RIMS 共同研究「非圧縮性粘性流体の数理解析」

日時： 2021 年 12 月 6 日（月）14:00 ～ 12 月 8 日（水）12:00
研究代表者：前川 泰則（京都大学）
副代表者：柴田 良弘（早稲田大学）

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プログラム

12 月 6 日（月）

14:00 - 14:50　大木谷　耕司　（京都大学）
Self-similar profiles of solutions to hypo-viscous fluid equations

15:05 - 15:55　齋藤　平和　（電気通信大学）
On the two-phase Navier-Stokes equations with a sharp interface

16:10 - 16:40　中里　亮介　（東北大学）
Well-posedness for the magnetohydrodynamics with Hall-effect near non-zero magnetic equilibrium states

16:50 - 17:20　顧　仲陽　（東京大学）
The Helmholtz decomposition of a space of vector fields with bounded mean oscillation in a bounded domain

12 月 7 日（火）

10:00 - 10:50　Tai-Peng Tsai　(The University of British Columbia)
Weak and mild solutions to the Navier-Stokes equations in Wiener amalgam spaces

11:05 - 11:55　阿部　健　（大阪市立大学）
Rigidity of Beltrami fields with a non-constant proportionality factor

14:00 - 14:50　Jan Brezina（九州大学）
On barotropic　Navier-Stokes system with general boundary conditions

15:05 - 15:55　牛越　惠理佳　（横浜国立大学・大阪大学）
Hadamard variational formula for the fundamental solution of the non stationary Stokes equations

16:10 - 16:40　三浦　達彦　（京都大学）
Enhanced dissipation for the two-jet Kolmogorov type flow on the unit sphere

12 月 8 日（水）

10:00 - 10:50　木村　芳文　（名古屋大学）
Vortex reconnection and a finite-time singularity of the Navier-Stokes equations

11:05 - 11:55　高田　了　（九州大学）
Fast rotation limit for the incompressible Navier-Stokes equations in a 3D layer

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Abstract 集

Self-similar profiles of solutions to hypo-viscous fluid equations

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1 Motivation

We consider the source-type solution to the hypoviscous Burgers equation and provide a heuristic argument for finding a "Cole-Hopf-like transform" for its possible linearisation.

2 Burgers equation with standard dissipativity (review)

We consider the Burgers equation

$$u_t + uu_x = \nu u_{xx}, \tag{1}$$

which satisfies static scale-invariance under $x \to \lambda x, t \to \lambda^2 t, u \to \lambda^{-1} u$, for any $\lambda(>0)$. This means that if $u(x,t)$ is a solution, so is $u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$. E.g. it is readily checked that $\|u_\lambda\|_{L^p} = \lambda^{\frac{p-1}{p}} \|u\|_{L^p}$, which shows that the L^1-norm is scale-invariant.

Let us clarify the two kinds of critical scale-invariance. *Type 1 critical scale-invariance* is achieved with the velocity potential ϕ, defined by $u = \phi_x$, which obeys

$$\phi_t + \frac{1}{2}\phi_x^2 = \nu \phi_{xx}, \tag{2}$$

where $[\phi] = [\nu] = L^2/T$. If $\phi(x,t)$ is a solution, so is $\phi(\lambda x, \lambda^2 t)$. Under dynamic scaling for the velocity potential $\phi(x,t) = \Phi(\xi,\tau)$, $\xi = \frac{x}{\sqrt{2at}}$, $\tau = \frac{1}{2a}\log t$, $(a > 0)$ we have

$$\Phi_\tau + \frac{1}{2}\Phi_\xi^2 = \nu \Phi_{\xi\xi} + a\xi\Phi_\xi. \tag{3}$$

Type 1 critical scale-invariance is deterministic in nature, where the number of additional terms is minimised, that is, only the drift term.

The other *type 2 critical scale-invariance* concerns the nth spatial derivative of the unknown for the type 1 scale-invariance (n is the spatial dimension.) For the Burgers equation, under dynamic scaling for velocity $u(x,t) = \frac{1}{\sqrt{2at}} U(\xi,\tau)$, we find

$$U_\tau + UU_\xi = \nu U_{\xi\xi} + a\left(\xi U\right)_\xi, \tag{4}$$

whose linearisation is the Fokker-Planck equation. The type 2 scale-invariance is statistical in nature where the number of additional terms is maximised in the sense that a divergence form is completed with the addition of aU term. A steady solution to (4)

$$U(\xi) = \frac{C\exp\left(-\frac{a\xi^2}{2\nu}\right)}{1 - \frac{C}{2\nu}\int_0^\xi \exp\left(-\frac{a\eta^2}{2\nu}\right) d\eta}. \tag{5}$$

is known as the source-type solution where $C = C(M)$ is a constant, see e.g. [1, 2]. Observe that (5) is a near-identity transformation of the Gaussian function.

3 Burgers equation with hypo-viscous dissipativity

We now turn our attention to the hypo-viscous Burgers equation of the following form

$$u_t + uu_x = -\nu' \Lambda u, \tag{6}$$

where $\Lambda \equiv (-\partial_{xx})^{1/2} = \partial_x H[\cdot]$ denotes the Zygmund operator and $H[\cdot]$ the Hilbert transform. The equation is known to be well-posed [3]. We are interested in deriving a source-type solution analogous to (5).

Type 1 critical scale-invariance: under dynamic scaling $u(x,t) = U(\xi), \xi = \frac{x}{at}, \tau = \frac{1}{a}\log t$, where $[\nu'] = [u] = L/T$, (6) is transformed to

$$U_t + UU_\xi = -\nu' \Lambda U + a\xi U_\xi. \tag{7}$$

Type 2 critical scale-invariance: the governing equation for velocity gradient $w = u_x$

$$w_t + uw_x = -w^2 - \nu' \Lambda w,$$

is transformed under dynamic scaling to

$$W_\tau + UW_\xi = -W^2 - \nu' \Lambda W + a(\xi W)_\xi, \tag{8}$$

where $W = U_\xi$. Note that the difference in dissipativity offsets the pair of critical variables (U, W) by one derivative from that of the standard Burgers equation's (Φ, U).

Self-similar profile: late-time behaviour is determined by the self-similar profile $W(\xi) = \lim_{\tau\to\infty} W(\xi,\tau)$, if it exists. Its equation is obtained by setting $W_\tau = 0$ in (8)

$$(UW)_\xi = -\nu' \Lambda W + a(\xi W)_\xi,$$

which is integrated to give

$$\frac{1}{\mu}\xi W - H[W] = \frac{1}{\nu'} UW, \tag{9}$$

where $\mu = \nu'/a$. This is the equation we ought to solve. It is clear that $W = \frac{\mu}{\xi^2+\mu^2}$ and $H[W] = \frac{\xi}{\xi^2+\mu^2}$ solves it, when the nonlinearity (that is, RHS) is discarded. We will handle (9) perturbatively assuming that the nonlinearity is small. A numerical approach for handling (8) will also discussed. Time permitting, the SQG equation will also be addressed together with comparisons to other works.

