Hitchin-Mochizuki morphism, Opers and Frobenius-destabilized vector bundles over curves

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Joint work with Christian Pauly

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RIMS talk June 3^{rd}, 2011

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Notations

- Let p be a prime number.
- Let k be an algebraically closed field of characteristic p.
- Let X/k be a smooth, projective curve over k.
- Let $g = g(X)$ be the genus of X.
- We will always assume $g \geq 2$.
- Let $\sigma : k \to k$ be the Frobenius map $x \mapsto x^p$,
- Let $S = \mathrm{Spec}(k)$
- Let Ω^1_X be the canonical bundle of X.
- Let T_X be the tangent bundle of X.

The Frobenius Morphism

- Let $F : X \to X$ be the absolute Frobenius morphism of X.
- This is the following morphism of schemes: it is identity on the topological space underlying X
- and on the sheaf of functions $\mathcal{O}_X \to \mathcal{O}_X$ it is the map $f \mapsto f^p$ (for local sections of $\mathcal{O}_X$).
- If we work over a base scheme S, we can also define a relative Frobenius morphism $F_{X/S} : X \to S$.

Semistability and stability

For any vector bundle V on X let

$$\mu(V) = \frac{\deg(V)}{\mathrm{rk}\,(V)},$$

where $\deg(V)$ is the degree of V and $\mathrm{rk}\,(V)$ is the rank of V. A vector bundle V on X is stable (resp. semi-stable) if for any non-zero sub-bundle $W \subset V$, we have

$$\mu(W) < \mu(V) \text{ (resp. } \mu(W) \leq \mu(V)).$$

A non-zero sub-bundle $(W \subset V)$ *with* $(\mu(W) \geq \mu(V))$ will be called a destabilizing sub-bundle.

The Harder-Narasimhan filtration

Recall that there exists, on every vector bundle V, a unique filtration $0 \subset V_1 \subset V_2 \subset \cdots \subset V_\ell = V$ with the following properties:

- every quotient V_{i+1}/V_i is semistable, and
- if $\mu_i = \mu(V_i/V_{i-1})$ then $\mu_1 > \mu_2 > \cdots > \mu_\ell$.

This is called the Harder-Narasimhan filtration of V.

Harder-Narasimhan polygons

The numerical data of Harder-Narasimhan filtration is conveniently encoded in a convex polygon with break-points at $(0, 0)$ and the points $(\mathrm{rk}\,(V_i), \deg(V_i))$ and the line segment joining $(\mathrm{rk}\,(V_i), \deg(V_i)), (\mathrm{rk}\,(V_{i-1}), \deg(V_{i-1}))$ has slope μ_i. This is called the Harder-Narasimhan polygon of V.

The p-curvature

Given a local system (V, ∇) over X, i.e. a pair (V, ∇) consisting of a vector bundle V over X and a connection ∇ on V, we have the *p*-curvature

$$\psi(V, \nabla) : T_X \to \mathrm{End}(V), \qquad D \mapsto \nabla(D)^p - \nabla(D^p).$$

Here D denotes a local vector field, D^p its p-th power (which again a vector field) and $\mathrm{End}(V)$ denotes the sheaf of $\mathcal{O}_X$-linear endomorphisms of V.

Cartier's Theorem

Theorem

1. *Let E be a vector bundle over X. The pull-back $F^*(E)$ under the Frobenius morphism carries a canonical connection ∇^{can}, which satisfies the equality $\psi(F^*(E), \nabla^{can}) = 0$.*
2. *Given a local system (V, ∇) over X, there exists a vector bundle E such that $(V, \nabla) = (F^*(E), \nabla^{can})$ if and only if $\psi(V, \nabla) = 0$.*

Nilpotence

A connection (V, ∇) is nilpotent if there is a filtration $W_\bullet$ on V which is preserved by ∇ and the induced connection on the associated graded has p-curvature zero.

Opers I

- Opers were introduced by A. Beilinson and V. Drinfeld in their study of Geometric Langlands correspondence.
- The local avatar goes back to Drinfel'd-Sokolov (study of Poisson reduction).
- Indigenous bundles on Riemann surfaces (rank two opers) appeared in works of Gunning, Mandelstam.
- In the seventies Ihara studied the Schwarzian differential equation in arithmetic.
- In 1993-96 S. Mochizuki studied indigenous bundles (always of rank two) (now called PGL_2-opers) in positive characteristic.
- General opers in characteristic $p > 0$ also appeared naturally in JRXY.

