotropic motion of viscous gas in the exterior domain Ω to a ball in $\mathbb{R}^n$ $(n \geq 2)$:

$$\begin{cases} \rho_t + \mathrm{div}(\rho U) = 0, \\ (\rho U)_t + \mathrm{div}(\rho U \otimes U) + \nabla p = \nu \triangle U + (\nu + \lambda)\nabla(\mathrm{div} U), \quad t > 0,\ x \in \Omega, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^n\ (n \geq 2); |x| > r_0\}$ (r_0 is a positive constant), $U = (u_1(t, x), \cdots, u_n(t, x))$ is the fluid velocity and ν and λ are constants satisfying $\nu > 0, 2\nu + n\lambda > 0$.

We show the existence of a unique radially stationary solution in a suitably small neighborhood of the far field state for both inflow and outflow problems for the problem (1). Furthermore, it is shown that the boundary layer of the density appears as the velocity data tend to zero in the inflow problem, but not in the outflow problem. The stationary problem corresponding to the problem (1) is written as

$$\begin{cases} (r^{n-1}\rho u)_r = 0, \\ \rho u u_r + p(\rho)_r = \mu\left(\frac{(r^{n-1}u)_r}{r^{n-1}}\right)_r, \qquad r > r_0, \\ \lim_{r\to\infty} (\rho, u)(r) = (\rho_+, u_+), \\ (\rho, u)(r_0) = (\rho_-, u_-) \ \ (u_- > 0), \quad u(r_0) = u_- \ \ (u_- \le 0). \end{cases} \tag{2}$$

The main theorem of this talk is as follows.

Theorem 1. *Let $n \ge 2$ and $u_+ = 0$. Then, for any $\rho_+ > 0$, there exist positive constants ϵ_0 and C satisfying the following:*

(I) Let $u_- > 0$. If $|u_-| + |\rho_- - \rho_+| \le \epsilon_0$, there exists a unique smooth solution (ρ, u) of the problem (2) satisfying

$$|\rho(r) - \rho_+| \le Cr^{-(n-1)}(|u_-|^2 + |\rho_- - \rho_+|),$$
$$C^{-1}r^{-(n-1)}|u_-| \le |u(r)| \le Cr^{-(n-1)}|u_-|, \qquad r \ge r_0.$$

Furthermore, for any positive constant h, there exists a positive constant C_h such that it holds

$$\sup_{r \ge r_0 + h} |\rho(r) - \rho_+| \le C_h |u_-|^2.$$

(II) Let $u_- \le 0$. If $|u_-| \le \epsilon_0$, there exists a unique smooth solution (ρ, u) of the problem (2) satisfying

$$|\rho(r) - \rho_+| \le Cr^{-2(n-1)}|u_-|^2,$$
$$C^{-1}r^{-(n-1)}|u_-| \le |u(r)| \le Cr^{-(n-1)}|u_-|, \qquad r \ge r_0.$$

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