References

[1] T.-P. Liu and M. Pierre, "Source-solutions and asymptotic behavior in conservation laws," *J. Diff. Eq.* **51** 419–441 (1984).

[2] M. Escobedo and E. Zuazua, "Large time behavior for convection-diffusion equations in R^N," *J. Func. Anal.* **100**, 119–161 (1991).

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On the two-phase Navier-Stokes equations with a shape interface

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Let $\Omega_\pm(t)$ and $\Gamma(t)$, $t > 0$, be given by

$$\Omega_\pm(t) = \{x = (x', x_3) \in \mathbf{R}^3 : x' = (x_1, x_2) \in \mathbf{R}^2, \pm(x_3 - \eta(x', t)) > 0\},$$
$$\Gamma(t) = \{x = (x', x_3) \in \mathbf{R}^3 : x' = (x_1, x_2) \in \mathbf{R}^2, x_3 = \eta(x', t)\}.$$

In this talk, we consider the motion of two immiscible, viscous, incompressible fluids, *fluid*$_+$ and *fluid*$_-$, which occupy $\Omega_+(t)$ and $\Omega_-(t)$, respectively. The positive constants $\rho_\pm$ and $\mu_\pm$ denote the densities and the viscosity coefficients of the respective fluids. Define $\rho = \rho_+ \mathbb{1}_{\Omega_+(t)} + \rho_- \mathbb{1}_{\Omega_-(t)}$ and $\mu = \mu_+ \mathbb{1}_{\Omega_+(t)} + \mu_- \mathbb{1}_{\Omega_-(t)}$, where $\mathbb{1}_A$ is the indicator function of $A \subset \mathbf{R}^3$.

Let $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))^\mathsf{T}$ be the fluid velocity and $\mathfrak{q} = \mathfrak{q}(x,t)$ the pressure at position $x \in \Omega_+(t) \cup \Omega_-(t)$ with time $t > 0$. The motion is governed by the two-phase Navier-Stokes equations with the sharp interface $\Gamma(t)$ as follows:

$$\begin{cases} \partial_t \eta - v_3 = -\mathbf{v}' \cdot \nabla' \eta & \text{on } \Gamma(t), \\ \rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = \mu \Delta \mathbf{v} - \nabla \mathfrak{q} & \text{in } \Omega_+(t) \cup \Omega_-(t), \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega_+(t) \cup \Omega_-(t); \end{cases} \tag{1}$$

the boundary conditions

$$\begin{cases} -[\![(\mu \mathbf{D}(\mathbf{v}) - \mathfrak{q}\mathbf{I})\mathbf{n}_{\Gamma(t)}]\!] - [\![\rho]\!]\gamma_a \eta \mathbf{n}_{\Gamma(t)} = \sigma \kappa_{\Gamma(t)} \mathbf{n}_{\Gamma(t)} & \text{on } \Gamma(t), \\ [\![\mathbf{v}]\!] = 0 & \text{on } \Gamma(t); \end{cases} \tag{2}$$

the initial conditions

$$\eta|_{t=0} = \eta_0 \quad \text{on } \mathbf{R}^2, \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega_{0+} \cup \Omega_{0-}. \tag{3}$$

The positive constants γ_a and σ denote the acceleration of gravity and the surface tension coefficient, respectively. Let $\partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$ for $j = 1, 2, 3$. Then $\mathbf{v}' \cdot \nabla' \eta = \sum_{j=1}^2 v_j \partial_j \eta$ and $\mathbf{D}(\mathbf{v}) = (\partial_i v_j + \partial_j v_i)_{1 \le i,j \le 3}$. Let $\mathbf{I}$ be the 3×3 identity matrix. In addition, $\mathbf{n}_{\Gamma(t)}$ is the unit normal vector on $\Gamma(t)$, pointing from $\Omega_-(t)$ into $\Omega_+(t)$, and $\kappa_{\Gamma(t)}$ the mean curvature of $\Gamma(t)$. For functions $f = f(x,t)$, $x \in \Omega_+(t) \cup \Omega_-(t)$ with $t > 0$, we set

$$[\![f]\!] = [\![f]\!](x_0, t) = \lim_{\varepsilon \to 0+0} \Big(f(x_0 + \varepsilon \mathbf{n}_{\Gamma(t)}, t) - f(x_0 - \varepsilon \mathbf{n}_{\Gamma(t)}, t) \Big) \quad (x_0 \in \Gamma(t)).$$

The $\eta_0 = \eta_0(x')$ and $\mathbf{v}_0 = \mathbf{v}_0(x)$ are given initial data, and

$$\Omega_{0\pm} = \{x = (x', x_3) \mid x' = (x_1, x_2), \pm(x_3 - \eta_0(x')) > 0\}.$$

Define the extension operators $\mathcal{E}_\pm$ by $\mathcal{E}_\pm\eta = \mathcal{F}^{-1}_{\xi'}[e^{\mp\sqrt{1+|\xi'|^2}x_3}\widehat{\eta}(\xi',t)](x')$, $\pm x_3 > 0$, where $\widehat{f}(\xi') = \int_{\mathbf{R}^2} e^{-ix'\cdot\xi'} f(x')\,dx'$ and $\mathcal{F}^{-1}_{\xi'}[g(\xi')](x') = (2\pi)^{-2}\int_{\mathbf{R}^2} e^{ix'\cdot\xi'} g(\xi')\,d\xi'$. Let $\Theta_\pm$ be diffeomorphisms such that

$$\begin{aligned}\Theta_\pm : \mathbf{R}^3_\pm \times (0,\infty) \ni (x,t) &= (x', x_3, t)\\ &\mapsto \Theta_\pm(x,t) := (x', x_3 + (\mathcal{E}_\pm\eta)(x',x_3,t), t) \in \bigcup_{\tau\in(0,\infty)} \Omega_\pm(\tau)\times\{\tau\},\end{aligned}$$

where $\mathbf{R}^3_\pm = \{\pm x_3 > 0\}$. One sets $\mathbf{u}_\pm = \mathbf{u}_\pm(x,t) = \mathbf{v}(\Theta_\pm(x,t))$ and $\mathfrak{p}_\pm = \mathfrak{p}_\pm(x,t) = \mathfrak{q}(\Theta_\pm(x,t))$, and then (1) becomes