Definition

An oper over a smooth algebraic curve X defined over an algebraically closed field k of characteristic $p > 0$ is a triple $(V, \nabla, V_\bullet)$, where

1. V is a vector bundle over X,
2. ∇ is a connection on V,
3. $V_\bullet : 0 = V_l \subset V_{l-1} \subset \cdots \subset V_1 \subset V_0 = V$ is a filtration by subbundles of V, called the oper flag.

These data have to satisfy the following conditions

1. $\nabla(V_i) \subset V_{i-1} \otimes \Omega^1_X$ for $1 \leq i \leq l-1$,
2. the induced maps $(V_i / V_{i+1}) \xrightarrow{\nabla} (V_{i-1}/V_i) \otimes \Omega^1_X$ are isomorphisms for $1 \leq i \leq l-1$.

Opers II: Nilpotent and Dormant opers

Definition

We say that an oper $(V, \nabla, V_\bullet)$ is nilpotent if (V, ∇) is nilpotent. We say an oper $(V, \nabla, V_\bullet)$ is dormant if (V, ∇) has p-curvature zero.

The term dormant is due to S. Mochizuki.

Indigenous bundles

An indigenous bundle (V, ∇) is an oper $(V, \nabla, V_\bullet)$ with $\mathrm{rk}\,(V) = 2$ and $\wedge^2(V, \nabla) = (\mathcal{O}_X, \nabla = d)$.

Fundamental Example

- Let θ be a line bundle with $\theta^{\otimes 2} \simeq \Omega^1_X$.
- Let $V \in \mathrm{Ext}^1(\theta^{-1}, \theta)$ be the unique (up to isom.) non-split extension.
- Then any connection ∇ on V makes $(V, \nabla, V_\bullet)$ into an oper (of rank two).
- By Weil's Theorem there are connections on V (this is true even in characteristic $p > \mathrm{rk}(V)$).

Let $\mathrm{Loc}_{\mathrm{GL}(r)}$ (resp. $\mathrm{Loc}_{\mathrm{SL}(r)}$) be stacks parameterizing rank-r local systems over X (resp. rank-r local systems with trivial determinant).We have a Hitchin morphism

$$\begin{array}{ccc} \mathrm{Loc}_{\mathrm{GL}(r)} & \longrightarrow & \oplus_{i=1}^{r} H^0(X, F^*(\Omega_X^1)^{\otimes i}), \\ (V, \nabla) & \mapsto & \mathrm{Char}(\psi(V, \nabla)), \end{array}$$

where $\mathrm{Char}(\psi(V,\nabla))$ is the characteristic polynomial of the p-curvature $\psi(V,\nabla) : V \to V \otimes F^*(\Omega_X^1)$. Let $\mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ be the stack of $\mathrm{PGL}(r)$-opers on X.

A beautiful observation of Mochizuki, rediscovered a few years later by Laszlo-Pauly, is that the components of $\mathrm{Char}(\psi(V, \nabla))$ descend under the Frobenius morphism.
That is the morphism $(V, \nabla) \mapsto \mathrm{Char}(\psi(V, \nabla))$ factorizes as

$$\mathrm{Loc}_{\mathrm{GL}(r)} \xrightarrow{\Phi} \oplus_{i=1}^{r} H^0(X, (\Omega_X^1)^{\otimes i}) \xrightarrow{F^*} \oplus_{i=1}^{r} H^0(X, F^*(\Omega_X^1)^{\otimes i}).$$

The Hitchin-Mochizuki morphism

The Hitchin-Mochizuki morphism is the morphism

$$\mathrm{HM} : \mathfrak{Op}_{\mathrm{PGL}(r)}(X) \to \oplus_{i=2}^{r} H^0(X, (\Omega_X^1)^{\otimes i})$$

which assigns to an oper $(V, \nabla, V_\bullet)$ the p^{th}-root $\psi(V, \nabla)^{1/p}$. This makes sense because of the factorization property stated earlier. The stack $\mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ is in fact an affine scheme, non-canonically isomorphic to the target of *HM*.

Finiteness of nilpotent opers

We will denote by $\mathrm{Nilp}_r(X) := \mathrm{HM}^{-1}(0) \subset \mathfrak{Op}_{\mathrm{PGL}(r)}(X)$ the fiber over 0 of the Hitchin-Mochizuki morphism.
It parameterizes nilpotent $\mathrm{PGL}(r)$-opers and contains in particular dormant $\mathrm{PGL}(r)$-opers.

Theorem (Main Theorem I)

The scheme $\mathrm{Nilp}_r(X)$ is finite.