$$\begin{cases} \partial_t\eta - \mathbf{u}_-\cdot\mathbf{e}_3 = \mathsf{D}(\eta,\mathbf{u}_-) & \text{on } \mathbf{R}^3_0\times(0,\infty),\\ \rho_\pm\partial_t\mathbf{u}_\pm - \mu_\pm\Delta\mathbf{u}_\pm + \nabla\mathfrak{p}_\pm = \mathsf{F}_\pm(\eta,\mathbf{u}_\pm) & \text{in } \mathbf{R}^3_\pm\times(0,\infty),\\ \operatorname{div}\mathbf{u}_\pm = \mathsf{G}_\pm(\eta,\mathbf{u}_\pm) = \operatorname{div}\widetilde{\mathsf{G}}_\pm(\eta,\mathbf{u}_\pm) & \text{in } \mathbf{R}^3_\pm\times(0,\infty),\end{cases} \tag{4}$$

where $\mathbf{R}^3_0 = \{x_3 = 0\}$; the boundary conditions in (2) become

$$\begin{cases} -\{(\mu_+\mathbf{D}(\mathbf{u}_+) - \mathfrak{p}_+\mathbf{I})\mathbf{e}_3 - (\mu_-\mathbf{D}(\mathbf{u}_-) - \mathfrak{p}_-\mathbf{I})\mathbf{e}_3\} - (\rho_+ - \rho_-)\gamma_a\eta\mathbf{e}_3 - \sigma\Delta'\eta\mathbf{e}_3\\ \qquad = \mathsf{H}_+(\eta,\mathbf{u}_+) - \mathsf{H}_-(\eta,\mathbf{u}_-) - \sigma\mathsf{H}_\kappa(\eta)\mathbf{e}_3 \quad \text{on } \mathbf{R}^3_0\times(0,\infty),\\ \mathbf{u}_+ - \mathbf{u}_- = 0 \quad \text{on } \mathbf{R}^3_0\times(0,\infty),\end{cases} \tag{5}$$

where $\mathbf{e}_3 = (0,0,1)^\mathsf{T}$ and $\Delta'\eta = \sum_{j=1}^2 \partial_j^2\eta$; the initial conditions in (3) become

$$\eta|_{t=0} = \eta_0 \quad \text{on } \mathbf{R}^2, \quad \mathbf{u}_\pm|_{t=0} = \mathbf{u}_{0\pm} \quad \text{in } \mathbf{R}^3_\pm. \tag{6}$$

Here $\mathsf{D}(\eta,\mathbf{u}_-)$, $\mathsf{F}_\pm(\eta,\mathbf{u}_\pm)$, $\mathsf{G}_\pm(\eta,\mathbf{u}_\pm)$, $\widetilde{\mathsf{G}}_\pm(\eta,\mathbf{u}_\pm)$, $\mathsf{H}_\pm(\eta,\mathbf{u}_\pm)$, and $\mathsf{H}_\kappa(\eta)$ are nonlinear terms. Our main result then reads as follows.

Theorem 1. *Suppose $\rho_- > \rho_+$. Then there exist a large number $p_0 > 2$ and small positive numbers q_0, r_0, ε_0 such that for any p, q_1, and q_2 satisfying*

$$p_0 \le p < \infty, \quad 2 < q_1 \le 2 + q_0, \quad 3 < q_2 < \infty,$$

and for any

$$(\eta_0, \mathbf{u}_{0+}, \mathbf{u}_{0-}) \in \bigcap_{r\in\{q_1/2, q_2\}} B^{3-1/p-1/r}_{r,p}(\mathbf{R}^2)\times B^{2-2/p}_{r,p}(\mathbf{R}^3_+)^3\times B^{2-2/p}_{r,p}(\mathbf{R}^3_-)^3$$

satisfying some compatibility condition and the smallness condition

$$\sum_{r\in\{q_1/2,q_2\}} \|(\eta_0,\mathbf{u}_{0+},\mathbf{u}_{0-})\|_{B^{3-1/p-1/r}_{r,p}(\mathbf{R}^2)\times B^{2-2/p}_{r,p}(\mathbf{R}^3_+)^3\times B^{2-2/p}_{r,p}(\mathbf{R}^3_-)^3} \le \varepsilon_0,$$

(4)–(6) *admits a unique solution $(\eta,\mathbf{u}_+,\mathbf{u}_-)$, with pressures $\mathfrak{p}_\pm$, satisfying*

$$\begin{aligned}\sum_{q\in\{q_1,q_2\}} \Big(&\|\langle t\rangle^{1/2}(\eta,\mathbf{u}_+,\mathbf{u}_-)\|_{L_p(\mathbf{R}_+,W^{3-1/q}_q(\mathbf{R}^2)\times H^2_q(\mathbf{R}^3_+)^3\times H^2_q(\mathbf{R}^3_-)^3)}\\ &+ \|\langle t\rangle^{1/2}(\partial_t\eta,\partial_t\mathbf{u}_+,\partial_t\mathbf{u}_-)\|_{L_p(\mathbf{R}_+,W^{2-1/q}_q(\mathbf{R}^2)\times L_q(\mathbf{R}^3_+)^3\times L_q(\mathbf{R}^3_-)^3)}\Big) \le r_0.\end{aligned}$$

Here $\langle t\rangle = \sqrt{1+t^2}$ and the compatibility condition is introduced in the talk.

Well-posedness for the magnetohydrodynamics with Hall-effect near non-zero magnetic equilibrium states

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We consider the initial-value problem for the incompressible magnetohydrodynamic system with the Hall-effect (we also call it the Hall-magnetohydrodynamic system) in the 3-dimensional Euclidean space $\mathbb{R}^3$:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = (\nabla \times \widetilde{B}) \times \widetilde{B}, & t > 0,\ x \in \mathbb{R}^3, \\ \partial_t \widetilde{B} - \Delta \widetilde{B} + \nabla \times \Big((\nabla \times \widetilde{B}) \times \widetilde{B} \Big) = \nabla \times (u \times \widetilde{B}), & t > 0,\ x \in \mathbb{R}^3, \\ \nabla \cdot u = \nabla \cdot \widetilde{B} = 0, & t > 0,\ x \in \mathbb{R}^3, \\ (u, \widetilde{B})|_{t=0} = (u_0, \widetilde{B}_0), & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$, $p = p(t,x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ and $\widetilde{B} = \widetilde{B}(t,x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ denote the velocity of the fluid, the pressure and the magnetic field, respectively. The third term of the second equations is often called the *Hall-term*. The system (1) is used to model the *magnetic reconnection phenomenon*, that is not able to be explained by the well-known magnetohydrodynamic system (namely, the system (1) without the Hall-term).

The aim of this talk is to obtain a solution as a perturbation from a constant equilibrium state $(0, \bar{B})$, where $\bar{B} \in \mathbb{R}^3$. In the case $\bar{B} = 0$, Danchin–Tan [1] have obtained a global-in-time solution of (1) in critical Besov spaces.

Under the assumption $\bar{B} \neq 0$, let us reformulate the system (1) by introducing the new unknown vector-valued function $B := \widetilde{B} - \bar{B}$ to get the following initial-value problem:

$$\begin{cases} \partial_t u - \Delta u + \nabla \left(p + \dfrac{|B|^2}{2} \right) - (\nabla \times B) \times \bar{B} = F, & t > 0, x \in \mathbb{R}^3, \\ \partial_t B - \Delta B + \nabla \times \Big((\nabla \times B) \times \bar{B} \Big) - \nabla \times (u \times \bar{B}) = \nabla \times G, & t > 0, x \in \mathbb{R}^3, \\ \nabla \cdot u = \nabla \cdot B = 0, & t > 0, x \in \mathbb{R}^3, \\ (u, B)|_{t=0} = (u_0, B_0), & x \in \mathbb{R}^3, \end{cases} \tag{2}$$

where the nonlinear terms F, G are given by

$$F := -(u \cdot \nabla) u + (B \cdot \nabla) B, \quad G := u \times B - (\nabla \times B) \times B, \tag{3}$$

respectively. In this talk, we shall state the results on the global well-posedness for the system (2) in a critical L^p-framework corresponding to Danchin–Tan [1].

References

[1] Danchin, R., Tan, J., *On the well-posedness of the Hall-magnetohydrodynamics system in critical spaces*, Comm. Partial Differential Equations, **46** (2021) 31–65.

The Helmholtz decomposition of a space of vector fields with bounded mean oscillation in a bounded domain

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The Helmholtz decomposition of a vector field is a fundamental tool to analyse the Stokes and the Navier-Stokes equations. It is formally a decomposition of a vector field v in a domain Ω of $\mathbf{R}^n$ into

$$v = v_0 + \nabla q$$

where v_0 is a divergence free vector field with supplemental condition like the boundary condition and ∇q denotes the gradient of a scalar field q. If v is in L^p in Ω for $1 < p < \infty$, such a decomposition is well-studied; see e.g.[1]. However, if the vector field is of bounded mean oscillation (BMO for short), such a problem is only studied when Ω is a half space $\mathbf{R}^n_+$ [2], where the boundary is flat. The goal of our result is to establish the Helmholtz decomposition of BMO vector fields in a bounded domain, which is a typical example of a domain with curved boundary.

Although the space of BMO functions in $\mathbf{R}^n$ is well-studied, the situation is less clear if we consider such a space in a domain since there are several possible definitions due to the presence of the boundary $\partial\Omega$. Here is our setting. Let Ω be a bounded C^3 domain. Let $\mu \in (0, \infty]$, for $f \in L^1_{loc}(\Omega)$ we define

$$[f]_{BMO^\mu(\Omega)} := \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}| \, dy \;\middle|\; B_r(x) \subset \Omega, \, r < \mu \right\}$$

where

$$f_{B_r(x)} := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, dy.$$

Then we define the space $BMO^\mu(\Omega)$ as

$$BMO^\mu(\Omega) := \{ f \in L^1_{loc}(\Omega) \mid [f]_{BMO^\mu(\Omega)} < \infty \}.$$

For $\nu \in (0, \infty]$, we set

$$[f]_{b^\nu} := \sup \left\{ r^{-n} \int_{B_r(x)\cap\Omega} |f(y)| \, dy \;\middle|\; x \in \partial\Omega, \, 0 < r < \nu \right\}$$

and let $d_\Omega(x)$ denote the distance of x from the boundary $\partial\Omega$, i.e.,

$$d_\Omega(x) = \inf \{ |x - y|, y \in \partial\Omega \}.$$

We then define the space

$$vBMO^{\mu,\nu}(\Omega) = \{v \in (BMO^{\mu}(\Omega))^n \mid [\nabla d_\Omega \cdot v]_{b^\nu} < \infty\}$$

where for $v \in vBMO^{\mu,\nu}(\Omega)$,

$$[v]_{vBMO^{\mu,\nu}(\Omega)} := [v]_{BMO^{\mu}(\Omega)} + [\nabla d_\Omega \cdot v]_{b^\nu}.$$

This space is introduced in our companion paper [3] in which the seminorm $[\cdot]_{vBMO^{\mu,\nu}(\Omega)}$ is shown to be equivalent as far as each index is finite. In particular, in the case for a bounded domain, $[\cdot]_{vBMO^{\mu,\nu}(\Omega)}$ is indeed a norm and the space $vBMO^{\mu,\nu}(\Omega)$ is a Banach space. Moreover, this space is independent of μ, ν including ∞. Hence, when Ω is bounded, without loss of generality we denote $vBMO^{\mu,\nu}(\Omega)$ by $vBMO(\Omega)$. The main theorem of our research reads as follow.

Theorem 1. *Let Ω be a bounded C^3 domain in $\mathbf{R}^n$. Then the topological direct sum decomposition*

$$vBMO(\Omega) = vBMO_\sigma(\Omega) \oplus GvBMO(\Omega)$$

holds with

$$vBMO_\sigma(\Omega) := \{v \in vBMO(\Omega) \mid \mathrm{div}\, v = 0 \text{ in } \Omega, v \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$GvBMO(\Omega) := \{\nabla q \in vBMO(\Omega) \mid q \in L^1_{\mathrm{loc}}(\Omega)\}$$

where $\mathbf{n}$ denotes the exterior unit normal vector field. In other words, for $v \in vBMO(\Omega)$, there exist unique $v_0 \in vBMO_\sigma(\Omega)$ and $\nabla q \in GvBMO(\Omega)$ satisfying $v = v_0 + \nabla q$. Moreover, the mappings $v \mapsto v_0$ and $v \mapsto \nabla q$ are bounded in $vBMO(\Omega)$.

For the space $vBMO(\Omega)$ that we consider, it is essential that we only require the normal component of v, i.e., $\nabla d_\Omega \cdot v$, to be b^ν bounded in order to have the Helmholtz decomposition. Requiring every component v_i of v to be b^ν bounded is too strict to have the Helmholtz decomposition.

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[2] Y. Giga and Z. Gu, On the Helmholtz decompositions of vector fields of bounded mean oscillation and in real Hardy spaces over the half space. Adv. Math. Sci. Appl. 29 (2020), 87-128.

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Weak and mild solutions to the Navier-Stokes equations in Wiener amalgam spaces

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For the three dimensional incompressible Navier-Stokes equations in the L^p setting, the classical theories give existence of weak solutions for data in L^2 and mild solutions for data in L^p, $p \geq 3$. These were extended to L^p_{uloc} spaces, the space of functions with uniform local L^p norms, by Lemarié-Rieusset for weak solutions, and by Maekawa and Terasawa for mild solutions. Our goal is to build existence theorems in intermediate spaces that bridge L^p and L^p_{uloc}.