For $r = 2$ this is due to S. Mochizuki and lies at the heart of his *p*-adic uniformization program.

The key observation we have is: the Hitchin-Mochizuki morphism is a also a Hitchin map–so it should be proper. On the other hand it is a proper map between two affine schemes. So it is finite.
We prove finiteness by proving properness (via a valuative criterion).
This provides a more conceptual philosophical framework for those interested in meditating up on such matters.

Valuative criterion for opers

Proposition

Let R be a discrete valuation ring and let s and η be the closed and generic point of $\mathrm{Spec}(R)$. For any nilpotent $\mathrm{SL}(r)$-oper $(V_\eta, \nabla_\eta, (V_\eta)_\bullet)$ over $X \times \mathrm{Spec}(K)$ there exists a nilpotent $\mathrm{SL}(r)$-oper $(V_R, \nabla_R, (V_R)_\bullet)$ over $X \times \mathrm{Spec}(R)$ extending $(V_\eta, \nabla_\eta, (V_\eta)_\bullet)$.

Stability and Frobenius

- In 1970s Mumford and later Gieseker found examples of vector bundles V on smooth, projective curves X such that $F^*(V)$ is not semistable.
- The question of understanding this phenomena was first raised by Mumford.

The sheaf locally of exact differentials

Now let $B^1_X = d(\mathcal{O}_X) \subset \Omega^1_X$, then Leibnitz rule

$$d(f^p g) = f^p dg$$

shows that d is linear with respect to the Frobenius and that B^1_X lives naturally as a locally free subsheaf of $F_*(\Omega^1_X)$.
Thus B^1_X is a vector bundle on X of rank $p - 1$ and slope $g - 1$.
Note that B^1_X is not a vector bundle in characteristic zero.

Raynaud's Theorem (1982)

The theorem is the following.

Theorem

For $p > 2$ the bundle B^1_X is semistable and $F^*(B^1_X)$ is not semistable.

In 2004 I proved that B^1_X is stable.

The instability locus

- Let $\mathcal{M}(r)$ denote the coarse moduli space of S-equivalence classes of semistable bundles of rank r and degree 0 over the curve X.
- Let $\mathcal{J}(r) \subset \mathcal{M}(r)$ be the locus of semistable bundles E which are destabilized by Frobenius pull-back, i.e. $F^*(E)$ is not semistable.
- Set-theoretically the locus $\mathcal{J}(r)$ is well-defined, since, given a strictly semistable bundle E with associated graded $\mathrm{gr}(E) = E_1 \oplus \cdots \oplus E_l$ with E_i stable, one observes that E is Frobenius-destabilized if and only if at least one of the stable summands E_i is Frobenius-destabilized.
- Moreover, $\mathcal{J}(r)$ is a closed subvariety of $\mathcal{M}(r)$. Let $\mathcal{J}^s(r) \subset \mathcal{J}(r)$ be the open subset corresponding to stable bundles

The case $p = 2, g = 2, r = 2$

This is rather special:

- The moduli of semistable bundles of trivial determinant is smooth and $\simeq \mathbb{P}^3$, with the locus of non-stable bundles embedded as the Kummer of the Jacobian of X.
- Mehta in the mid 1990s observed (in private conversations) that the instability locus is finite as its complement contains the Kummer–an ample divisor in $\mathbb{P}^3$.
- In 2000 with Eugene Xia we gave a complete classification of all vector bundles of degree one and $p = 2, g = 2, r = 2$ which are destabilized by Frobenius.

- In 2004 that Laszlo-Pauly (ordinary X) and later (2005-2008) Laszlo-Pauly, Ducrohet completed the analysis (for non-ordinary $g = 2$).
- Later Osserman (2006,2008) provided analysis for $g = 2, r = 2, p \leq 5$ (and X general).
- The Laszlo-Pauly, Ducrohet, Osserman approaches are quite computational and involves explicit equations which cannot be generalized beyond $g = 2$ as the equations become increasingly complicated with p (for $g = 2$).

The case $p = 2, g \geq 2, r = 2$

- This case was completely dealt with from a completely intrinsic viewpoint in a joint work (JRXY) of Joshi, S. Ramanan, Eugene Xia and Jiu-Kang Yu.
- We gave a complete construction of all Frobenius destabilized bundles in all degrees and all genus $g \geq 2$ and gave a complete construction of the instability locus.
- We also proved: if $p \leq 5$ and M a line bundle, then $F_*(M)$ is stable.