The Wiener amalgam space $E(p,q)$ consists of functions whose local L^p-norms at lattice points are globally in ℓ^q. Thus a function in $E(p,q)$ has local integrability in L^p and global decay in ℓ^q.

For weak solutions, we establish global existence in $E(2,q)$ for q between 2 and ∞. For q close to 2, the solutions are shown to satisfy some properties known in the Leray class but not the Lemarié-Rieusset class, namely eventual regularity and long time estimates on the growth of the local energy.

For mild solutions, we establish local existence in $E(p,q)$ for $p \in (3,\infty]$ and $q \in [2,\infty]$. When $p = 3$, a further smallness assumption on the initial data ensures local existence if $q > 3$, and global existence if $q \leq 3$. We also prove local spacetime integral bounds of the solutions using Giga's estimates.

This talk is based on joint work with Zachary Bradshaw and Chen-Chih Lai.

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Rigidity of Beltrami fields with a non-constant proportionality factor

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Beltrami fields curl $u = fu$, div $u = 0$ appear as steady states of ideal incompressible flows or plasma equilibria. I will discuss existence and non-existence issues on them for non-constant factor f. In the first half of the talk, I will explain existence of axisymmetric Beltrami fields forming vortex rings and their construction via a variational principle. In the second half, I will discuss a rigidity problem on symmetry of u for symmetric f and a relation with Grad's conjecture. This talk is based on preprints arXiv:2008.09345, arXiv:2108.03870.

On barotropic Navier-Stokes system with general boundary conditions

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We consider the barotropic Navier–Stokes system describing the motion of a compressible Newtonian fluid in a bounded domain with in and out flux boundary conditions. That is, the time evolution of the mass density $\varrho = \varrho(t, x)$ and the velocity $\mathbf{u} = \mathbf{u}(t, x)$ is governed by

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0, \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) &= \operatorname{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \varrho \nabla_x G, \\
\mathbb{S}(\mathbb{D}_x \mathbf{u}) &= \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \lambda \geq 0, \\
\text{with } \mathbb{D}_x \mathbf{u} &\equiv \frac{1}{2} \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} \right),
\end{aligned}
\tag{1}
$$

on a bounded domain $\Omega \subset R^d$, $d = 1, 2, 3$ and we consider the realistic situation with a given boundary velocity,

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_b, \tag{2}$$

and, decomposing the boundary as

$$\partial\Omega = \Gamma_{in} \cup \Gamma_{out}, \ \Gamma_{in} = \left\{ x \in \partial\Omega \ \middle| \ \text{the outer normal } \mathbf{n}(x) \text{ exists, and } \mathbf{u}_b(x) \cdot \mathbf{n}(x) < 0 \right\},$$

we prescribe the density on the in–flow component,

$$\varrho|_{\Gamma_{in}} = \varrho_b. \tag{3}$$

Note that Γ_{out} includes the part of the boundary on which the field $\mathbf{u}_b$ is tangential, meaning $\mathbf{u}_b \cdot \mathbf{n} = 0$.

We show that if the boundary velocity coincides with that of a rigid motion, all solutions converge to an equilibrium state for large times. In other words, we study stability and convergence to the static states in the multi–dimensional case, with the velocity $\mathbf{u}_E$ associated to a *rigid motion*, meaning

$$\mathbb{D}_x \mathbf{u}_E = 0. \tag{4}$$

The corresponding density ϱ_E satisfies

$$
\begin{aligned}
\operatorname{div}_x(\varrho_E \mathbf{u}_E) &= 0, \\
\operatorname{div}_x(\varrho_E \mathbf{u}_E \otimes \mathbf{u}_E) + \nabla_x p(\varrho_E) &= \varrho_E \nabla_x G.
\end{aligned}
\tag{5}
$$

Accordingly, we consider the problem (1)–(3) with the boundary conditions

$$\mathbf{u}_b = \mathbf{u}_E \text{ on } \partial\Omega, \ \ \varrho_b = \varrho_E \text{ on } \Gamma_{in}. \tag{6}$$

Under the hypothesis (6), and if the stationary density ϱ_E is strictly positive, the problem (1)–(3) admits a Lyapunov function, namely the relative energy

$$\int_\Omega E\left(\varrho, \mathbf{u} \Big| \varrho_E, \mathbf{u}_E\right), \ E\left(\varrho, \mathbf{u} \Big| \varrho_E, \mathbf{u}_E\right) \equiv \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{u}_E|^2 + P(\varrho) - P'(\varrho_E)(\varrho - \varrho_E) - P(\varrho_E)\right].$$

The situation becomes more delicate if ϱ_E vanishes on a non–trivial part of Ω. In that case, the stationary problem may admit more (infinitely many) solutions even if the total mass is prescribed.

Our main result asserts that any *weak* solution of the problem (1)–(3), satisfying a suitable form of energy inequality, approaches the equilibrium solution $[\varrho_E, \mathbf{u}_E]$ as $t \to \infty$ as long as the stationary problem (5) admits a unique solution. To the best of our knowledge, this is the first result of this kind in the multi–dimensional case under the non–zero in/out flow boundary conditions. Note that such a result does not follow from "standard" arguments, even if $\varrho_E > 0$, as the Lyapunov function

$$t \mapsto \int_\Omega E\left(\varrho, \mathbf{u} \Big| \varrho_E, \mathbf{u}_E\right)(t, \cdot)$$

is not continuous on the trajectories generated by weak solutions.

Hadamard variational formula for the fundamental solution of the nonstationary Stokes equations

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$. We consider the nonstationary Stokes equations with the Dirichlet boundary conditions in Ω;

$$\begin{cases} \dfrac{d\boldsymbol{v}}{dt}(x,t) - \Delta_x \boldsymbol{v}(x,t) + \nabla_x p(x,t) = \boldsymbol{f}(x,t), & x \in \Omega,\ t>0, \\ \operatorname{div}_x \boldsymbol{v}(x,t) = 0, & x \in \Omega,\ t>0, \\ \boldsymbol{v}(x,t) = 0, & x \in \partial\Omega,\ t>0, \\ \boldsymbol{v}(x,0) = \boldsymbol{a}_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\boldsymbol{v}(x,t) = (v^1(x,t), v^2(x,t), v^3(x,t))$ and $p(x,t)$ denote unknown velocity and pressure, respectively. Furthermore, $\boldsymbol{f}(x) = (f^1(x), f^2(x), f^3(x))$ is a given external force and $\boldsymbol{a}(x) = (a^1(x), a^2(x), a^3(x))$ is an initial data. It is known that the solution of (1) can be expressed by

$$v^m(x,t) = \int_\Omega \boldsymbol{a}_0(y)\cdot \boldsymbol{U}_m(y-x,t)\,dy + \int_0^t \int_\Omega \boldsymbol{f}(x,\tau)\cdot \boldsymbol{U}_m(y-x,t-\tau)\,dyd\tau, \quad m=1,2,3$$

for a given $\boldsymbol{f}$ and $\boldsymbol{a}_0$, where $\boldsymbol{U}_m(x,t)$ is the fundamental solution for that.