The following is natural after JRXY:

Question

Given a semistable vector bundle M on X, is it true that $F_*(M)$ (the push-forward) is semistable?

- In 2007 Lange-Pauly proved that for any line bundle M, $F_*(M)$ is stable.
- Soon Mehta-Pauly showed that if M (of degree zero) is semi-stable then $F_*(M)$ is semi-stable.
- The Mehta-Pauly approach is, philosophically speaking, similar to Raynaud's approach: show a bundle is semistable by showing that it has a theta divisor.
- On the other hand my approach to stability of B^1_X did not use theta divisors at all.

In 2008 Xiao-Tao Sun showed that

Theorem

If $g \geq 2$ and if M is any stable (resp. semistable) bundle then $F_*(M)$ is stable (resp. semistable).

Sun's beautiful proof: improves certain slope-bounds proved for proving stability of B^1_X, $F_*(L)$ and uses a critical construction of JRXY: that $F^*(F_*(M))$ carries a canonical filtration.

The JRXY filtration alluded to here is the following: Let $V_p = V = F^*(F_*(E))$,

$$V_{p-1} = \ker(V_p = F^*(F_*(E)) \to E),$$

for $0 \leq i \leq p-2$ let

$$V_i = \ker(V_{i+1} \to V \otimes \Omega^1_X \to (V/V_{i+1}) \otimes \Omega^1_X).$$

Opers III

The following result which combines the results of JRXY and Sun provides the basic example of opers in characteristic $p > 0$.

Theorem

Let E be any stable vector bundle over X and let $F : X \to X$ be the absolute Frobenius of X. Then the triple

$$(V = F^*(F_*(E)), \nabla^{can}, V_\bullet),$$

where $V_\bullet$ is the canonical filtration defined by JRXY is a dormant oper of type $\mathrm{rk}\,(E)$ and length p. Moreover, there is an equality $V_0/V_1 = E$.

Main Theorem II

Theorem

Let $r \geq 2$ be an integer and put $C(r, g) = r(r-1)(r-2)(g-1)$. If $p > C(r, g)$, then we have

1. *Every stable, Frobenius-destabilized vector bundle V of rank r and slope $\mu(V) = \mu$ over X is a subsheaf $V \hookrightarrow F_*(Q)$ for some stable vector bundle Q of rank $\mathrm{rk}\,(Q) < r$ and $\mu(Q) < p\mu$.*
2. *Conversely, given a semistable vector bundle Q with $\mathrm{rk}\,(Q) < r$ and $\mu(Q) < p\mu$, every subsheaf $V \hookrightarrow F_*(Q)$ of rank $\mathrm{rk}\,(V) = r$ and slope $\mu(V) = \mu$ is semistable and destabilized by Frobenius.*

- For $p = 2$ this result was proved by JRXY.
- The road, from $p = 2$, to the above theorem involves considerable issues which simply do not show up for $p = 2$.
- For instance: one major difficulty is this: suppose $F^*(V)$ is not semi-stable and say $F^*(V) \to Q$ is a destabilizing quotient of minimal slope, then by adjunction one obtains a morphism $V \to F_*(Q)$.
- But is this mapping injective? This is a major headache but relatively simpler to handle for $p = 2$ or $r = 2$ (as was shown by JRXY).

The degree zero case

In the case when the degree of the Frobenius-destabilized bundle equals 0, we the following.

Theorem (Main Theorem III)

Let X be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic p. If $p > C(r, g)$, then every stable, Frobenius-destabilized vector bundle V of rank r and of degree 0 over X is a subsheaf

$$V \hookrightarrow F_*(Q)$$

for some stable vector bundle Q of rank $\mathrm{rk}\,(Q) < r$ and degree $\deg(Q) = -1$.

A quot-scheme

- Let $1 \leq q \leq r-1$ be an integer and let $\mathcal{M}(q,-1)$ be the moduli space of semistable bundles of rank q and degree -1 over the curve X.
- As $\gcd(q,-1)=1$ we are in the coprime case and so every semistable bundle $Q \in \mathcal{M}(q,-1)$ is stable.
- So there exists a universal Poincaré bundle $\mathcal{U}$ on $\mathcal{M}(q,-1) \times X$.
- Let $\mathcal{Q}\mathrm{uot}(q,r,0)$ be the relative Quot-scheme:

$$\alpha : \mathrm{Quot}^{r,0}((F \times \mathrm{id}_{\mathcal{M}(q,-1)})_* \mathcal{U}) \longrightarrow \mathcal{M}(q,-1)$$

- The fibre $\alpha^{-1}(Q)$ over a point $Q \in \mathcal{M}(q,-1)$ equals $\mathrm{Quot}^{r,0}(F_*(Q))$ of quotients $F_*(Q)$ with associated kernel of rank r and degree 0.