The purpose of this article is to analyze the domain dependence of $\boldsymbol{U}_m(x,t)$. Such a problem was firstly considered in Hadamard [1] for the Green function of the Laplace equation, and in recent years, Kozono-Ushikoshi [2] and Ushikoshi [5] investigated that for the stationary Stokes equations. On the other hand, Ozawa [3] presented the variational formula for the fundamental solution of the heat equation, which was applied to determine the topological type of the domain by the variation of the eigenvalues for the Laplace operator in Ozawa [4]. We establish the Hadamard variationl formula for the fundamental solution of the nonstationary Stokes equations with the Dirichlet boundary conditions. This is the joint work with Mr. Masaru Kamiya who is a graduate of Yokohama National University.

2 Main Result

We assume that for every $\varepsilon \geq 0$, there is a diffeomorphism $\boldsymbol{\Phi}_\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ satisfying the following conditions.

(A.1) $\Phi_\varepsilon = (\phi_\varepsilon^1, \phi_\varepsilon^2, \phi_\varepsilon^3) \in C^\infty(\overline{\Omega})^3$.

(A.2) $\Phi_0(x) = x$ for all $x \in \overline{\Omega}$.

(A.3) There exists $\boldsymbol{S} = (S^1, S^2, S^3) \in C^\infty(\overline{\Omega})^3$ such that $K(x;\varepsilon) := \Phi_\varepsilon(x) - x - S(x)\varepsilon$ satisfies $\sup_{x\in\overline{\Omega}} |K(x;\varepsilon)| + \sup_{x\in\overline{\Omega}} |\nabla K(x;\varepsilon)| = O(\varepsilon^2)$ as $\varepsilon \to 0$.

(A.4) It holds that

$$\det \left(\frac{\partial \phi_\varepsilon^i(x)}{\partial x^j} \right)_{i,j=1,2,3} = 1 \qquad \text{for all } x \in \overline{\Omega} \text{ and all } \varepsilon \geq 0.$$

For the fundamental solution $\boldsymbol{U}_{\varepsilon,m}(x,t)$ of (1) in $\Omega_\varepsilon \times (0,T)$, the following theorem holds;

Theorem 1. *Let $\{\boldsymbol{U}_{\varepsilon,m}\}_{m=1,2,3}$ be the fundamental solution of (1) in $\Omega_\varepsilon \times (0,T)$. Then for any $y, z \in \Omega$ with $y \neq z$, there exists*

$$\delta U_m^k(y,z,t) := \lim_{\varepsilon\to 0} \frac{U_{\varepsilon,m}^k(y,z,t) - U_m^k(y,z,t)}{\varepsilon}$$

for all $t > 0$. Moreover, it is expressed by

$$\delta U_m^k(y,z,t) = \int_0^t \int_{\partial\Omega} \boldsymbol{S}(x)\cdot \nu_x \sum_{i=1}^3 \left(\frac{\partial U_m^i}{\partial \nu_x}(x,z,\tau) \frac{\partial U_k^i}{\partial \nu_x}(x,y,t-\tau) \right) d\sigma_x d\tau$$

for $k, m = 1,2,3$, where S is as in (A.3), $\nu_x = (\nu_x^1, \nu_x^2, \nu_x^3)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and σ_x denotes the surface element of $\partial\Omega$.

Remark 1. We assume that the domain is smoothly perturbed with keeping its volume. In order to remove this assumption, we need to simplify the method to construct its formula and make use of the piola transform.

The key lemma is as follows.

Lemma 2.1. *Let $\{\boldsymbol{V}_{\varepsilon,m}\}_{m=1,2,3}$ be the function defined in $\Omega_T := \Omega \times (0,T)$ by*

$$V_{\varepsilon,m}^k(x,y,t) := \sum_{j=1}^3 \frac{\partial x^k}{\partial \tilde{x}^j} U_{\varepsilon,m}^j(\boldsymbol{\Phi}_\varepsilon(x), \boldsymbol{\Phi}_\varepsilon(y), t), \qquad k = 1,2,3.$$

Then, for any $0 < \theta < 1$ and for any $y \in \Omega$, it holds that

$$\|(\boldsymbol{V}_{\varepsilon,m} - \boldsymbol{U}_m)(\cdot - y, \cdot)\|_{\Omega_T}^{2+\theta,\frac{\theta}{2}} \to 0 \quad \text{as } \varepsilon \to 0$$

for $m = 1,2,3$, where $\{\boldsymbol{U}_m\}_{m=1,2,3}$ is the fundamental solution of (1) in Ω_T and $\|\cdot\|_{\Omega_T}^{2+\theta,\frac{\theta}{2}}$ denotes the norm of $C^{2+\theta,\frac{\theta}{2}}(\Omega_T)$.

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Enhanced dissipation for the two-jet Kolmogorov type flow on the unit sphere

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A part of this talk is based on a joint work with Professor Yasunori Maekawa.

We consider the vorticity equation for a viscous fluid on the 2D unit sphere S^2 in $\mathbb{R}^3$ of the form

$$\partial_t\omega + \mathbf{u}\cdot\nabla\omega - \nu(\Delta\omega + 2\omega) = f, \quad \mathbf{u} = \mathbf{n}_{S^2}\times\nabla\Delta^{-1}\omega \quad \text{on} \quad S^2\times(0,\infty). \tag{1}$$

Here ω is the scalar vorticity, $\mathbf{u}$ is the tangential velocity field, and f is a given external force. Also, $\nu > 0$ is the viscosity coefficient, ∇ is the gradient on S^2, Δ is the Laplace–Beltrami operator on S^2 which is invertible on the space $L^2_0(S^2)$ of L^2 functions on S^2 with vanishing mean, $\mathbf{n}_{S^2}$ is the unit outward normal vector field of S^2, and $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are the inner and vector products of $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$. Note that the zeroth order term $2\nu\omega$ appears in (1) since we take the viscous them in the Navier–Stokes equations for the velocity field $\mathbf{u}$ as twice of the divergence of the deformation tensor for $\mathbf{u}$ (see [6]).