Theorem (Main Theorem IV)

If $p > C(r, g)$, then the image of the forgetful morphism

$$\pi : \coprod_{q=1}^{r-1} \mathcal{Q}\mathrm{uot}(q, r, 0) \longrightarrow \mathcal{M}(r), \qquad [E \subset F_*(Q)] \mapsto E$$

is contained in the locus $\mathcal{J}(r)$ and contains the closure $\overline{\mathcal{J}^s(r)}$ of the stable locus $\mathcal{J}^s(r)$.

Oper polygons

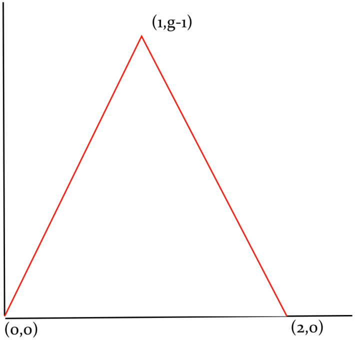
We introduce the oper-polygon

$$\mathscr{P}_r^{oper} : \quad \text{with vertices } (i, i(r-i)(g-1)) \text{ for } 0 \leq i \leq r.$$

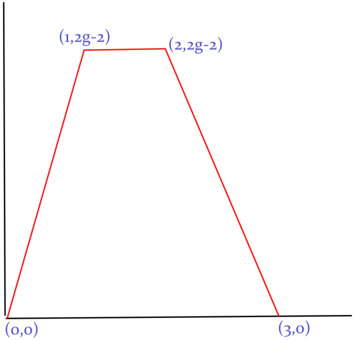
This is a convex polygon with line segments of integer slopes.

Examples

For $r = 2$, $\mathscr{P}_2^{oper}$ is



For $r = 3$, $\mathscr{P}_3^{oper}$ is



Oper Polygons are maximal

Theorem (Main Theorem V)

Let (V, ∇) *be a semistable local system of rank* r *and degree* 0. *Then*

1. $\mathscr{P}_V$ *lies on or below* $\mathscr{P}_r^{oper}$.
2. *And the equality*

$$\mathscr{P}_r^{oper} = \mathscr{P}_V$$

holds if and only if the triple $(V, \nabla, V_\bullet^{HN})$ *is an oper.*

- This theorem is, in some sense an analogue of Mazur's Theorem (Katz' conjecture for F-crystals).
- The oper polygon is a polygon with integer slopes and integer break points. It plays the role of Hodge polygon.
- The theorem is equivalent to a bunch of a complicated inequalities for slopes of $F^*(V)$.

Theorem (Main Theorem VI)

Let $r \geq 2$ be an integer and assume $p > C(r, g)$. Then we have

1. *Given a line bundle Q of degree $\deg(Q) = -(r-1)(g-1)$, the Quot-scheme $\mathrm{Quot}^{r,0}(F_*(Q))$ is non-empty and any vector bundle $W \in \mathrm{Quot}^{r,0}(F_*(Q))$ we have*
$$(F^*(W), \nabla^{can}) \qquad \text{with} \qquad \mathscr{P}_{F^*W} = \mathscr{P}_r^{oper},$$
i.e., the triple $(F^*(W), \nabla^{can}, (F^*(W))_\bullet^{HN})$ is a dormant oper.
2. *Conversely, any dormant oper of degree 0 is of the form $(F^*(W), \nabla^{can}, (F^*(W))_\bullet^{HN})$ with $W \in \mathrm{Quot}^{r,0}(F_*(Q))$ for some line bundle Q of degree $\deg(Q) = -(r-1)(g-1)$.*

As a corollary we deduce:

Corollary

Assume $p > C(g, r)$. Then the locus of semistable bundles V of degree zero with $\mathscr{P}_V = \mathscr{P}_r^{oper}$ is finite. This is the zero dimensional stratum of the instability locus $\mathcal{J}(r)$.

The $r = 2$ case

- In this case $C(2, g) = 0$, so results are valid for all $p \geq 2$,
- and $\mathcal{J}^s(2) = \mathcal{J}(2)$ as there are no strictly semistable rank-2 Frobenius-destabilized vector bundles.
- A formal consequence of results of S. Mochizuki, pointed out by B. Osserman, is that $\dim \mathcal{J}(2) = 2g - 4$ for a general curve X under the assumption $p > 2g - 2$.
- JRXY have shown that this is true for any X ($g \geq 2$) if $p = 2$.

As an application of our results on opers we obtain the following information on the locus of Frobenius-destabilized bundles $\mathcal{J}(2)$.

Theorem (Main Theorem VII)

For any X with $g \geq 2$ and $p \geq 2$, any irreducible component of $\mathcal{J}(2)$ containing a dormant oper has dimension $2g - 4$.

Dormant opers always exists and so there is at least one irreducible component which satisfies the conditions of the theorem.

The key estimate

The key technical tool in the the proof of Main Theorem II is the following:

Theorem (Key Estimate)

Let Q be a semistable vector bundle over the curve X. Let $\delta \in \mathbb{R}^{>0}$ and let n be a positive integer. Assume that $p > \frac{(n-1)(g-1)}{\delta}$. Then any subbundle $W \subset F_*(Q)$ of rank $\mathrm{rk}\,(W) \leq n$ has slope

$$\mu(W) < \frac{\mu(Q)}{p} + \delta.$$

Key Estimate $\Rightarrow$ Main Theorem II

Let V be a stable and Frobenius-destabilized vector bundle of rank r and slope $\mu(V) = \mu$. Consider the first quotient Q of the Harder-Narasimhan filtration of $F^*(V)$. So we have a stable Q such that

$$F^*(V) \to Q \qquad \text{and} \qquad p\mu = \mu(F^*(V)) > \mu(Q).$$

Moreover $\mathrm{rk}\,(Q) < \mathrm{rk}\,(V) = r$. By adjunction we obtain a non-zero map

$$V \to F_*(Q).$$

So to prove Main Theorem II it will suffice to prove that this map is injective.

Key Estimate ⇒ Main Theorem II

Suppose that this is not the case. Then the image of $V \to F_*(Q)$ generates a subbundle, say, $W \subset F_*(Q)$ and one has $1 \leq \mathrm{rk}\,(W) \leq r-1$ and by the stability of V, we have

$$\mu(V) = \mu < \mu(W).$$

Now we observe that we can bound $\mu(W)$ from below

$$\mu(W) \geq \mu + \frac{1}{r(r-1)} > \frac{\mu(Q)}{p} + \frac{1}{r(r-1)}.$$

The proof of Main Theorem II now follows from the Key Estimate with $\delta = \frac{1}{r(r-1)}$ and $n = r-1$.

Proof of the Key Estimate

Here are the important issues:

- The Key Estimate is a substantial strengthening of bounds for subbundles of $F_*(Q)$ proved (and improved) by various people starting with JRXY, Joshi, Lange-Pauly, Sun.
- Sun's bound are useful for proving stability of $F_*(Q)$ (for Q stable) but are not strong enough to prove stability of the subsheaves (of the sort which come up in proving injectivity of $V \hookrightarrow F_*(Q)$).
- The Proof of the Key Estimate uses the fact (combining result of JRXY and Sun) that $F^*(F_*(Q))$ carries a structure of a dormant oper in a critical way.

Proof of Main Theorem III: The degree zero case

- By Main Theorem II, every V of rank r and degree zero is a subsheaf of $F_*(Q)$ for some stable Q with $\mu(Q) < 0$.
- We observe that if $Q \subset Q'$ is a subsheaf then $F_*(Q) \subset F_*(Q')$, so to find a Q' of degree -1, starting with Q, we perform upper modifications on Q.
- This needs a delicate technical argument because a general modification will disturb stability of Q but we will not give its proof here.

Canonical sections for small p

The results of Joshi and Raynaud can be stated as follows:

Theorem

For $p > 2$, the bundle $\mathbb{P}(F^*(B^1_X))$ together with its Cartier connection and the Harder-Narasimhan filtration is a dormant oper.

Equivalently: The morphism $\mathrm{Nil}_{p-1} \to \mathfrak{M}_g$ has a canonical section whose image lies in the dormant locus.

Even for $p = 2$ we have a canonical section (this is implicit in JRXY). The prescription is as follows. For $p = 2$, B^1_X is a line bundle of degree $g - 1$ and is a canonical theta line bundle. Then $(F^*(F_*((B^1_X)^{-1}), \nabla^{can}, HN_\bullet)$ is a canonical dormant oper (on any genus $g \geq 2$ curve).
In contrast if $p > r + 1$ then there are no canonical sections of $\mathrm{Nil}_r \to \mathfrak{M}_g$.