Let Y_n^m with $n \geq 0$ and $|m| \leq n$ be the spherical harmonics and $\lambda_n = n(n+1)$ the eigenvalue of $-\Delta$ corresponding to Y_n^m with $|m| \leq n$. For $n \geq 1$ and $a \in \mathbb{R}$, the vorticity equation (1) with external force $f = a\nu(\lambda_n - 2)Y_n^0$ has a stationary solution $\omega_n^a = aY_n^0$. Such a stationary flow can be seen as a spherical version of the Kolmogorov flow on the 2D flat torus and we call it the n-jet Kolmogorov type flow. In this talk, we focus on the case $n = 2$. The linearized equation of (1) around the two-jet Kolmogorov type flow $\omega_2^a = ac_2^0(3\cos^2\theta - 1)$ (here c_2^0 is a constant) is of the form (after relabeling $a \in \mathbb{R}$)

$$\begin{aligned} \partial_t\omega = \mathcal{L}^{\nu,a}\omega = \nu A\omega - ia\Lambda\omega, \quad \omega|_{t=0} = \omega_0 \quad \text{in} \quad L^2_0(S^2), \\ A = \Delta + 2, \quad \Lambda = \cos\theta(-i\partial_\varphi)(I + 6\Delta^{-1}), \end{aligned} \tag{2}$$

where θ and φ are the colatitude and longitude.

We are interested in the behavior of a solution $\omega(t) = e^{t\mathcal{L}^{\nu,a}}\omega_0$ to (2) as $\nu \to 0$. When $(\omega_0, Y_1^m)_{L^2(S^2)} = 0$ for $m = 0, \pm1$, a standard energy method shows that $e^{t\mathcal{L}^{\nu,a}}\omega_0$ decays at the rate $O(e^{-\nu t})$. In the case of the plane Kolmogorov flow [1, 3, 2, 8, 9], however, it is shown that a solution to the linearized equation decays at a rate faster than $O(e^{-\nu t})$ when ν is sufficiently small. Such a phenomenon is called the enhanced dissipation, and our aim is to study the enhanced dissipation for the solution $e^{t\mathcal{L}^{\nu,a}}\omega_0$ to (2).

In fact, we obtained the enhanced dissipation for $e^{t\mathcal{L}^{\nu,a}}\omega_0$ first without a precise decay rate in [5] and then with the decay rate $O(e^{-\sqrt{\nu}t})$ in [4], which is the same as in the plane case [1, 2, 8, 9]. There results themselves, however, are proved just by applications of the abstract results given by Ibrahim–Maekawa–Masmoudi [2] and of the Gearhart–Prüss type theorem shown by Wei [7]. So in this talk we would like to focus on a somewhat new idea for the spectral analysis of the perturbation operator Λ arising in the study of the enhanced dissipation. To apply the abstract results of [2], we need to show that Λ

does not have nonzero eigenvalues. In order to prove it, one typically analyzes an ODE associated with Λ and applies the uniqueness of a (smooth) solution to the ODE to show that a solution to the eigenvalue problem identically vanishes. Such an ODE approach is used in the plane case [3, 2, 9], but in our case we encounter a difficulty due to the size of the coefficient of Δ^{-1} in Λ, and it seems to be too difficult to deal with this difficulty by the ODE approach. Instead, to overcome this difficulty, we make use of the *mixing* property of Λ expressed by the recurrence relation for the spherical harmonics

$$\cos\theta\, Y_n^m = a_n^m Y_{n-1}^m + a_{n+1}^m Y_{n+1}^m \tag{3}$$

with nonnegative coefficients a_n^m. In the actual proof, we use (3), a Hardy type inequality on S^2, and $\lambda_n \to \infty$ as $n \to \infty$ to show that the high frequency part (with respect to the index n in the expansion by Y_n^m) of a solution to the eigenvalue problem vanishes, and then apply (3) again to find that the low frequency part also vanishes.

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Vortex reconnection and a finite-time singularity of the Navier-Stokes equations

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As a fundamental process in both classical and quantum turbulence, vortex reconnection has intensively been studied over recent decades. Recently we have developed an analytical model of vortex reconnection challenging to study the finite singularity problem for the Naiver-Stokes equations [1],[2]. In this model, two circular vortex rings of circulation $\pm\Gamma$ and radius $R = 1/\kappa$ are symmetrically placed on two planes inclined to the plane $x = 0$ at angles $\pm\alpha$. Under an assumption that the vortex Reynolds number, $R_\Gamma = \Gamma/\nu$, is very large, we have derived a nonlinear dynamical system for the local behavior near the points of closest approach of the vortices (tipping points). Careful numerical investigation of the dynamical system reveals that the magnitude of vorticity could take any large value for small viscosity but remains finite since the minimum core radius never becomes zero.

In this talk, the analytic model will be illustrated after a brief review of the problem and some preliminary numerical model [3] are presented. The assumptions for the analysis are far beyond the ones that the current DNS could attain, but we try to compare the results of the analytic model and the DNS and other numerical simulations. Finally some new development of the problem will be introduced [4].

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Fast rotation limit for the incompressible Navier-Stokes equations in a 3D layer

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Let $\mathbb{D} := \mathbb{R}^2 \times \mathbb{T}$ be a three-dimensional layer. Here, $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1]$ is the one-dimensional torus, and the point of $\mathbb{D}$ is denoted by (x,z) with the horizontal variable $x = (x_1, x_2) \in \mathbb{R}^2$ and the vertical variable $z \in \mathbb{T}$.

In this talk, we consider the initial value problem for the rotating Navier-Stokes equations, describing the motion of incompressible viscous fluids around the rotating vector field $\Omega/2(-x_2, x_1, 0)$ in $\mathbb{D}$:

$$\begin{cases} \partial_t u - \Delta u + \Omega(e_3 \times u) + (u \cdot \nabla)u + \nabla p = 0 & t > 0,\ (x,z) \in \mathbb{D}, \\ \nabla \cdot u = 0 & t \geqslant 0,\ (x,z) \in \mathbb{D}, \\ u(0,x,z) = u_0(x,z) & (x,z) \in \mathbb{D}. \end{cases} \tag{1}$$

Here, $u = u(t,x,z) = (u_1(t,x,z), u_2(t,x,z), u_3(t,x,z))$ and $p = p(t,x,z)$ denote the unknown velocity field and the unknown pressure, respectively, while $u_0 = u_0(x,z) = (u_{0,1}(x,z), u_{0,2}(x,z), u_{0,3}(x,z))$ denotes the initial velocity field. The constant $\Omega \in \mathbb{R}$ represents the rotating speed around the vertical unit vector $e_3 = (0,0,1)$.

The main purpose of this talk is to prove the unique existence of global in time solutions to (1) for the initial data in scaling critical spaces, and study the asymptotics of solutions when the rotating speed $|\Omega|$ tends to infinity.

Before stating our results, we review the known results on the global existence of solutions to (1). In the whole space $\mathbb{R}^3$, Chemin, Desjardins, Gallagher and Grenier [1, 2] proved that for given initial velocity $u_0 = v_0 + w_0 \in L^2(\mathbb{R}^2)^3 + \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$, there exists a positive parameter $\Omega_0 = \Omega_0(u_0)$ such that for every $\Omega \in \mathbb{R}$ with $|\Omega| \geqslant \Omega_0$ the rotating Navier-Stokes equations (1) possesses a unique global solution. Furthermore, they [1, 2] showed that the global solution to (1) converges to that of the 2D Navier-Stokes equations with the initial data v_0 in the local in time norm $L^2_{\mathrm{loc}}(0,\infty; L^q(\mathbb{R}^3))$ for $2 < q < 6$ as $|\Omega| \to \infty$. In the 3D infinite layer $\mathbb{D}$, Gallay and Roussier-Michon [3] proved the global existence and the long-time asymptotics of infinite-energy solutions to (1) for large $|\Omega|$. They [3] decomposed the initial data as $u_0 = \bar{u}_0 + \tilde{u}_0$ with $\bar{u}_0(x) = \int_{\mathbb{T}} u_0(x,z)\,dz$ and $\tilde{u}_0 = u_0 - \bar{u}_0$, and showed that for given initial data $u_0 \in H^1_{\mathrm{loc}}(\mathbb{D})^3$ satisfying $\tilde{u}_0 \in H^1(\mathbb{D})^3, \bar{u}_{0,3} \in H^1(\mathbb{R}^2), \partial_1 \bar{u}_{0,2} - \partial_2 \bar{u}_{0,1} \in (L^1 \cap L^2)(\mathbb{R}^2)$, there exists a $\Omega_0 = \Omega_0(u_0) > 0$ such that (1) has a unique global solution u provided that $|\Omega| \geqslant \Omega_0$. Moreover, it is shown in [3] that the global solution converges to the two-dimensional Lamb-Oseen vortex in $L^1(\mathbb{R}^2)$ as $t \to \infty$.

Following the idea in [3], we decompose the velocity fields as $u(t,x,z) = \bar{u}(t,x) + \tilde{u}(t,x,z)$, where

$$\bar{u}(t,x) = (\mathcal{Q}u)(t,x) := \int_{\mathbb{T}} u(t,x,z)\,dz$$

is the average of u with respect to the vertical variable z, and we set $\tilde{u} := (1 - \mathcal{Q})u$. Note that $\tilde{u}$ has zero vertical average $\int_{\mathbb{T}} \tilde{u}(t,x,z)\,dz = 0$. Similarly to the whole space $\mathbb{R}^3$ case in [1, 2], the limit equation is the 2D incompressible Navier-Stokes equations for the three-components velocity fields:

$$\begin{cases} \partial_t u^\infty - \Delta_h u^\infty + (u_h^\infty \cdot \nabla_h) u^\infty + (\nabla_h q, 0) = 0 & t > 0,\ x \in \mathbb{R}^2, \\ \nabla_h \cdot u_h^\infty = 0 & t \geqslant 0,\ x \in \mathbb{R}^2, \\ u^\infty(0, x) = \bar{u}_0(x) & x \in \mathbb{R}^2, \end{cases} \tag{2}$$

where $\nabla_h = (\partial_1, \partial_2)$, $\Delta_h = \partial_1^2 + \partial_2^2$, $u_h^\infty = (u_1^\infty, u_2^\infty)$ and $q = q(t, x)$ is a 2D pressure.

The main result of this talk reads as follows:

Theorem 1. *For $u_0 = \bar{u}_0 + \tilde{u}_0 \in L^2(\mathbb{R}^2)^3 + (1 - \mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})^3$ with $\nabla_h \cdot \bar{u}_{0,h} = \nabla \cdot \tilde{u}_0 = 0$, there exists a $\Omega_0 = \Omega_0(\bar{u}_0, \tilde{u}_0) > 0$ such that, for all $\Omega \in \mathbb{R}$ with $|\Omega| \geqslant \Omega_0$, the rotating Navier-Stokes equation (1) admits a unique global solution $u = \bar{u} + \tilde{u}$ satisfying*

$$\bar{u} \in C([0, \infty); L^2(\mathbb{R}^2))^3 \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^2))^3,$$
$$\tilde{u} \in C([0, \infty); (1 - \mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D}))^3 \cap L^2(0, \infty; (1 - \mathcal{Q})\dot{H}^{\frac{3}{2}}(\mathbb{D}))^3.$$

Furthermore, for $2 < p, q < \infty$ with $2/p + 2/q = 1$, there holds

$$\lim_{|\Omega| \to \infty} \|u - u^\infty(t)\|_{L^p(0,\infty; L^q(\mathbb{D}))} = 0. \tag{3}$$

Here, u^∞ is the unique global solution of the limit equation (2) with the initial data $\bar{u}_0$ in the class $u^\infty \in C([0, \infty); L^2(\mathbb{R}^2))^3 \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^2))^3$.

Remark 1. Let p, q satisfy $2 < p, q < \infty$ and $2/p + 2/q = 1$. Then, by the Sobolev embeddings $\dot{H}^{1-\frac{2}{q}}(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$, $H^{\frac{3}{2}(1-\frac{2}{q})}(\mathbb{D}) \hookrightarrow L^q(\mathbb{D})$ and the Poincaré inequality in $\mathbb{D}$, we have the following relations:

$$L^\infty(0, \infty; L^2(\mathbb{R}^2)) \cap L^2(0, \infty; \dot{H}^1(\mathbb{R}^2)) \hookrightarrow L^p(0, \infty; L^q(\mathbb{R}^2)),$$
$$L^\infty(0, \infty; (1 - \mathcal{Q})\dot{H}^{\frac{1}{2}}(\mathbb{D})) \cap L^2(0, \infty; \dot{H}^{\frac{3}{2}}(\mathbb{D})) \hookrightarrow L^p(0, \infty; (1 - \mathcal{Q})L^q(\mathbb{D})).$$

In the proof of Theorem 1, we adapt the ideas in [1, 2, 3], and decompose the solution to (1) as $u = \bar{u} + \tilde{u} = u^\infty + (\bar{u} - u^\infty) + \lambda + (\tilde{u} - \lambda)$, where λ is the linear solution associated with the initial data $\tilde{u}_0$ whose Fourier transform is compactly supported in some ball. The key ingredients of the proof are the Strichartz estimates for the linear solution λ, and the global energy estimates for the perturbations $\bar{u} - u^\infty$ and $\tilde{u} - \lambda$.

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